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Research Article

Coincidence Theorems for Certain Classes of Hybrid Contractions

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Coincidence and fixed point theorems for a new class of hybrid contractions consisting of a pair of single-valued and multivalued maps on an arbitrary nonempty set with values in a metric space are proved. In addition, the existence of a common solution for certain class of functional equations arising in dynamic programming, under much weaker conditions are discussed. The results obtained here in generalize many well known results.

1. Introduction

Nadler's multivalued contraction theorem [1] (see also Covitz and Nadler, Jr. [2]) was subsequently generalized among others by Reich [3] and Ćirić [4]. For a fundamental development of fixed point theory for multivalued maps, one may refer to Rus [5]. Hybrid contractive conditions, that is, contractive conditions involving single-valued and multivalued maps are the further addition to metric fixed point theory and its applications. For a comprehensive survey of fundamental development of hybrid contractions and historical remarks, refer to Singh and Mishra [6] (see also Naimpally et al. [7] and Singh and Mishra [8]).

Recently Suzuki [9, Theorem 2] obtained a forceful generalization of the classical Banach contraction theorem in a remarkable way. Its further outcomes by Kikkawa and Suzuki [10, 11], Moţ and Petruşel [12] and Dhompongsa and Yingtaweesittikul [13], are important contributions to metric fixed point theory. Indeed, [10, Theorem 2] (see Theorem 2.1 below) presents an extension of [9, Theorem 2] and a generalization of the multivalued contraction theorem due to Nadler, Jr. [1]. In this paper we obtain a coincidence theorem (Theorem 3.1) for a pair of single-valued and multivalued maps on an arbitrary nonempty set with values in a metric space and derive fixed point theorems which generalize Theorem 2.1 and certain results of Reich [3], Zamfirescu [14], Moţ and Petruşel [12], and others. Further, using a corollary of Theorem 3.1, we obtain another fixed point theorem for multivalued maps. We also deduce the existence of a common solution for Suzuki-Zamfirescu type class of functional equations under much weaker contractive conditions than those in Bellman [15], Bellman and Lee [16], Bhakta and Mitra [17], Baskaran and Subrahmanyam [18], and Pathak et al. [19].

2. Suzuki-Zamfirescu Hybrid Contraction

For the sake of brevity, we follow the following notations, wherein P and T are maps to be defined specifically in a particular context while x, and y are the elements of specific domains:

$$
\begin{gathered}
M(P;x,y) = \left\{d(x,y), \frac{d(x,Px)+d(y,Py)}{2}, \frac{d(x,Py)+d(y,Px)}{2}\right\}, \\
M(P;Tx,Ty) = \left\{d(Tx,Ty), \frac{d(Tx,Px)+d(Ty,Py)}{2}, \frac{d(Tx,Py)+d(Ty,Px)}{2}\right\}, \\
m(P;x,y) = \left\{d(x,y), d(x,Px), d(y,Py), \frac{d(x,Py)+d(y,Px)}{2}\right\}.
\end{gathered}
\tag{2.1}
$$

Consistent with Nadler, Jr. [20, page 620], Y will denote an arbitrary nonempty set, (X,d) a metric space, and $CL(X)$ (resp. $CB(X)$) the collection of nonempty closed (resp., closed and bounded) subsets of X. For $A,B \in CL(X)$ and $\epsilon > 0$,

$$
\begin{gathered}
N(\epsilon, A) = \{x \in X : d(x,a) < \epsilon \text{ for some } a \in A\}, \\
E_{A,B} = \{\epsilon > 0 : A \subseteq N(\epsilon,B),\ B \subseteq N(\epsilon,A)\}, \\
H(A,B) = \begin{cases} \inf E_{A,B}, & \text{if } E_{A,B} \neq \phi \\ +\infty, & \text{if } E_{A,B} = \phi. \end{cases}
\end{gathered}
\tag{2.2}
$$

The hyperspace $(CL(X),\ H)$ is called the generalized Hausdorff metric space induced by the metric d on X.

