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Research Article

ε-Optimal Solutions in Nonconvex Semi-Infinite Programs with Support Functions

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Approximate optimality conditions for a class of nonconvex semi-infinite programs involving support functions are given. The objective function and the constraint functions are locally Lipschitz functions on $\mathbb{R}^n$. By using a Karush-Kuhn-Tucker (KKT) condition, we deduce a necessary optimality condition for local approximate solutions. Then, generalized KKT conditions for the problems are proposed. Based on properties of ε-semiconvexity and semiconvexity applied to locally Lipschitz functions and generalized KKT conditions, we establish sufficient optimality conditions for another kind of local approximate solutions of the problems. Obtained results in case of nonconvex semi-infinite programs and nonconvex infinite programs are discussed.

1. Introduction

There were several papers concerning approximate solutions of convex/nonconvex problems published over years such as [1–10]. Recently, optimization problems which have a number of infinite constraints were considered in several papers such as [9–15]. In particular, approximate optimality conditions of nonconvex problems with infinite constraints were investigated in [9, 10]. On the other side, finite optimization problems which have objective functions involving support functions also attract several authors such as [16–23].

In this paper we deal with approximate optimality conditions of a class of nonconvex optimization problems which have objective functions containing support functions and have a number of infinite constraints. We consider the following semi-infinite programming problem:

$$\begin{aligned} &\text{Minimize} && f(x)+s(x\mid D)\\ &\text{subject to} && f_t(x)\le 0,\ t\in T,\\ &&& x\in C, \end{aligned} \tag{P}$$

where $f, f_t : X \to \mathbb{R}, t \in T$, are locally Lipschitz functions, X is a normed space, T is an index set (possibly in infinite), C and D are nonempty closed convex subsets of X, and $s(\cdot \mid D)$ is support function corresponding to D. In the case of $X = \mathbb{R}^n$, T is finite, the convex set C is suppressed, and the functions involved are continuously differentiable, the problem (P) becomes the one considered in [16, 17]. In case X is a Banach space and $s(\cdot \mid D)$ is suppressed, the problem (P) becomes the one considered recently in [10].

Our results on approximate optimality conditions in this paper are established based on properties of ϵ-semiconvexity and of semiconvexity applied to locally Lipschitz functions proposed by Loridan [1] and Mifflin [24], respectively (the property of ϵ-semiconvexity is an extension of the one of semiconvexity), and based on the calculus rules of subdifferentials of nonconvex functions introduced in a well-known book of Clarke [25]. We focus on sufficient optimality conditions for a kind of locally approximate solutions. Concretely, we deal with *almost ϵ-quasisolutions* of (P). Recently, there were several papers dealed with ϵ-quasisolutions or almost ϵ-quasisolutions [3, 7, 9, 10, 26]. While an ϵ-solution has a global property, an ϵ-quasisolution has a local one. Naturally, it is suitable for nonconvex problems. On the other hand, we can see that the concept of almost ϵ-quasisolutions introduced by Loridan (see [1]) is relaxed from the one of ϵ-quasisolutions when we expand a feasible set of an optimization problem to an ϵ-feasible set.

We now describe the content of the paper. In the preliminaries, besides basic concepts, we recall definitions of several kinds of approximate solutions of (P) and an necessary optimality condition for obtaining exact solutions of nonconvex infinite problems. Applying this result into the case of a finite setting space, in Section 3, we deduce a necessary optimality condition of a kind of approximate solutions of (P), *ϵ-quasisolution*. Then a concept of generalized Karush-Kuhn-Tucker pair up to ϵ is presented. Our main results are stated by three sufficient optimality conditions for another kind of approximate solutions of (P), *almost ϵ-quasisolution* (see Definition 2.7 in Section 2). Section 4 is devoted to discuss approximate sufficient optimality conditions for (P) in the case the support function is suppressed. Several sufficient conditions for almost ϵ-quasisolutions of nonconvex semi-infinite programs are given. Concerning the class of nonconvex infinite programs considered in [10], we also state that some new versions of sufficient optimality conditions for approximate solutions of the problems can be established.

2. Preliminaries

Let $f : X \to \mathbb{R}$ be a locally Lipschitz function at $x \in X$, where X is a Banach space. The *generalized directional derivative* of f at x in the direction $d \in X$ (see [25, page 25]) is defined by

$$f^\circ(x; d) := \limsup_{\substack{h\to 0\\ \theta\downarrow 0}} \frac{f(x+h+\theta d) - f(x+h)}{\theta}, \tag{2.1}$$

and the *Clarke's subdifferential* of f at x, denoted by $\partial^c f(x)$, is

$$\partial^c f(x) := \{u \in X^* \mid \langle u, d\rangle \le f^\circ(x; d),\ \forall d \in X\}, \tag{2.2}$$

where X^* denotes the dual of X. When f is convex, $\partial^c f(x)$ coincides with $\partial f(x)$, the subdifferential of f at x, in the sense of convex analysis. If the limit

$$\lim_{\theta \downarrow 0} \frac{f(x+\theta d) - f(x)}{\theta} \tag{2.3}$$

exists for $d \in X$ then it is called the *directional derivative* of f at x in direction d and it is denoted by $f'(x; d)$. The function f is said to be *quasidifferentiable* or *regular* (in the sense of Clarke [25]) at x if $f'(x; d)$ exists and $f'(x; d) = f^\circ(x; d)$ for every $d \in X$.

