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Research Article

Some Fixed-Point Theorems for Multivalued Monotone Mappings in Ordered Uniform Space

Duran Turkoglu and Demet Binbasioglu

Department of Mathematics, Faculty of Science, University of Gazi, Teknikokullar 06500, Ankara, Turkey

Correspondence should be addressed to Duran Turkoglu, dturkoglu@gazi.edu.tr

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We use the order relation on uniform spaces defined by Altun and Imdad (2009) to prove some new fixed-point and coupled fixed-point theorems for multivalued monotone mappings in ordered uniform spaces.

1. Introduction

There exists considerable literature of fixed-point theory dealing with results on fixed or common fixed-points in uniform space (e.g., between [1–14]). But the majority of these results are proved for contractive or contractive type mapping (notice from the cited references). Also some fixed-point and coupled fixed-point theorems in partially ordered metric spaces are given in [15–20]. Recently, Aamri and El Moutawakil [2] have introduced the concept of E-distance function on uniform spaces and utilize it to improve some well-known results of the existing literature involving both E-contractive or E-expansive mappings. Lately, Altun and Imdad [21] have introduced a partial ordering on uniform spaces utilizing E-distance function and have used the same to prove a fixed-point theorem for single-valued nondecreasing mappings on ordered uniform spaces. In this paper, we use the partial ordering on uniform spaces which is defined by [21], so we prove some fixed-point theorems of multivalued monotone mappings and some coupled fixed-point theorems of multivalued mappings which are given for ordered metric spaces in [22] on ordered uniform spaces.

Now, we recall some relevant definitions and properties from the foundation of uniform spaces. We call a pair (X, ϑ) to be a uniform space which consists of a nonempty set X together with an uniformity ϑ wherein the latter begins with a special kind of filter on $X \times X$ whose all elements contain the diagonal $\Delta = \{(x, x) : x \in X\}$. If $V \in \vartheta$ and $(x, y) \in V$, $(y, x) \in V$ then x and y are said to be V-close. Also a sequence $\{x_n\}$ in X, is said to be a Cauchy sequence with regard to uniformity ϑ if for any $V \in \vartheta$, there exists $N \geq 1$ such that x_n and x_m are V-close for $m, n \geq N$. An uniformity ϑ defines a unique topology $\tau(\vartheta)$ on X for which the neighborhoods of $x \in X$ are the sets $V(x) = \{y \in X : (x, y) \in V\}$ when V runs over ϑ.

A uniform space (X, ϑ) is said to be Hausdorff if and only if the intersection of all the $V \in \vartheta$ reduces to diagonal Δ of X, that is, $(x, y) \in V$ for $V \in \vartheta$ implies $x = y$. Notice that Hausdorffness of the topology induced by the uniformity guarantees the uniqueness of limit of a sequence in uniform spaces. An element of uniformity ϑ is said to be symmetrical if $V = V^{-1} = \{(y, x) : (x, y) \in V\}$. Since each $V \in \vartheta$ contains a symmetrical $W \in \vartheta$ and if $(x, y) \in W$ then x and y are both W and V-close and then one may assume that each $V \in \vartheta$ is symmetrical. When topological concepts are mentioned in the context of a uniform space (X, ϑ), they are naturally interpreted with respect to the topological space $(X, \tau(\vartheta))$.

2. Preliminaries

We will require the following definitions and lemmas in the sequel.

Definition 2.1 (see [2]). Let (X, ϑ) be a uniform space. A function $p : X \times X \to \mathbb{R}^+$ is said to be an E-distance if

(p_1) for any $V \in \vartheta$, there exists $\delta > 0$, such that $p(z, x) \leq \delta$ and $p(z, y) \leq \delta$ for some $z \in X$ imply $(x, y) \in V$,

(p_2) $p(x, y) \leq p(x, z) + p(z, y)$, for all $x, y, z \in X$.

The following lemma embodies some useful properties of E-distance.