For any subsets A,B of X, $d(A,B)$ denotes the ordinary distance between the subsets A and B, while

$$
\begin{gathered}
\rho(A,B) = \sup\{d(a,b) : a \in A, b \in B\}, \\
BN(X) = \left\{A : \phi \neq A \subseteq X \text{ and the diameter of } A \text{ is finite}\right\}.
\end{gathered}
\tag{2.3}
$$

As usual, we write $d(x,B)$ (resp., $\rho(x,B)$) for $d(A,B)$ (resp., $\rho(A,B)$) when $A = \{x\}$.

In all that follows η is a strictly decreasing function from $[0,1)$ onto $(1/2,1]$ defined by

$$\eta(r) = \frac{1}{1+r}. \tag{2.4}$$

Recently Kikkawa and Suzuki [10] obtained the following generalization of Nadler, Jr. [1].

Theorem 2.1. *Let (X,d) be a complete metric space and $P : X \to CB(X)$. Assume that there exists $r \in [0,1)$ such that*

(KSC) $\eta(r)d(x,Px) \le d(x,y)$ *implies* $H(Px,Py) \le rd(x,y)$

for all $x,y \in X$. Then P has a fixed point.

For the sake of brevity and proper reference, the assumption (KSC) will be called Kikkawa-Suzuki multivalued contraction.

Definition 2.2. Maps $P : Y \to CL(X)$ and $T : Y \to X$ are said to be Suzuki-Zamfirescu hybrid contraction if and only if there exists $r \in [0,1)$ such that

(S-Z) $\eta(r)d(Tx,Px) \le d(Tx,Ty)$ implies $H(Px,Py) \le r \cdot \max M(P;Tx,Ty)$

for all $x,y \in Y$.

A map $P : X \to CL(X)$ satisfying

(CG) $H(Px,Py) \le r \cdot \max m(P;x,y)$

for all $x,y \in X$, where $0 \le r < 1$, is called Ćirić-generalized contraction. Indeed, Ćirić [4] showed that a Ćirić generalized contraction has a fixed point in a P-orbitally complete metric space X.

It may be mentioned that in a comprehensive comparison of 25 contractive conditions for a single-valued map in a metric space, Rhoades [21] has shown that the conditions (CG) and (Z) are, respectively, the conditions (21′) and (19″) when P is a single-valued map, where

(Z) $H(Px,Py) \le r \cdot \max M(P;x,y)$ for all $x,y \in X$.

Obiviously, (Z) implies (CG). Further, Zamfirescu's condition [14] is equivalent to (Z) when P is single-valued (see Rhoades [21, pages 259 and 266]).

The following example indicates the importance of the condition (S-Z).

Example 2.3. Let $X = \{1,2,3\}$ be endowed with the usual metric and let P and T be defined by

$$Px = \begin{cases} 2,3 & \text{if } x \neq 3, \\ 3 & \text{if } x = 3, \end{cases}$$
$$Tx = \begin{cases} 1 & \text{if } x \neq 1, \\ 3 & \text{if } x = 1. \end{cases} \tag{2.5}$$

Then P does not satisfy the condition (KSC). Indeed, for $x = 2,\ y = 3$,

$$\eta(r)d(2, P2) = 0 \leq d(2, 3), \tag{2.6}$$

and this does not imply

$$1 = H(P2, P3) \leq d(2, 3) = r. \tag{2.7}$$

Further, as easily seen, P does not satisfy (CG) for $x = 2,\ y = 3$. However, it can be verified that the pair P and T satisfies the assumption (S-Z). Notice that P does not satisfy the condition (S-Z) when $Y = X$ and T is the identity map.

We will need the following definitions as well.