For a closed subset D of X, the Clarke tangent cone to D is defined by

$$T(x) = \{v \in X \mid d_D^\circ(x; v) = 0\}, \tag{2.4}$$

where d_D denotes the distance function to D (see [25, page 11]) and $d_D^\circ(x; v)$ is the generalized directional derivative of d_D at x in direction v. The normal cone to D is defined by

$$N_D(x) = \{x^* \in X^* \mid \langle x^*, v\rangle \le 0,\ \forall v \in T_C(x)\}. \tag{2.5}$$

If D is convex, then the normal cone to D coincides with the one in the sense of convex analysis, that is,

$$N_D(x) = \{x^* \in X^* \mid \langle x^*, y - x\rangle \le 0,\ \forall y \in D\}. \tag{2.6}$$

Let us denote by $\mathbb{R}^{(T)}$ the linear space of generalized finite sequences $\lambda = (\lambda)_{t\in T}$ such that $\lambda_t \in \mathbb{R}$ for all $t \in T$ but only finitely many $\lambda_t \neq 0$,

$$\mathbb{R}^{(T)} := \{\lambda = (\lambda_t)_{t\in T} \mid \lambda_t = 0,\ \forall\ t \in T \text{ but only finitely many } \lambda_t \neq 0\}. \tag{2.7}$$

For each $\lambda \in \mathbb{R}^{(T)}$, the corresponding supporting set $T(\lambda) := \{t \in T \mid \lambda_t \neq 0\}$ is a finite subset of T. We denote the nonnegative cone of $\mathbb{R}^{(T)}$ by

$$\mathbb{R}_+^{(T)} := \left\{\lambda = (\lambda_t)_{t\in T} \in \mathbb{R}^{(T)} \mid \lambda_t \ge 0, t \in T\right\}. \tag{2.8}$$

It is easy to see that this cone is convex. For $\lambda \in \mathbb{R}^{(T)}$, $\{z_t\}_{t\in T} \subset Z$, Z being a real linear space and the sequence $(f_t)_t$, $t \in T$, we understand that

$$\sum_{t\in T}\lambda_t z_t = \begin{cases} \sum_{t\in T(\lambda)}\lambda_t z_t & \text{if } T(\lambda) \neq \emptyset, \\ 0 & \text{if } T(\lambda) = \emptyset, \end{cases}$$
$$\sum_{t\in T}\lambda_t f_t = \begin{cases} \sum_{t\in T(\lambda)}\lambda_t f_t & \text{if } T(\lambda) \neq \emptyset, \\ 0 & \text{if } T(\lambda) = \emptyset. \end{cases} \tag{2.9}$$

We now recall necessary optimality condition for a class of nonconvex infinite problems with a Banach setting space. Let us consider the following problem:

$$\begin{aligned} &\text{Minimize} && f(x) \\ &\text{subject to} && f_t(x) \leq 0, t \in T, \\ & && x \in C, \end{aligned} \tag{Q}$$

where $f, f_t : X \to \mathbb{R}$, $t \in T$, are locally Lipschitz on a Banach space X and C is a closed convex subset of X.

We denote by $(\mathcal{A})$ the fact that at least one of the following conditions is satisfied:

(a1) X is separable, or

(a2) T is metrizable and $\partial^c f_t(x)$ is upper semicontinuous (w^*) in $t \in T$ for each $x \in X$.

In the following proposition, $\overline{\text{co}}(\cdot)$ denotes a closed convex hull with the closure taken in the weak* topology of the dual space.

Proposition 2.1 ([10, Proposition 2.1]). *Let x be a feasible point of (Q), and let $I(x) = \{t \in T \mid f_t(x) = 0\}$. Suppose that the condition $(\mathcal{A})$ holds. If the following condition is satisfied:*

$$\exists d \in T_C(x) : f_t^\circ(x; d) < 0, \quad \forall t \in I(x), \tag{2.10}$$

then

$$x \text{ is a local solution of } (Q) \Longrightarrow 0 \in \partial^c h(x) + \mathbb{R}_+ \overline{\text{co}}\left\{\cup\, \partial^c f_t(x) \mid t \in I(x)\right\} + N_C(x). \tag{2.11}$$

In order to obtain results in the next sections, we need the following preliminary concept and results with $X = \mathbb{R}^n$. Let C be a nonempty closed convex subset of X. The support function $s(\cdot \mid C) : X \to \mathbb{R}$ is defined by

$$s(x \mid C) := \max\left\{x^T y \mid y \in C\right\}. \tag{2.12}$$

Its subdifferential is given by

$$\partial s(x \mid C) := \left\{z \in C \mid z^T x = s(x \mid C)\right\}. \tag{2.13}$$

It is easy to see that $s(\cdot \mid C)$ is convex and finite everywhere. Since $s(\cdot \mid C)$ is a Lipschitz function with Lipschitz rate K, where $K = \sup\{\|v\|, v \in C\}$, we can show that it is a regular function by using Proposition 2.3.6 of [25]. The normal cone to C at $x \in C$ is

$$N_C(x) := \left\{ y \in \mathbb{R}^n \mid y^T(z - x) \leq 0,\ \forall z \in C \right\}. \tag{2.14}$$

In this case we can verify that

$$\begin{aligned} y \in N_C(x) &\iff s(y \mid C) = x^T y \\ &\iff x \in \partial s(y \mid C). \end{aligned} \tag{2.15}$$

Definition 2.2 (see [24]). Let C be a subset of X. A function $f : X \to \mathbb{R}$ is said to be *semiconvex at* $x \in C$ if it is locally Lipschitz at x, quasidifferentiable at x, and satisfies the following condition:

$$\left(d \in X, x + d \in C, f'(x; d) \geq 0\right) \Longrightarrow f(x + d) \geq f(x). \tag{2.16}$$

The function f is said to be *semiconvex on* C if f is semiconvex at every point $x \in C$.