Lemma 2.2 (see [1, 2]). *Let (X, ϑ) be a Hausdorff uniform space and p be an E-distance on X. Let $\{x_n\}$ and $\{y_n\}$ be arbitrary sequences in X and $\{\alpha_n\}$, $\{\beta_n\}$ be sequences in $\mathbb{R}^+$ converging to 0. Then, for $x, y, z \in X$, the following holds:*

(a) *if $p(x_n, y) \leq \alpha_n$ and $p(x_n, z) \leq \beta_n$ for all $n \in \mathbb{N}$, then $y = z$. In particular, if $p(x, y) = 0$ and $p(x, z) = 0$, then $y = z$,*

(b) *if $p(x_n, y_n) \leq \alpha_n$ and $p(x_n, z) \leq \beta_n$ for all $n \in \mathbb{N}$, then $\{y_n\}$ converges to z,*

(c) *if $p(x_n, x_m) \leq \alpha_n$ for all $m > n$, then $\{x_n\}$ is a Cauchy sequence in (X, ϑ).*

Let (X, ϑ) be a uniform space equipped with E-distance p. A sequence in X is p-Cauchy if it satisfies the usual metric condition. There are several concepts of completeness in this setting.

Definition 2.3 (see [1, 2]). Let (X, ϑ) be a uniform space and p be an E-distance on X. Then

(i) X said to be S-complete if for every p-Cauchy sequence $\{x_n\}$ there exists $x \in X$ with $\lim_{n\to\infty} p(x_n, x) = 0$,

(ii) X is said to be p-Cauchy complete if for every p-Cauchy sequence $\{x_n\}$ there exists $x \in X$ with $\lim_{n\to\infty} x_n = x$ with respect to $\tau(\vartheta)$,

(iii) $f : X \to X$ is p-continuous if $\lim_{n\to\infty} p(x_n, x) = 0$ implies

$$\lim_{n\to\infty} p(fx_n, fx) = 0, \tag{2.1}$$

(iv) $f : X \to X$ is $\tau(\vartheta)$-continuous if $\lim_{n\to\infty} x_n = x$ with respect to $\tau(\vartheta)$ implies $\lim_{n\to\infty} f x_n = f x$ with respect to $\tau(\vartheta)$.

Remark 2.4 (see [2]). Let (X, ϑ) be a Hausdorff uniform space and let $\{x_n\}$ be a p-Cauchy sequence. Suppose that X is S-complete, then there exists $x \in X$ such that $\lim_{n\to\infty} p(x_n, x) = 0$. Then Lemma 2.2(b) gives that $\lim_{n\to\infty} x_n = x$ with respect to the topology $\tau(\vartheta)$ which shows that S-completeness implies p-Cauchy completeness.

Lemma 2.5 (see [15]). *Let (X, ϑ) be a Hausdorff uniform space, p be E-distance on X and $\varphi : X \to \mathbb{R}$. Define the relation "$\preceq$" on X as follows:*

$$x \preceq y \Longleftrightarrow x = y \quad \text{or} \quad p(x, y) \le \varphi(x) - \varphi(y). \tag{2.2}$$

Then "$\preceq$" is a (partial) order on X induced by φ.

3. The Fixed-Point Theorems of Multivalued Mappings

Theorem 3.1. *Let (X, ϑ) a Hausdorff uniform space and p is an E-distance on X, $\varphi : X \to \mathbb{R}$ be a function which is bounded below and "$\preceq$" the order introduced by φ. Let X be also a p-Cauchy complete space, $T : X \to 2^X$ be a multivalued mapping, $[x, +\infty) = \{y \in X : x \preceq y\}$ and $M = \{x \in X \mid T(x) \cap [x, +\infty) \ne \emptyset\}$. Suppose that:*

(i) *T is upper semicontinuous, that is, $x_n \in X$ and $y_n \in T(x_n)$ with $x_n \to x_0$ and $y_n \to y_0$, implies $y_0 \in T(x_0)$,*

(ii) *$M \ne \emptyset$,*

(iii) *for each $x \in M$, $T(x) \cap M \cap [x, +\infty) \ne \emptyset$.*

Then T has a fixed-point x^* and there exists a sequence $\{x_n\}$ with

$$x_{n-1} \preceq x_n \in T(x_{n-1}), \quad n = 1, 2, 3, \ldots \tag{3.1}$$

such that $x_n \to x^*$. Moreover if φ is lower semicontinuous, then $x_n \preceq x^*$ for all n.