Definition 2.4 (see [4]). An orbit for $P : X \to CL(X)$ at $x_0 \in X$ is a sequence $\{x_n : x_n \in Px_{n-1}\},\ n = 1,\ 2, \ldots$. A space X is called P-orbitally complete if and only if every Cauchy sequence of the form $\{x_{n_i} : x_{n_i} \in Px_{n_i-1}\},\ i = 1,\ 2, \ldots$ converges in X.

Definition 2.5. Let $P : Y \to CL(X)$ and $T : Y \to X$. If for a point $x_0 \in Y$, there exists a sequence $\{x_n\}$ in Y such that $Tx_{n+1} \in Px_n,\ n = 0, 1, 2, \ldots$, then

$$O_T(x_0) = \{Tx_n : n = 1, 2, \ldots\} \tag{2.8}$$

is the orbit for (P, T) at x_0. We will use $O_T(x_0)$ as a set and a sequence as the situation demands. Further, a space X is (P, T)-orbitally complete if and only if every Cauchy sequence of the form $\{Tx_{n_i} : Tx_{n_i} \in Px_{n_i-1}\}$ converges in X.

As regards the existence of a sequence $\{Tx_n\}$ in the metric space X, the sufficient condition is that $P(Y) \subseteq T(Y)$. However, in the absence of this requirement, for some $x_0 \in Y$, a sequence $\{Tx_n\}$ may be constructed some times. For instance, in the above example, the range of P is not contained in the range of T, but we have the sequence $\{Tx_n\}$ for $x_0 = 2,\ x_1 = x_2 = \cdots = 1$. So we have the following definition.

Definition 2.6. If for a point $x_0 \in Y$, there exists a sequence $\{x_n\}$ in Y such that the sequence $O_T(x_0)$ converges in X, then X is called (P, T)-orbitally complete with respect to x_0 or simply (P, T, x_0)-orbitally complete.

We remark that Definitions 2.5 and 2.6 are essentially due to Rhoades et al. [22] when $Y = X$. In Definition 2.6, if $Y = X$ and T is the identity map on X, the (P, T, x_0)-orbital completeness will be denoted simply by (P, x_0)-orbitally complete.

Definition 2.7 ([23], see also [8]). Maps $P : X \to CL(X)$ and $T : X \to X$ are IT-commuting at $z \in X$ if $TPz \subseteq PTz$.

We remark that IT-commuting maps are more general than commuting maps, weakly commuting maps and weakly compatible maps at a point. Notice that if P is also single-valued, then their IT-commutativity and commutativity are the same.

3. Coincidence and Fixed Point Theorems

Theorem 3.1. *Assume that the pair of maps* $P : Y \to CL(X)$ *and* $T : Y \to X$ *is a Suzuki-Zamfirescu hybrid contraction such that* $P(Y) \subseteq T(Y)$*. If there exists an* $u_0 \in Y$ *such that* $T(Y)$ *is* (P, T, u_0)*-orbitally complete, then* P *and* T *have a coincidence point; that is, there exists* $z \in Y$ *such that* $Tz \in Pz$*.*

Further, if $Y = X$*, then* P *and* T *have a common fixed point provided that* P *and* T *are IT-commuting at* z *and* Tz *is a fixed point of* T*.*

Proof. Without any loss of generality, we may take $r > 0$ and T a nonconstant map. Let $q = r^{-1/2}$. Pick $u_0 \in Y$. We construct two sequences $\{u_n\} \subseteq Y$ and $\{y_n = Tu_n\} \subseteq T(Y)$ in the following manner. Since $P(Y) \subseteq T(Y)$, we take an element $u_1 \in Y$ such that $Tu_1 \in Pu_0$. Similarly, we choose $Tu_2 \in Pu_1$ such that

$$d(Tu_1, Tu_2) \le qH(Pu_0, Pu_1). \tag{3.1}$$

If $Tu_1 = Tu_2$, then $Tu_1 \in Pu_1$ and we are done as u_1 is a coincidence point of T and P. So we take $Tu_1 \ne Tu_2$. In an analogous manner, choose $Tu_3 \in Pu_2$ such that