It is easy to verify that if a locally Lipschitz function f is semiconvex at $x \in C$ and there exists $u \in \partial^c f(x)$ such that $\langle u, z - x\rangle \geq 0$, then $f(z) \geq f(x)$.

Lemma 2.3 (see [24, Theorem 8]). *Suppose that f is semiconvex on a convex set $C \subset X$. Then, for $x \in C$ and $x + d \in C$ with $d \in X$,*

$$f(x + d) \leq f(x) \Longrightarrow f'(x; d) \leq 0. \tag{2.17}$$

The notion of semiconvexity presented in [24] was used in several papers such as [10, 14, 27]. We also note that Definition 2.2 and/or Lemma 2.3 utilized in the papers above with X a Banach space or a reflexive Banach space. We now recall an extension of this notion called ϵ-semiconvexity.

Definition 2.4 (see [1]). Let C be a subset of X, and let $\epsilon \geq 0$. A function $f : X \to \mathbb{R}$ is said to be *ϵ-semiconvex at* $x \in C$ if it is locally Lipschitz at x, regular at x, and satisfies the following condition:

$$\left(d \in X, x + d \in C, f'(x; d) + \sqrt{\epsilon}\|d\| \geq 0\right) \Longrightarrow f(x + d) + \sqrt{\epsilon}\|d\| \geq f(x). \tag{2.18}$$

The function f is said to be *ϵ-semiconvex on* C if f is ϵ-semiconvex at every point $x \in C$.

Remark 2.5. It is worth mentioning that a convex function on X is the ϵ-semiconvex function with respect to X for any $\epsilon \geq 0$ (see [1, 3, 12]). When $\epsilon = 0$, this concept coincides with the semiconvexity defined by Mifflin [24].

We now concern with concepts of approximate solution. The most common concept of an approximate solution of a function f from X to $\mathbb{R}$ is that of an ϵ-solution, that is, the function f satisfies the following inequality:

$$f(z) \leq f(x) + \epsilon, \quad \forall x \in X, \tag{2.19}$$

where $\epsilon \geq 0$ is a given number. This concept is used usually for approximate minimum of a convex function. For nonconvex functions, it is suitable for concepts of approximate local minimums. We deal with ϵ-quasisolutions. A point z is an ϵ-quasisolution of f on X if z is a solution of the function $x \mapsto f(x) + \sqrt{\epsilon}\|x - z\|$. In this case, if x belongs to a ball B around z with the radius is less or equal to $\sqrt{\epsilon}$, then we have $f(z) \leq f(x) + \epsilon$. So, we can see that an ϵ-quasisolution is a local ϵ-solution. We recall several definitions of approximate solutions of a function f defined on a subset of X. Consider the problem (R) given by

$$\begin{aligned} &\text{Minimize} \quad && f(x) \\ &\text{subject to} \quad && x \in S, \end{aligned} \tag{R}$$

where $f : X \to \mathbb{R}$ and S is a subset of X.

Definition 2.6. Let $\epsilon \geq 0$. A point $z \in S$ is said to be

(i) an ϵ-solution of (R) if $f(z) \leq f(x) + \epsilon$ for all $x \in S$,

(ii) an ϵ-quasisolution of (R) if $f(z) \leq f(x) + \sqrt{\epsilon}\|x - z\|$ for all $x \in S$,

(iii) a regular ϵ-solution of (R) if it is an ϵ-solution and an ϵ-quasisolution of (R).

Denote by S the feasible set of (P), $S := \{x \in C \mid f_t(x) \leq 0, \ \forall t \in T\}$. Set $S_\epsilon := \{x \in C \mid f_t(x) \leq \sqrt{\epsilon}, \ \forall t \in T\}$ with $\epsilon \geq 0$. S_ϵ is called an ϵ-feasible set of (P).

Definition 2.7. Let $\epsilon \geq 0$. A point $z \in S_\epsilon$ is said to be

(i) an almost ϵ-solution of (R) if $f(z) \leq f(x) + \epsilon$ for all $x \in S$,

(ii) an almost ϵ-quasisolution of (R) if $f(z) \leq f(x) + \sqrt{\epsilon}\|x - z\|$ for all $x \in S$,

(iii) an almost regular ϵ-solution of (R) if it is an almost ϵ-solution and an almost ϵ-quasisolution of (R).