Proof. By the condition (ii), take $x_0 \in M$. From (iii), there exist $x_1 \in T(x_0) \cap M$ and $x_0 \preceq x_1$. Again from (iii), there exist $x_2 \in T(x_1) \cap M$. Thus $x_1 \preceq x_2$.

Continuing this procedure we get a sequence $\{x_n\}$ satisfying

$$x_{n-1} \preceq x_n \in T(x_{n-1}), \quad n = 1, 2, 3, \ldots. \tag{3.2}$$

So by the definition of "$\preceq$", we have $\cdots \varphi(x_2) \le \varphi(x_1) \le \varphi(x_0)$, that is, the sequence $\{\varphi(x_n)\}$ is a nonincreasing sequence in $\mathbb{R}$. Since φ is bounded from below, $\{\varphi(x_n)\}$ is convergent and hence it is Cauchy, that is, for all $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that for all $m > n > n_0$ we have $|\varphi(x_m) - \varphi(x_n)| < \varepsilon$. Since $x_n \preceq x_m$, we have $x_n = x_m$ or $p(x_n, x_m) \le \varphi(x_n) - \varphi(x_m)$. Therefore,

$$\begin{aligned} p(x_n, x_m) &\le \varphi(x_n) - \varphi(x_m) \\ &= \left|\varphi(x_n) - \varphi(x_m)\right| \\ &< \varepsilon, \end{aligned} \tag{3.3}$$

which shows that (in view of Lemma 2.2(c)) that $\{x_n\}$ is p-Cauchy sequence. By the p-Cauchy completeness of X, $\{x_n\}$ converges to x^*. Since T is upper semicontinuous, $x^* \in T(x^*)$.

Moreover, when φ is lower semicontinuous, for each n

$$\begin{aligned} p(x_n, x^*) &= \lim_{m\to\infty} p(x_n, x_m) \\ &\le \lim_{m\to\infty} \sup\left(\varphi(x_n) - \varphi(x_m)\right) \\ &= \varphi(x_n) - \lim_{m\to\infty} \inf \varphi(x_m) \\ &\le \varphi(x_n) - \varphi(x^*). \end{aligned} \tag{3.4}$$

So $x_n \preceq x^*$, for all n. □

Similarly, we can prove the following.

Theorem 3.2. *Let (X, ϑ) a Hausdorff uniform space and p an E-distance on X, $\varphi : X \to \mathbb{R}$ be a function which is bounded above and "$\preceq$" the order introduced by φ. Let X be also a p-Cauchy complete space, $T : X \to 2^X$ be a multivalued mapping, $(-\infty, x] = \{y \in X : y \preceq x\}$ and $M = \{x \in X \mid T(x) \cap (-\infty, x] \neq \emptyset\}$. Suppose that*

(i) *T is upper semicontinuous, that is, $x_n \in X$ and $y_n \in T(x_n)$ with $x_n \to x_0$ and $y_n \to y_0$, implies $y_0 \in T(x_0)$,*

(ii) *$M \neq \emptyset$,*

(iii) *for each $x \in M$, $T(x) \cap M \cap (-\infty, x] \neq \emptyset$.*

Then T has a fixed-point x^* and there exists a sequence $\{x_n\}$ with

$$x_{n-1} \succeq x_n \in T(x_{n-1}), \quad n = 1, 2, 3, \ldots \tag{3.5}$$

such that $x_n \to x^*$. Moreover, if φ is upper semicontinuous, then $x^* \preceq x_n$ for all n.