$$d(Tu_2, Tu_3) \le qH(Pu_1, Pu_2). \tag{3.2}$$

If $Tu_2 = Tu_3$, then $Tu_2 \in Pu_2$ and we are done. So we take $Tu_2 \ne Tu_3$, and continue the process. Inductively, we construct sequences $\{u_n\}$ and $\{Tu_n\}$ such that $Tu_{n+2} \in Pu_{n+1}$, $Tu_{n+1} \ne Tu_{n+2}$ and

$$d(Tu_{n+1}, Tu_{n+2}) \le qH(Pu_n, Pu_{n+1}). \tag{3.3}$$

Now we see that

$$\eta(r)d(Tu_n, Pu_n) \le \eta(r)d(Tu_n, Tu_{n+1}) \le d(Tu_n, Tu_{n+1}). \tag{3.4}$$

Therefore by the condition (S-Z),

$$\begin{aligned} d(y_{n+1}, y_{n+2}) &\le qH(Pu_n, Pu_{n+1}) \\ &\le qr \cdot \max\left\{ d(Tu_n, Tu_{n+1}), \frac{d(Tu_n, Pu_n) + d(Tu_{n+1}, Pu_{n+1})}{2}, \right. \\ &\qquad\qquad \left. \frac{d(Tu_n, Pu_{n+1}) + d(Tu_{n+1}, Pu_n)}{2} \right\} \\ &\le qr \cdot \max\left\{ \begin{matrix} d(y_n, y_{n+1}), \dfrac{d(y_n, y_{n+1}) + d(y_{n+1}, y_{n+2})}{2}, \\ \dfrac{1}{2} d(y_n, y_{n+2}) \end{matrix} \right\}. \end{aligned} \tag{3.5}$$

This yields

$$d(y_{n+1}, y_{n+2}) \leq r_1 d(y_n, y_{n+1}), \tag{3.6}$$

where $r_1 = qr < 1$.

Therefore the sequence $\{y_n\}$ is Cauchy in $T(Y)$. Since $T(Y)$ is (P, T, u_0)-orbitally complete, it has a limit in $T(Y)$. Call it u. Let $z \in T^{-1}u$. Then $z \in Y$ and $u = Tz$.

Now as in [10], we show that

$$d(Tz, Px) \leq rd(Tz, Tx) \tag{3.7}$$

for any $Tx \in T(Y) - \{Tz\}$. Since $y_n \to Tz$, there exists a positive integer n_0 such that

$$d(Tz, Tu_n) \leq \frac{1}{3} d(Tz, Tx) \quad \forall n \geq n_0. \tag{3.8}$$

Therefore for $n \geq n_0$,

$$\begin{aligned} \eta(r) d(Tu_n, Pu_n) &\leq d(Tu_n, Pu_n) \leq d(Tu_n, Tu_{n+1}) \\ &\leq d(Tu_n, Tz) + d(Tu_{n+1}, Tz) \\ &\leq \frac{2}{3} d(Tz, Tx) = d(Tz, Tx) - \frac{1}{3} d(Tz, Tx) \\ &\leq d(Tz, Tx) - d(Tz, Tu_n) \leq d(Tu_n, Tx). \end{aligned} \tag{3.9}$$

Therefore by the condition (S-Z),

$$\begin{aligned} d(y_{n+1}, Px) &\leq H(Pu_n, Px) \\ &\leq r \cdot \max\left\{ d(y_n, Tx), \frac{d(y_n, Pu_n) + d(\mathrm{T}x, Px)}{2}, \frac{d(y_n, Px) + d(Tx, Pu_n)}{2} \right\} \\ &\leq r \cdot \max\left\{ d(y_n, Tx), \frac{d(y_n, y_{n+1}) + d(Tx, Px)}{2}, \frac{d(y_n, Px) + d(Tx, y_{n+1})}{2} \right\}. \end{aligned} \tag{3.10}$$

Making $n \to \infty$,

$$d(Tz, Px) \leq r \cdot \max\left\{ d(Tz, Tx), \frac{1}{2} d(Tx, Px), \frac{d(Tz, Px) + d(Tx, Tz)}{2} \right\}. \tag{3.11}$$

This yields (3.7); $Tx \neq Tz$.