Throughout the paper, $X = \mathbb{R}^n$, T is a compact topological space, $f : X \to \mathbb{R}$ is locally Lipschitz function, and $f_t : X \to \mathbb{R}$, $t \in T$, are locally Lipschitz function with respect to x uniformly in t, that is,

$$\forall x \in X, \quad \exists U(x), \quad \exists K > 0, \quad \left| f_t(u) - f_t(v) \right| \leq K\|u - v\|, \quad \forall u, v \in U(x), \ \forall t \in T. \tag{2.20}$$

3. Approximate Optimality Conditions

In this section, several approximate optimality conditions will be established based on concepts of ϵ-semiconvexity and semiconvexity applied to locally Lipschitz functions. Firstly, we need to introduce a necessary condition for ϵ-quasisolution of (P).

Theorem 3.1. *Let $\epsilon \geq 0$, and let z_ϵ be an ϵ-quasisolution of (P). If the assumption (2.10) is satisfied corresponding to z_ϵ, then there exist $\lambda \in \mathbb{R}_+^{(T)}$, $v \in D$ such that $\langle v, z_\epsilon \rangle = s(z_\epsilon \mid D)$ and*

$$-v \in \partial^c f(z_\epsilon) + \sum_{t\in T} \lambda_t \partial^c f_t(z_\epsilon) + N_C(z_\epsilon) + \sqrt{\epsilon}B^*, \quad f_t(z_\epsilon) = 0, \ \forall t \in T(\lambda). \tag{3.1}$$

Proof. Let $h(x) := f(x) + s(x \mid D)$. It is easy to see that h is locally Lipschitz since f is locally Lipschitz and $s(\cdot \mid D)$ is Lipschitz with Lipschitz rate $K = \sup\{\|v\|, v \in D\}$. Since $X = \mathbb{R}^n$, X is separable. So, the condition $(\mathcal{A})$ is fulfilled. Let $\epsilon \geq 0$. Suppose that z_ϵ is an ϵ-quasisolution of (P). Set

$$h_1(x) = h(x) + \sqrt{\epsilon}\|x - z_\epsilon\|. \tag{3.2}$$

It is obvious that z_ϵ is an exact solution of the following problem:

$$\begin{aligned} &\text{Minimize} && h_1(x) \\ &\text{subject to} && x \in S, \end{aligned} \tag{3.3}$$

where S is the feasible set of (P). Since the assumption (2.10) is satisfied for z_ϵ then, by applying Proposition 2.1, we obtain

$$0 \in \partial^c h_1(z_\epsilon) + \mathbb{R}_+ \overline{\text{co}}\{\cup\, \partial^c f_t(z_\epsilon) \mid t \in I(z_\epsilon)\} + N_C(z_\epsilon), \tag{3.4}$$

where $I(z_\epsilon) = \{t \in T \mid f_t(z_\epsilon) = 0\}$. Note that

$$\partial^c(h_1)(z_\epsilon) = \partial^c\big(f + \sqrt{\epsilon}\|\cdot - z_\epsilon\|\big)(z_\epsilon) \subset \partial^c f(z_\epsilon) + \sqrt{\epsilon}B^*. \tag{3.5}$$

Since X is a finite dimensional space, the set $\{\cup\partial^c f_t(z_\epsilon) \mid t \in I(z_\epsilon)\}$ is compact, and, consequently, its convex hull $\text{co}\{\cup\partial^c f_t(z_\epsilon) \mid t \in I(z_\epsilon)\}$ is closed. Moreover, by the convexity property of the function $s(\cdot \mid D)$, we get $\partial^c s(\cdot \mid D)(z_\epsilon) = \partial s(\cdot \mid D)(z_\epsilon)$. Hence, from (3.4), we obtain

$$\begin{gathered} 0 \in \partial^c f(z_\epsilon) + \partial s(\cdot \mid D)(z_\epsilon) + \sum_{t\in T} \lambda_t \partial^c f_t(z_\epsilon) + N_C(z_\epsilon) + \sqrt{\epsilon}B^*, \\ f_t(z_\epsilon) = 0, \quad \forall t \in T(\lambda). \end{gathered} \tag{3.6}$$

Furthermore, by (2.13), $v \in \partial s(\cdot \mid D)(z_\epsilon)$ is equivalent to the fact that $v \in D$ and $\langle v, z_\epsilon \rangle = s(z_\epsilon \mid D)$. Consequently,

$$-v \in \partial^c f(z_\epsilon) + \sum_{t\in T} \lambda_t \partial^c f_t(z_\epsilon) + N_C(z_\epsilon) + \sqrt{\epsilon}B^*, f_t(z_\epsilon) = 0, \quad \forall t \in T(\lambda), \tag{3.7}$$

where $\langle v, z_\epsilon \rangle = s(z_\epsilon \mid D)$. We obtain the desired conclusion. □

Condition (3.1) with $z_\epsilon \in S$ may be strict. We expand the set S to the ϵ-feasible set, S_ϵ, and give a definition for an approximate generalized Karush-Kuhn-Tucker (KKT) pair as follows.

Definition 3.2. Let $\epsilon \geq 0$. A pair $(z_\epsilon, \lambda) \in S_\epsilon \times \mathbb{R}_+^{(T)}$ is called a generalized Karush-Kuhn-Tucker (KKT) pair up to ϵ corresponding to (P) if the following condition is satisfied:

$$\text{(KKT)}: \quad -v \in \partial^c f(z_\epsilon) + \sum_{t\in T} \lambda_t \partial^c f_t(z_\epsilon) + N_C(z_\epsilon) + \sqrt{\epsilon} B^*, \quad f_t(z_\epsilon) \geq 0, \forall t \in T(\lambda), \tag{3.8}$$

where $v \in D$ and $\langle v, z_\epsilon \rangle = s(z_\epsilon \mid D)$.