Corollary 3.3. *Let (X, ϑ) a Hausdorff uniform space and p is an E-distance on X, $\varphi : X \to \mathbb{R}$ be a function which is bounded below and "$\preceq$" the order introduced by φ. Let X be also a p-Cauchy complete space, $T : X \to 2^X$ be a multivalued mapping and $[x, +\infty) = \{y \in X : x \preceq y\}$. Suppose that:*

(i) *T is upper semicontinuous, that is, $x_n \in X$ and $y_n \in T(x_n)$ with $x_n \to x_0$ and $y_n \to y_0$, implies $y_0 \in T(x_0)$,*

(ii) *T satisfies the monotonic condition: for any $x,y \in X$ with $x \preceq y$ and any $u \in T(x)$, there exists $v \in T(y)$ such that $u \preceq v$,*

(iii) *there exists an $x_0 \in X$ such that $T(x_0) \cap [x_0, +\infty) \neq \emptyset$.*

Then T has a fixed-point x^* and there exists a sequence $\{x_n\}$ with

$$x_{n-1} \preceq x_n \in T(x_{n-1}), \quad n = 1, 2, 3, \ldots, \tag{3.6}$$

such that $x_n \to x^*$. Moreover if φ is lower semicontinuous, then $x_n \preceq x^*$ for all n.

Proof. By (iii), $x_0 \in M = \{x \in X : T(x) \cap [x, +\infty) \neq \emptyset\}$. For $x \in M$, take $y \in T(x)$ and $x \preceq y$. By the monotonicity of T, there exists $z \in T(y)$ such that $y \preceq z$. So $y \in M$, and $T(x) \cap M \cap [x, +\infty) \neq \emptyset$. The conclusion follows from Theorem 3.1. □

Corollary 3.4. *Let (X, ϑ) a Hausdorff uniform space and p is an E-distance on X, $\varphi : X \to \mathbb{R}$ be a function which is bounded above and "$\preceq$" the order introduced by φ. Let X be also a p-Cauchy complete space, $T : X \to 2^X$ be a multivalued mapping and $(-\infty, x] = \{y \in X : y \preceq x\}$. Suppose that:*

(i) *T is upper semicontinuous,*

(ii) *T satisfies the monotonic condition; for any $x, y \in X$ with $x \preceq y$ and any $v \in T(y)$, there exists $u \in T(x)$ such that $u \preceq v$,*

(iii) *there exists an $x_0 \in X$ such that $T(x_0) \cap (-\infty, x_0] \neq \emptyset$.*

Then T has a fixed-point x^* and there exists a sequence $\{x_n\}$ with

$$x_{n-1} \succeq x_n \in T(x_{n-1}), \quad n = 1, 2, \ldots, \tag{3.7}$$

such that $x_n \to x^*$. Moreover if φ is upper semicontinuous, then $x_n \succeq x^*$ for all n.

Corollary 3.5. *Let (X, ϑ) a Hausdorff uniform space and p is an E-distance on X, $\varphi : X \to \mathbb{R}$ be a function which is bounded below and "$\preceq$" the order introduced by φ. Let X be also a p-Cauchy complete space, $f : X \to X$ be a map and $M = \{x \in X : x \preceq f(x)\}$. Suppose that:*

(i) *f is $\tau(\vartheta)$-continuous,*

(ii) *$M \neq \emptyset$,*

(iii) *for each $x \in M$, $f(x) \in M$.*

Then f has a fixed-point x^* and the sequence

$$x_{n-1} \preceq x_n = f(x_{n-1}), \quad n = 1, 2, 3, \ldots \tag{3.8}$$

converges to x^*. Moreover if φ is lower semicontinuous, then $x_n \preceq x^*$ for all n.

Corollary 3.6. *Let (X, ϑ) be a Hausdorff uniform space, p is an E-distance on X, $\varphi : X \to \mathbb{R}$ be a function which is bounded above, and "$\preceq$" the order introduced by φ. Let X be also a p-Cauchy* *complete space, $f : X \to X$ be a map and $M = \{x \in X : x \succeq f(x)\}$. Suppose that:*

(i) *f is $\tau(\vartheta)$-continuous,*

(ii) *$M \neq \emptyset$,*

(iii) *for each $x \in M$, $f(x) \in M$.*

Then f has a fixed-point x^*. And the sequence

$$x_{n-1} \succeq x_n = f(x_{n-1}), \quad n = 1, 2, 3, \ldots \tag{3.9}$$

converges to x^*. Moreover, if φ is upper semicontinuous, then $x_n \succeq x^*$ for all n.