Next we show that

$$H(Px, Pz) \leq r \cdot \max\left\{ d(Tx, Tz), \frac{d(Tx, Px) + d(Tz, Pz)}{2}, \frac{d(Tx, Pz) + d(Tz, Px)}{2} \right\} \tag{3.12}$$

for any $x \in Y$. If $x = z$, then it holds trivially. So we suppose $x \neq z$ such that $Tx \neq Tz$. Such a choice is permissible as T is not a constant map.

Therefore using (3.7),

$$\begin{aligned} d(Tx, Px) &\leq d(Tx, Tz) + d(Tz, Px) \\ &\leq d(Tx, Tz) + rd(Tx, Tz). \end{aligned} \tag{3.13}$$

Hence

$$\frac{1}{(1+r)} d(Tx, Px) \leq d(Tx, Tz). \tag{3.14}$$

This implies (3.12), and so

$$\begin{aligned} d(y_{n+1}, Pz) &\leq H(Pu_n, Pz) \\ &\leq r \cdot \max\left\{ d(Tu_n, Tz), \frac{d(Tu_n, Pu_n) + d(Tz, Pz)}{2}, \frac{d(Tu_n, Pz) + d(Tz, Pu_n)}{2} \right\} \\ &\leq r \cdot \max\left\{ d(y_n, Tz), \frac{d(y_n, y_{n+1}) + d(Tz, Pz)}{2}, \frac{d(y_n, Pz) + d(Tz, y_{n+1})}{2} \right\}. \end{aligned} \tag{3.15}$$

Making $n \to \infty$,

$$d(Tz, Pz) \leq rd(Tz, Pz). \tag{3.16}$$

So $Tz \in Pz$, since Pz is closed.

Further, if $Y = X$, $TTz = Tz$, and P, T are IT-commuting at z, that is, $TPz \subseteq PTz$, then $Tz \in Pz \Rightarrow TTz \in TPz \subseteq PTz$, and this proves that Tz is a fixed point of P. □

We remark that, in general, a pair of continuous commuting maps at their coincidences need not have a common fixed point unless T has a fixed point (see, e.g., [6–8]).

Corollary 3.2. *Let $P : X \to CL(X)$. Assume that there exists $r \in [0, 1)$ such that*

$$\eta(r) d(x, Px) \leq d(x, y) \quad \text{implies } H(Px, Py) \leq r \cdot \max M(P; x, y) \tag{3.17}$$

for all $x, y \in X$. If there exists a $u_0 \in X$ such that X is (P, u_0)-orbitally complete, then P has a fixed point.

Proof. It comes from Theorem 3.1 when $Y = X$ and T is the identity map on X. □

The following two results are the extensions of Suzuki [9, Theorem 2]. Corollary 3.3 also generalizes the results of Kikkawa and Suzuki [10, Theorem 3] and Jungck [24].

Corollary 3.3. *Let $f, T : Y \to X$ be such that $f(Y) \subseteq T(Y)$ and $T(Y)$ is an (f,T)-orbitally complete subspace of X. Assume that there exists $r \in [0,1)$ such that*

$$\eta(r)d(Tx, fx) \le d(Tx, Ty) \tag{3.18}$$

implies

$$d(fx, fy) \le r \cdot \max M(f; Tx, Ty) \tag{3.19}$$

for all $x, y \in Y$. Then f and T have a coincidence point; that is, there exists $z \in Y$ such that $fz = Tz$.

Further, if $Y = X$ and f and T commute at z, then f and T have a unique common fixed point.