The pair is called strict if $f_t(z_\epsilon) > 0$ for all $t \in T(\lambda)$, equivalently, $\lambda_t = 0$ if $f_t(z_\epsilon) \leq 0$.

To show that the definition above is reasonable, we need to show that there exists generalized KKT pair for (P). This work is done following the idea of Theorem 4.2 in [10].

Lemma 3.3. *Let $\epsilon > 0$. There exists an almost regular ϵ-solution z_ϵ for (P) and $\lambda \in \mathbb{R}_+^{(T)}$ such that (z_ϵ, λ) is a strict generalized KKT pair up to ϵ.*

Proof. Firstly, we note that the space $\mathbb{R}^n$ is separable, and, for every $x \in S_\epsilon$, the set $\{\cup \partial^c f_t(x) \mid t \in I(x)\}$ is compact. Consequently, the convex hull $\text{co}\{\cup \partial^c f_t(x) \mid t \in I(x)\}$ is closed. By applying Theorem 4.2 in [10], there exists an almost regular ϵ-solution z_ϵ for (P) and $\lambda \in \mathbb{R}_+^{(T)}$ such that (z_ϵ, λ) satisfy the following condition:

$$0 \in \partial^c h(z_\epsilon) + \sum_{t\in T} \lambda_t \partial^c f_t(z_\epsilon) + N_C(z_\epsilon) + \sqrt{\epsilon} B^* \tag{3.9}$$

with $f_t(z_\epsilon) > 0$ for all $t \in T(\lambda)$, where $h = f + s(\cdot \mid D)$. Hence, we obtain the desired result by noting that

$$\partial^c(h)(z_\epsilon) \subset \partial^c f(z_\epsilon) + \partial s(\cdot \mid D)(z_\epsilon), \tag{3.10}$$

and $v \in \partial s(\cdot \mid D)(z_\epsilon)$ is equivalent to $v \in D$ and $\langle v, z_\epsilon \rangle = s(z_\epsilon \mid D)$. □

We now are at position to give some sufficient conditions for almost ϵ-quasisolutions of (P).

Theorem 3.4. *Let $\epsilon \geq 0$, and let $(z_\epsilon, \lambda) \in S_\epsilon \times \mathbb{R}_+^{(T)}$ satisfy condition (3.8). Suppose that $f, f_t, t \in T$, are quasidifferentiable at z_ϵ. If $f + s(\cdot \mid D) + \sum_{t\in T} \lambda_t f_t$ is ϵ-semiconvex at z_ϵ, then z_ϵ is an almost ϵ-quasisolution of (P).*

Proof. Suppose that $(z_\epsilon, \lambda) \in S_\epsilon \times \mathbb{R}_+^{(T)}$ satisfies condition (3.8). Then there exist $u \in \partial^c f(z_\epsilon)$, $v \in \partial s(z_\epsilon \mid D)$, $w_t \in \partial^c f_t(z_\epsilon)$, $t \in T$, $r \in B^*$, and $w \in N_C(z_\epsilon)$ such that

$$u + v + \sum_{t\in T} \lambda_t w_t + \sqrt{\epsilon} r = -w. \tag{3.11}$$

Since $-w(x - z_\epsilon) \geq 0$ for all $x \in C$,

$$\left(u + v + \sum_{t\in T}\lambda_t w_t\right)(x - z_\epsilon) + \sqrt{\epsilon}\|x - z_\epsilon\| \geq 0, \quad \forall x \in C. \tag{3.12}$$

Since $f, f_t, t \in T$, are quasidifferentiable and $s(\cdot \mid D)$ is also quasidifferentiable (discussed above),

$$\left(u + v + \sum_{t\in T}\lambda_t w_t\right) \in \partial^c\left(f + s(\cdot \mid D) + \sum_{t\in T}\lambda_t f_t\right)(z_\epsilon). \tag{3.13}$$

Since $f + s(\cdot \mid D) + \sum_{t\in T}\lambda_t f_t$ is ϵ-semiconvex at z_ϵ, from (3.12), we deduce that

$$\left(f + s(\cdot \mid D) + \sum_{t\in T}\lambda_t f_t\right)(x) + \sqrt{\epsilon}\|x - z_\epsilon\| \geq \left(f + s(\cdot \mid D) + \sum_{t\in T}\lambda_t f_t\right)(z_\epsilon), \quad \forall x \in C. \tag{3.14}$$

When $x \in S$, we have $f_t(x) \leq 0$ for all $t \in T$. Furthermore, since $(z_\epsilon, \lambda) \in S_\epsilon \times \mathbb{R}_+^{(T)}$ satisfies condition (3.8), $f_t(z_\epsilon) \geq 0$ for all $t \in T(\lambda)$. These, together with the inequality above, imply that

$$f(x) + s(x \mid D) + \sqrt{\epsilon}\|x - z_\epsilon\| \geq f(z_\epsilon) + s(z_\epsilon \mid D), \quad \forall x \in S. \tag{3.15}$$

Since $z_\epsilon \in S_\epsilon$, z_ϵ is an almost ϵ-quasisolution of (P). □

Theorem 3.5. *Let $\epsilon \geq 0$, and let $(z_\epsilon, \lambda) \in S_\epsilon \times \mathbb{R}_+^{(T)}$ satisfy condition* (3.8). *Suppose that $f + s(\cdot \mid D)$ is ϵ-semiconvex at z_ϵ and $f_t, t \in T$, are semiconvex at z_ϵ then z_ϵ, is an almost ϵ-quasisolution of* (P).