Corollary 3.7. *Let (X, ϑ) be a Hausdorff uniform space, p is an E-distance on X, $\varphi : X \to \mathbb{R}$ be a function which is bounded below, and "$\preceq$" the order introduced by φ. Let X be also a p-Cauchy complete space, $f : X \to X$ be a map and $M = \{x \in X : x \succeq f(x)\}$. Suppose that:*

(i) *f is $\tau(\vartheta)$-continuous,*

(ii) *f is monotone increasing, that is, for $x \preceq y$ we have $f(x) \preceq f(y)$,*

(iii) *there exists an x_0, with $x_0 \preceq f(x_0)$.*

Then f has a fixed-point x^* and the sequence

$$x_{n-1} \preceq x_n = f(x_{n-1}), \quad n = 1, 2, 3, \ldots \tag{3.10}$$

converges to x^*. Moreover if φ is lower semicontinuous, then $x_n \preceq x^*$ for all n.

Example 3.8. Let $X = \{k, l, m\}$ and $\vartheta = \{V \subset X \times X : \Delta \subset V\}$. Define $p : X \times X \to \mathbb{R}^+$ as $p(x, x) = 0$ for all $x \in X$, $p(k, l) = p(l, k) = 2$, $p(k, m) = p(m, k) = 1$ ve $p(l, m) = p(m, l) = 3$. Since definition of ϑ, $\bigcap_{V \in \vartheta} V = \Delta$ and this show that the uniform space (X, ϑ) is a Hausdorff uniform space. On the other hand, $p(k, l) \leq p(k, m) + p(m, l)$, $p(k, m) \leq p(k, l) + p(l, m)$ and $p(l, m) \leq p(l, k) + p(k, m)$ for $k, l, m \in X$ and thus p is an E-distance as it is a metric on X. Next define $\varphi : X \to \mathbb{R}$ $\varphi(k) = 3$, $\varphi(l) = 2$, $\varphi(m) = 1$. Since $p(k, m) = p(m, k) = 1 \leq \varphi(k) - \varphi(m)$, therefore $k \preceq m$. But as $p(l, k) = p(k, l) = 2 \not\leq |\varphi(k) - \varphi(l)|$ therefore $k \npreceq l$ and $l \npreceq k$. Again similarly $l \npreceq m$ and $m \npreceq l$ which show that this ordering is partial and hence X is a partially ordered uniform space. Define $f : X \to X$ as $f(k) = k$, $f(l) = l$ and $f(m) = m$, then by a routine calculation one can verify that all the conditions of Corollary 3.7 are satisfied and f has a fixed-point. Notice that $p(f(k), f(l)) = p(k, l)$ which shows that f is neither E-contractive nor E expansive, therefore the results of [2] are not applicable in the context of this example. Thus, this example demonstrates the utility of our result.

Corollary 3.9. *Let (X, ϑ) be a Hausdorff uniform space, p is an E-distance on X, $\varphi : X \to \mathbb{R}$ be a function which is bounded above and "$\preceq$" the order introduced by φ. Let X be also a p-Cauchy complete space and $f : X \to X$ be a map. Suppose that*

(i) *f is $\tau(\vartheta)$-continuous,*

(ii) *f is monotone increasing, that is, for $x \preceq y$ we have $f(x) \preceq f(y)$,*

(iii) *there exists an x_0 with $x_0 \succeq f(x_0)$.*

Then f has a fixed-point x^*. And the sequence

$$x_{n-1} \succeq x_n = f(x_{n-1}), \quad n = 1, 2, 3, \dots \tag{3.11}$$

converges to x^*. Moreover if φ is upper semicontinuous, then $x_n \succeq x^*$ for all n.