Proof. Set $Px = \{fx\}$ for every $x \in Y$. Then it comes from Theorem 3.1 that there exists $z \in Y$ such that $fz = Tz$. Further, if $Y = X$ and f, and T commute at z, then $ffz = fTz = Tfz$. Also, $\eta(r)d(Tz, fz) = 0 \le d(Tz, Tfz)$, and this implies

$$\begin{aligned} d(fz, ffz) &\le r \cdot \max M(f; Tz, Tfz) \\ &= rd(fz, ffz). \end{aligned} \tag{3.20}$$

This yields that fz is a common fixed point of f and T. The uniqueness of the common fixed point follows easily. □

Corollary 3.4. *Let $f : X \to X$ be such that X is f-orbitally complete. Assume that there exists $r \in [0,1)$ such that*

$$\eta(r)d(x, fx) \le d(x, y) \text{ implies } d(fx, fy) \le r \cdot \max M(f; x, y) \tag{3.21}$$

for all $x, y \in X$. Then f has a unique fixed point.

Proof. It comes from Corollary 3.2 that f has a fixed point. The uniqueness of the fixed point follows easily. □

Theorem 3.5. *Let $P : Y \to BN(X)$ and $T : Y \to X$ be such that $P(Y) \subseteq T(Y)$ and let $T(Y)$ be (P,T)-orbitally complete. Assume that there exists $r \in [0,1)$ such that*

$$\eta(r)\rho(Tx, Px) \le d(Tx, Ty) \tag{3.22}$$

implies

$$\rho(Px, Py) \le r \cdot \max\left\{d(Tx, Ty), \frac{\rho(Tx, Px) + \rho(Ty, Py)}{2}, \frac{d(Tx, Py) + d(Ty, Px)}{2}\right\} \tag{3.23}$$

for all $x, y \in Y$. Then there exists $z \in Y$ such that $Tz \in Pz$.

Proof. Choose $\lambda \in (0,1)$. Define a single-valued map $f : Y \to X$ as follows. For each $x \in Y$, let fx be a point of Px, which satisfies

$$d(Tx, fx) \geq r^{\lambda}\rho(Tx, Px). \tag{3.24}$$

Since $fx \in Px, d(Tx, fx) \leq \rho(Tx, Px)$. So (3.22) gives

$$\eta(r)d(Tx, fx) \leq \eta(r)\rho(Tx, Px) \leq d(Tx, Ty), \tag{3.25}$$

and this implies (3.23). Therefore

$$\begin{aligned} d(fx, fy) &\leq \rho(Px, Py) \\ &\leq r \cdot r^{-\lambda} \cdot \max\left\{ r^{\lambda}d(Tx, Ty), \frac{r^{\lambda}\rho(Tx, Px) + r^{\lambda}\rho(Ty, Py)}{2}, \right. \\ &\qquad \left. \frac{r^{\lambda}d(Tx, Py) + r^{\lambda}d(Ty, Px)}{2} \right\} \\ &\leq r^{1-\lambda} \cdot \max\left\{ d(Tx, Ty), \frac{d(Tx, fx) + d(Ty, fy)}{2}, \frac{d(Tx, fy) + d(Ty, fx)}{2} \right\}. \end{aligned} \tag{3.26}$$

This means that Corollary 3.3 applies as

$$f(Y) = \cup\{fx \in Px\} \subseteq P(Y) \subseteq T(Y). \tag{3.27}$$

Hence f and T have a coincidence at $z \in Y$. Clearly $fz = Tz$ implies $Tz \in Pz$. □

Now we have the following.

Theorem 3.6. *Let $P : X \to BN(X)$ and let X be P-orbitally complete. Assume that there exists $r \in [0,1)$ such that $\eta(r)\rho(x, Px) \leq d(x, y)$ implies*

$$\rho(Px, Py) \leq r \cdot \max\left\{ d(x, y), \frac{\rho(x, Px) + \rho(y, Py)}{2}, \frac{d(x, Py) + d(y, Px)}{2} \right\} \tag{3.28}$$

for all $x, y \in X$. Then P has a unique fixed point.