Proof. Suppose that $(z_\epsilon, \lambda) \in S_\epsilon \times \mathbb{R}_+^{(T)}$ satisfy condition (3.8). Then there exist $u \in \partial^c f(z_\epsilon)$, $w_t \in \partial^c f_t(z_\epsilon)$, $t \in T$, $w \in N_C(z_\epsilon)$, $r \in B^*$, $v \in D$ such that $\langle v, z_\epsilon\rangle = s(z_\epsilon \mid D)$ (i.e., $v \in \partial^c s(z_\epsilon \mid D)$), and

$$-v = u + \sum_{t\in T}\lambda_t w_t + w + \sqrt{\epsilon} r, \tag{3.16}$$

or, equivalently,

$$u + v + \sqrt{\epsilon}\, r = -w - \sum_{t\in T}\lambda_t w_t. \tag{3.17}$$

Since C is convex subset of X, $w(x - z_\epsilon) \leq 0$ for all $x \in C$. Since $f_t, t \in T$, are semiconvex at z_ϵ and $f_t(z_\epsilon) \geq 0$ for all $t \in T(\lambda)$, by Lemma 2.3, it follows that $f_t'(z_\epsilon, x - z_\epsilon) \leq 0$ for all $x \in S$. Under the property of regularity of f_t for all $t \in T$, $f_t'(z_\epsilon, x - z_\epsilon) = f_t^\circ(z_\epsilon, x - z_\epsilon)$, we deduce that $w_t(x-z_\epsilon) \leq 0$ for all $x \in S$, $w_t \in \partial^c f(z_\epsilon)$ (in fact, we only need $w_t(x-z_\epsilon) \leq 0$ for all $t \in T(\lambda)$). Combining these with (3.17), we get

$$\left(u+v+\sqrt{\epsilon}\,r\right)(x-z_\epsilon) \geq 0, \quad \forall x \in S, \tag{3.18}$$

that is,

$$(u+v)(x-z_\epsilon) + \sqrt{\epsilon}\,\|x-z_\epsilon\| \geq 0, \quad \forall x \in S. \tag{3.19}$$

Since $s(\cdot \mid D)$ is Lipschitz and convex, by Proposition 2.3.6 of [25], it is quasidifferentiable at z_ϵ. Moreover, since f is quasidifferentiable at z, by Corollary 3 of [25],

$$\partial^c f(z_\epsilon) + \partial^c s(z_\epsilon \mid D) = \partial^c\left(f + s(\cdot \mid D)\right)(z_\epsilon). \tag{3.20}$$

It follows that $(u+v) \in \partial^c(f + s(\cdot \mid D))(z)$. Combining (3.19) and the assumption that $f + s(\cdot \mid D)$ is ϵ-semiconvex at z_ϵ, we deduce that

$$f(x) + s(\cdot \mid D)(x) + \sqrt{\epsilon}\|x-z_\epsilon\| \geq f(z_\epsilon) + s(\cdot \mid D)(z_\epsilon), \quad \forall x \in S. \tag{3.21}$$

Since $z_\epsilon \in S_\epsilon$, z_ϵ is an almost ϵ-quasisolution of (P). □

Corollary 3.6. *Let $\epsilon \geq 0$, and let $(z_\epsilon, \lambda) \in S_\epsilon \times \mathbb{R}_+^{(T)}$ satisfy condition* (3.8)*. Suppose that $f + s(\cdot \mid D)$ is ϵ-semiconvex at z_ϵ and $f_t, t \in T$, are convex on C, then z_ϵ is an almost ϵ-quasisolution of (P).*

Proof. The desired conclusion follows by using Remark 2.5. □

Theorem 3.7. *Let $\epsilon \geq 0$ and let $(z_\epsilon, \lambda) \in S_\epsilon \times \mathbb{R}_+^{(T)}$ satisfy condition* (3.8)*. Suppose that $f_t, t \in T$, are quasidifferentiable at z_ϵ. If $f + s(\cdot \mid D)$ is ϵ-semiconvex at z_ϵ, the set S_ϵ is convex, and $f_t(z_\epsilon) = \sqrt{\epsilon}$ for all $t \in T(\lambda)$, then z_ϵ is an almost ϵ-quasisolution of (P).*

Proof. The proof is similar to the one of Theorem 3.5 except for the argument to show that $w_t(x-z_\epsilon) \leq 0$ for all $x \in S$ and for all $t \in T(\lambda)$, where $w_t \in \partial^c f_t(z_\epsilon)$. Note that

$$w_t(x-z_\epsilon) \leq f_t^\circ(z_\epsilon; x-z_\epsilon) = f_t'(z_\epsilon; x-z_\epsilon). \tag{3.22}$$

Hence,

$$w_t(x-z_\epsilon) \leq \lim_{\theta \downarrow 0} \frac{f_t(z_\epsilon + \theta(x-z_\epsilon)) - f_t(z_\epsilon)}{\theta}. \tag{3.23}$$