Theorem 3.10. *Let (X, ϑ) be a Hausdorff uniform space, p is an E-distance on X, $\varphi : X \to \mathbb{R}$ be a continuous function bounded below and "$\preceq$" the order introduced by φ. Let X be also a p-Cauchy complete space, $T : X \to 2^X$ be a multivalued mapping and $[x, +\infty) = \{y \in X : x \preceq y\}$. Suppose that*

(i) *T satisfies the monotonic condition: for each $x \preceq y$ and each $u \in T(x)$ there exists $v \in T(y)$ such that $u \preceq v$,*

(ii) *$T(x)$ is compact for each $x \in X$,*

(iii) *$M = \{x \in X : T(x) \cap [x, +\infty) \neq \emptyset\} \neq \emptyset$.*

Then T has a fixed-point x_0.

Proof. We will prove that M has a maximum element. Let $\{x_v\}_{v \in \Lambda}$ be a totally ordered subset in M, where Λ is a directed set. For $v, \mu \in \Lambda$ and $v \le \mu$, one has $x_v \preceq x_\mu$, which implies that $\varphi(x_v) \ge \varphi(x_\mu)$ for $v \le \mu$. Since φ is bounded below, $\{\varphi(x_v)\}$ is a convergence net in $\mathbb{R}$. From $p(x_v, x_\mu) \le \varphi(x_v) - \varphi(x_\mu)$, we get that $\{x_v\}$ is a p-cauchy net in X. By the p-Cauchy completeness of X, let x_v converge to z in X.

For given $\mu \in \Lambda$
$p(x_\mu, z) = \lim_v p(x_\mu, x_v) \le \lim_v (\varphi(x_\mu) - \varphi(x_v)) = \varphi(x_\mu) - \varphi(x_z)$. So $x_\mu \preceq z$ for all $\mu \in \Lambda$.

For $\mu \in \Lambda$, by the condition (i), for each $u_\mu \in T(x_\mu)$, there exists a $v_\mu \in T(z)$ such that $u_\mu \preceq v_\mu$. By the compactness of $T(z)$, there exists a convergence subnet $\{v_{\mu^|}\}$ of $\{v_\mu\}$. Suppose that $\{v_{\mu^|}\}$ converges to $w \in T(z)$. Take $\Lambda^|$ such that $\mu^| \ge \Lambda^|$ implies $u_\mu \preceq v_\mu \preceq v_{\mu^|}$.

We have

$$p(u_\mu, w) = \lim_{\mu^|} p\left(u_\mu, v_{\mu^|}\right) \le \lim_{\mu^|}\left(\varphi(u_\mu) - \varphi\left(v_{\mu^|}\right)\right) = \varphi(u_\mu) - \varphi(w). \tag{3.12}$$

So $u_\mu \preceq w$ for all μ and

$$p(z, w) = \lim_{\mu} p(u_\mu, w) \le \lim_{\mu}(\varphi(u_\mu) - \varphi(w)) = \varphi(z) - \varphi(w). \tag{3.13}$$

So $z \preceq w$ and this gives that $z \in M$. Hence we have proven that $\{x_\mu\}$ has an upper bound in M.

By Zorn's Lemma, there exists a maximum element x_0 in M. By the definition of M, there exists a $y_0 \in T(x_0)$ such that $x_0 \preceq y_0$. By the condition (i), there exists a $z_0 \in T(y_0)$ such that $y_0 \preceq z_0$. Hence $y_0 \in M$. Since x_0 is the maximum element in M, it follows that $y_0 = x_0$ and $x_0 \in T(x_0)$. So x_0 is a fixed-point of T. □

Theorem 3.11. *Let (X, ϑ) be a Hausdorff uniform space, p is an E-distance on X, $\varphi : X \to \mathbb{R}$ be a continuous function bounded above and "$\preceq$" the order introduced by φ. Let X be also a p-Cauchy complete space, $T : X \to 2^X$ be a multivalued mapping and $(-\infty, x] = \{y \in X : y \preceq x\}$. Suppose* *that*

(i) *T satisfies the following condition; for each* $x \preceq y$ *and* $v \in T(x)$, *there exists* $u \in T(y)$ *such that* $u \preceq v$,

(ii) $T(x)$ *is compact for each* $x \in X$,

(iii) $M = \{x \in X : T(x) \cap (-\infty, x] \neq \emptyset\} \neq \emptyset$.