Proof. For $\lambda \in (0,1)$, define a single-valued map $f : X \to X$ as follows. For each $x \in X$, let fx be a point of Px such that

$$d(x, fx) \geq r^{\lambda}\rho(x, Px). \tag{3.29}$$

Now following the proof technique of Theorem 3.5 and using Corollary 3.4, we conclude that f has a unique fixed point $z \in X$. Clearly $z = fz$ implies that $z \in Pz$. □

Now we close this section with the following.

Question 1. Can we replace Assumption (3.17) in Corollary 3.2 by the following:

$$\eta(r)d(x,Px) \leq d(x,y) \tag{3.30}$$

implies

$$H(Px,Py) \leq r \cdot \max\left\{d(x,y), d(x,Px), d(y,Py), \frac{1}{2}[d(x,Py)+d(y,Px)]\right\} \tag{3.31}$$

for all $x, y \in X$?

4. Applications

Throughout this section, we assume that U and V are Banach spaces, $W \subseteq U$, and $D \subseteq V$. Let $\mathbb{R}$ denote the field of reals, $\tau : W \times D \to W$, $g, g' : W \times D \to \mathbb{R}$, and $G, F : W \times D \times \mathbb{R} \to \mathbb{R}$. Viewing W and D as the state and decision spaces respectively, the problem of dynamic programming reduces to the problem of solving the functional equations:

$$p := \sup_{y \in D}\{g(x,y) + G(x,y,p(\tau(x,y)))\}, \quad x \in W, \tag{4.1}$$

$$q := \sup_{y \in D}\{g'(x,y) + F(x,y,q(\tau(x,y)))\}, \quad x \in W. \tag{4.2}$$

In the multistage process, some functional equations arise in a natural way (cf. Bellman [15] and Bellman and Lee [16]); see also [17–19, 25]). In this section, we study the existence of the common solution of the functional equations (4.1), (4.2) arising in dynamic programming.

Let $B(W)$ denote the set of all bounded real-valued functions on W. For an arbitrary $h \in B(W)$, define $\|h\| = \sup_{x \in W}|h(x)|$. Then $(B(W), \|\cdot\|)$ is a Banach space. Suppose that the following conditions hold:

(DP-1) G, F, g and g' are bounded.

(DP-2) Let η be defined as in the previous section. There exists $r \in [0,1)$ such that for every $(x,y) \in W \times D$, $h, k \in B(W)$ and $t \in W$,

$$\eta(r)|Kh(t) - Jh(t)| \leq |Jh(t) - Jk(t)| \tag{4.3}$$

implies

$$\begin{aligned} &\left|G(x,y,h(t)) - G(x,y,k(t))\right| \\ &\quad \leq r \cdot \max\left\{|Jh(t) - Jk(t)|, \frac{|Jh(t) - Kh(t)| + |Jk(t) - Kk(t)|}{2},\right. \\ &\qquad \left.\frac{|Jh(t) - Kk(t)| + |Jk(t) - Kh(t)|}{2}\right\}, \end{aligned} \tag{4.4}$$

where K and J are defined as follows:

$$Kh(x) = \sup_{y \in D}\{g(x,y) + G(x,y,h(\tau(x,y)))\}, \quad x \in W,\ h \in B(W), \tag{$*$}$$

$$Jh(x) = \sup_{y \in D}\{g'(x,y) + F(x,y,h(\tau(x,y)))\}, \quad x \in W,\ h \in B(W). \tag{4.5}$$