Since S_ϵ is convex, $z_\epsilon + \theta(x-z_\epsilon) \in S_\epsilon$ when $\theta > 0$ is small enough. Hence, $f_t(z_\epsilon + \theta(x-z_\epsilon)) \leq \sqrt{\epsilon}$

for all $t \in T$ when $\theta > 0$ is small enough. Note that $f_t(z_\epsilon) = \sqrt{\epsilon}$ for all $t \in T(\lambda)$. These imply that

$$\lim_{\theta \downarrow 0} \frac{f_t(z_\epsilon + \theta(x - z_\epsilon)) - f_t(z_\epsilon)}{\theta} \leq 0, \quad t \in T(\lambda). \tag{3.24}$$

So, $w_t(x - z_\epsilon) \leq 0$ for all $t \in T(\lambda)$. The proof is complete. □

Remark 3.8. To obtain the conclusions for ϵ-quasisolution of (P), it needs a minor to change in the hypothesis without any change in the proofs. Concretely, let (z_ϵ, λ) belong to $S \times \mathbb{R}_+^{(T)}$ instead of $S_\epsilon \times \mathbb{R}_+^{(T)}$.

4. Applications and Discussions

We now discuss the previous results applied to a class of semi-infinite programs. For the problem (P), in case D is suppressed, we have the following problem

$$\begin{aligned} &\text{Minimize} && f(x) \\ &\text{subject to} && f_t(x) \leq, \quad t \in T, \\ & && x \in C. \end{aligned} \tag{P_1}$$

Similar to Definition 3.2, a pair $(z_\epsilon, \lambda) \in S_\epsilon \times \mathbb{R}_+^{(T)}$ is called a generalized Karush-Kuhn-Tucker pair up to ϵ corresponding to (P_1) if the following condition is satisfied

$$(\mathrm{KKT}_1) : 0 \in \partial^c f(z_\epsilon) + \sum_{t \in T} \lambda_t \partial^c f_t(z_\epsilon) + N_C(z_\epsilon) + \sqrt{\epsilon} B^*, \quad f_t(z_\epsilon) \geq 0, \ \forall t \in T(\lambda). \tag{4.1}$$

Next, we can obtain some corollaries on sufficient optimality conditions for almost ϵ-quasisolutions of (P_1) directly from Theorems 3.4, 3.5, and 3.7 with the proofs omitted.

Corollary 4.1. *For the problem* (P_1), *let* $(z_\epsilon, \lambda) \in S_\epsilon \times \mathbb{R}_+^{(T)}$ *satisfy condition* (4.1). *Suppose that* $f, f_t, t \in T$, *are quasidifferentiable at* z_ϵ. *If* $f + \sum_{t \in T} \lambda_t f_t$ *is* ϵ-*semiconvex at* z_ϵ, *then* z_ϵ *is an almost* ϵ-*quasisolution of* (P_1).

Corollary 4.2. *For the problem* (P_1), *let* $(z_\epsilon, \lambda) \in S_\epsilon \times \mathbb{R}_+^{(T)}$ *satisfy condition* (4.1). *Suppose that* f *is* ϵ-*semiconvex at* z_ϵ *and* $f_t, t \in T$, *are semiconvex at* z_ϵ *then* z_ϵ, *is an almost* ϵ-*quasisolution of* (P_1).

Corollary 4.3. *For the problem* (P_1), *let* $(z_\epsilon, \lambda) \in S_\epsilon \times \mathbb{R}_+^{(T)}$ *satisfy condition* (4.1). *Suppose that* f *is* ϵ-*semiconvex at* z_ϵ *and* $f_t, t \in T$, *are convex on* C, *then* z_ϵ *is an almost* ϵ-*quasisolution of* (P_1).

Corollary 4.4. *For the problem* (P_1), *let* $(z_\epsilon, \lambda) \in S_\epsilon \times \mathbb{R}_+^{(T)}$ *satisfy condition* (4.1). *Suppose that* $f_t, t \in T$, *are quasidifferentiable at* z_ϵ. *If* f *is* ϵ-*semiconvex at* z_ϵ, $f_t(z_\epsilon) = \sqrt{\epsilon}$ *for all* $t \in T(\lambda)$, *and the set* S_ϵ *is convex, then* z_ϵ *is an almost* ϵ-*quasisolution of* (P_1).

We note that if X is a Banach space, then the problem (P_1) becomes the problem (Q) (considered recently in [10]). In this case, we can see that Corollary 4.3 is Theorem 4.3

presented in [10], and similar technique could also be adopted to give the proofs for the four corollaries above when X is a Banach space. Hence, for the nonconvex-infinite programs considered in [10], besides the sufficient optimality condition for a point to be an almost ϵ-quasisolution, we can establish some new versions of it.

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References

[1] P. Loridan, "Necessary conditions for ε-optimality," *Mathematical Programming Study*, no. 19, pp. 140–152, 1982.

[2] J.-J. Strodiot, V. H. Nguyen, and N. Heukemes, "ε-optimal solutions in nondifferentiable convex programming and some related questions," *Mathematical Programming*, vol. 25, no. 3, pp. 307–328, 1983.

[3] J. C. Liu, "ε-duality theorem of nondifferentiable nonconvex multiobjective programming," *Journal of Optimization Theory and Applications*, vol. 69, no. 1, pp. 153–167, 1991.