(DP-3) For any $h \in B(W)$, there exists $k \in B(W)$ such that

$$Kh(x) = Jk(x), \quad x \in W. \tag{4.6}$$

(DP-4) There exists $h \in B(W)$ such that

$$Jh(x) = Kh(x) \quad \text{implies } JKh(x) = KJh(x). \tag{4.7}$$

Theorem 4.1. *Assume that the conditions (DP-1)–(DP-4) are satisfied. If $J(B(W))$ is a closed convex subspace of $B(W)$, then the functional equations* (4.1) *and* (4.2) *have a unique common bounded solution.*

Proof. Notice that $(B(W), d)$ is a complete metric space, where d is the metric induced by the supremum norm on $B(W)$. By (DP-1), J and K are self-maps of $B(W)$. The condition (DP-3) implies that $K(B(W)) \subseteq J(B(W))$. It follows from (DP-4) that J and K commute at their coincidence points.

Let λ be an arbitrary positive number and $h_1, h_2 \in B(W)$. Pick $x \in W$ and choose $y_1, y_2 \in D$ such that

$$Kh_j < g(x,y_j) + G(x,y_j,h_j(x_j)) + \lambda, \tag{4.8}$$

where $x_j = \tau(x,y_j)$, $j = 1,2$.

Further,

$$Kh_1(x) \geq g(x,y_2) + G(x,y_2,h_1(x_2)), \tag{4.9}$$

$$Kh_2(x) \geq g(x,y_1) + G(x,y_1,h_2(x_1)). \tag{4.10}$$

Therefore, the first inequality in (DP-2) becomes

$$\eta(r)|Kh_1(x) - Jh_1(x)| \leq |Jh_1(x) - Jh_2(x)|, \tag{4.11}$$

and this together with (4.8) and (4.10) implies

$$\begin{aligned} Kh_1(x) - Kh_2(x) &< G(x, y_1, h_1(x_1)) - G(x, y_1, h_2(x_1)) + \lambda \\ &\le \left| G(x, y_1, h_1(x_1)) - G(x, y_1, h_2(x_1)) \right| + \lambda \\ &\le r \cdot \max M(K; Jh_1, Jh_2) + \lambda. \end{aligned} \tag{4.12}$$

Similarly, (4.8), (4.9), and (4.11) imply

$$Kh_2(x) - Kh_1(x) \le r \cdot \max M(K; Jh_1, Jh_2) + \lambda. \tag{4.13}$$

So, from (4.12) and (4.13), we have

$$|Kh_1(x) - Kh_2(x)| \le r \cdot \max M(K; Jh_1, Jh_2) + \lambda. \tag{4.14}$$

Since the above inequality is true for any $x \in W$, and $\lambda > 0$ is arbitrary, we find from (4.17) that

$$\eta(r) d(Kh_1, Jh_1) \le d(Jh_1, Jh_2) \tag{4.15}$$

implies

$$d(Kh_1, Kh_2) \le r \cdot \max M(K; Jh_1, Jh_2). \tag{4.16}$$

Therefore Corollary 3.3 applies, wherein K and J correspond, respectively, to the maps f and T, Therefore, K and J have a unique common fixed point h^*, that is, $h^*(x)$ is the unique bounded common solution of the functional equations (4.1) and (4.2). □

Corollary 4.2. *Suppose that the following conditions hold.*

(i) *G and g are bounded.*

(ii) *For η defined earlier (cf. (DP-2) above), there exists $r \in [0, 1)$ such that for every $(x, y) \in W \times D$, $h, k \in B(W)$ and $t \in W$,*

$$\eta(r) |h(t) - Kh(t)| \le |h(t) - k(t)| \tag{4.17}$$

implies

$$\left| G(x, y, h(t)) - G(x, y, k(t)) \right| \le r \cdot \max M(K; h(t), k(t)), \tag{4.18}$$

where K is defined by $(*)$. Then the functional equation (4.1) *possesses a unique bounded solution in W.*

Proof. It comes from Theorem 4.1 when $q = p$, $F = G$, and $g = g'$ as the conditions (DP-3) and (DP-4) become redundant in the present context. □

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