[4] A. Hamel, "An ε-Lagrange multiplier rule for a mathematical programming problem on Banach spaces," *Optimization*, vol. 49, no. 1-2, pp. 137–149, 2001.

[5] K. Yokoyama, "ε-optimality criteria for convex programming problems via exact penalty functions," *Mathematical Programming*, vol. 56, no. 2, pp. 233–243, 1992.

[6] C. Scovel, D. Hush, and I. Steinwart, "Approximate duality," Tech. Rep. La-UR-05-6755, Los Alamos National Laboratory, September 2005.

[7] J. Dutta, "Necessary optimality conditions and saddle points for approximate optimization in Banach spaces," *Top*, vol. 13, no. 1, pp. 127–143, 2005.

[8] N. Dinh and T. Q. Son, "Approximate optimality condition and duality for convex infinite programming problems," *Journal of Science & Technology for Development*, vol. 10, pp. 29–38, 2007.

[9] T. Q. Son, J. J. Strodiot, and V. H. Nguyen, "ε-optimality and ε-lagrangian duality for a nonconvex programming problem with an infinite number of constraints," in *Proceedings of Vietnam-Korea Workshop on Optimization and Applied Mathematics*, Nhatrang, Vietnam, 2008.

[10] T. Q. Son, J. J. Strodiot, and V. H. Nguyen, "ε-optimality and ε-Lagrangian duality for a nonconvex programming problem with an infinite number of constraints," *Journal of Optimization Theory and Applications*, vol. 141, no. 2, pp. 389–409, 2009.

[11] N. Dinh, M. A. Goberna, and M. A. López, "From linear to convex systems: consistency, Farkas' lemma and applications," *Journal of Convex Analysis*, vol. 13, no. 1, pp. 113–133, 2006.

[12] N. Dinh, M. A. Goberna, M. A. López, and T. Q. Son, "New Farkas-type constraint qualifications in convex infinite programming," *ESAIM. Control, Optimisation and Calculus of Variations*, vol. 13, no. 3, pp. 580–597, 2007.

[13] N. Dinh, B. Mordukhovich, and T. T. A. Nghia, "Subdifferentials of value functions and optimality conditions for DC and bilevel infinite and semi-infinite programs," *Mathematical Programming*, vol. 123, no. 1, pp. 101–138, 2010.

[14] T. Q. Son and N. Dinh, "Characterizations of optimal solution sets of convex infinite programs," *Top*, vol. 16, no. 1, pp. 147–163, 2008.

[15] J.-J. Rückmann and A. Shapiro, "Augmented Lagrangians in semi-infinite programming," *Mathematical Programming*, vol. 116, no. 1-2, pp. 499–512, 2009.

[16] M. Schechter, "A subgradient duality theorem," *Journal of Mathematical Analysis and Applications*, vol. 61, no. 3, pp. 850–855, 1977.

[17] M. Schechter, "More on subgradient duality," *Journal of Mathematical Analysis and Applications*, vol. 71, no. 1, pp. 251–262, 1979.

[18] B. Mond and M. Schechter, "Nondifferentiable symmetric duality," *Bulletin of the Australian Mathematical Society*, vol. 53, no. 2, pp. 177–188, 1996.

[19] X. M. Yang, K. L. Teo, and X. Q. Yang, "Duality for a class of nondifferentiable multiobjective programming problems," *Journal of Mathematical Analysis and Applications*, vol. 252, no. 2, pp. 999–1005, 2000.

[20] I. Husain, Abha, and Z. Jabeen, "On nonlinear programming with support functions," *Journal of Applied Mathematics & Computing*, vol. 10, no. 1-2, pp. 83–99, 2002.

[21] I. Husain and Z. Jabeen, "On fractional programming containing support functions," *Journal of Applied Mathematics & Computing*, vol. 18, no. 1-2, pp. 361–376, 2005.

[22] D. S. Kim and K. D. Bae, "Optimality conditions and duality for a class of nondifferentiable multiobjective programming problems," *Taiwanese Journal of Mathematics*, vol. 13, no. 2, pp. 789–804, 2009.

[23] D. S. Kim, S. J. Kim, and M. H. Kim, "Optimality and duality for a class of nondifferentiable multiobjective fractional programming problems," *Journal of Optimization Theory and Applications*, vol. 129, no. 1, pp. 131–146, 2006.

[24] R. Mifflin, "Semismooth and semiconvex functions in constrained optimization," *SIAM Journal on Control and Optimization*, vol. 15, no. 6, pp. 959–972, 1977.

[25] F. H. Clarke, *Optimization and Nonsmooth Analysis*, Canadian Mathematical Society Series of Monographs and Advanced Texts, John Wiley & Sons, New York, NY, USA, 1983.

[26] M. Beldiman, E. Panaitescu, and L. Dogaru, "Approximate quasi efficient solutions in multiobjective optimization," *Bulletin Mathematique de la Societe des Sciences Mathematiques de Roumanie*, vol. 51, no. 2, pp. 109–121, 2008.

[27] V. Jeyakumar, G. M. Lee, and N. Dinh, "New sequential Lagrange multiplier conditions characterizing optimality without constraint qualification for convex programs," *SIAM Journal on Optimization*, vol. 14, no. 2, pp. 534–547, 2003.