Then T has a fixed-point.

Corollary 3.12. *Let* (X, ϑ) *be a Hausdorff uniform space,* p *is an E-distance on* X, $\varphi : X \to \mathbb{R}$ *be a continuous function bounded below and "*$\preceq$*" the order introduced by* φ. *Let X be also a p-Cauchy complete space and* $f : X \to X$ *be a map. Suppose that;*

(i) f *is monotone increasing, that is for* $x \preceq y$, $f(x) \preceq f(y)$,

(ii) *there is an* $x_0 \in X$ *such that* $x_0 \preceq f(x_0)$.

Then f has a fixed-point.

Corollary 3.13. *Let* (X, ϑ) *be a Hausdorff uniform space,* p *is an E-distance on* X, $\varphi : X \to \mathbb{R}$ *be a continuous function bounded above and "*$\preceq$*" the order introduced by* φ. *Let X be also a p-Cauchy complete space and* $f : X \to X$ *be a map. Suppose that;*

(i) f *is monotone increasing, that is, for* $x \preceq y$, $f(x) \preceq f(y)$;

(ii) *there is an* $x_0 \in X$ *such that* $x_0 \succeq f(x_0)$.

Then f has a fixed-point.

4. The Coupled Fixed-Point Theorems of Multivalued Mappings

Definition 4.1. An element $(x, y) \in X \times X$ is called a coupled fixed-point of the multivalued mapping $T : X \times X \to 2^X$ if $x \in T(x, y)$, $y \in T(y, x)$.

Theorem 4.2. *Let* (X, ϑ) *be a Hausdorff uniform space,* p *is an E-distance on* X, $\varphi : X \to \mathbb{R}$ *be a function bounded below and "*$\preceq$*" be the order in X introduced by* φ. *Let X be also a p-Cauchy complete space,* $T : X \times X \to 2^X$ *be a multivalued mapping,* $[x, +\infty) = \{y \in X : x \preceq y\}$, $(-\infty, y] = \{x \in X : x \preceq y\}$, *and* $M = \{(x, y) \in X \times X : x \preceq y,\ T(x, y) \cap [x, +\infty) \neq \emptyset$ *and* $T(y, x) \cap (-\infty, y] \neq \emptyset\}$. *Suppose that:*

(i) *T is upper semicontinuous, that is,* $x_n \in X$, $y_n \in X$ *and* $z_n \in T(x_n, y_n)$, *with* $x_n \to x_0$, $y_n \to y_0$ *and* $z_n \to z_0$ *implies* $z_0 \in T(x_0, y_0)$,

(ii) $M \neq \emptyset$,

(iii) *for each* $(x, y) \in M$, *there is* $(u, v) \in M$ *such that* $u \in T(x, y) \cap [x, +\infty)$ *and* $v \in T(y, x) \cap (-\infty, y]$.

Then T has a coupled fixed-point (x^*, y^*), *that is,* $x^* \in T(x^*, y^*)$ *and* $y^* \in T(y^*, x^*)$. *And there exist two sequences* $\{x_n\}$ *and* $\{y_n\}$ *with*

$$x_{n-1} \preceq x_n \in T(x_{n-1}, y_{n-1}), \quad y_{n-1} \succeq y_n \in T(y_{n-1}, x_{n-1}), \quad n = 1, 2, 3, \ldots \tag{4.1}$$

such that $x_n \to x^*$ *and* $y_n \to y^*$.

Proof. By the condition (ii), take $(x_0, y_0) \in M$. From (iii), there exist $(x_1, y_1) \in M$ such that $x_1 \in T(x_0, y_0)$, $x_0 \preceq x_1$ and $y_1 \in T(y_0, x_0)$, $y_1 \preceq y_0$. Again from (iii), there exist $(x_2, y_2) \in M$ such that $x_2 \in T(x_1, y_1)$, $x_1 \preceq x_2$ and $y_2 \in T(y_1, x_1)$, $y_2 \preceq y_1$.

Continuing this procedure we get two sequences $\{x_n\}$ and $\{y_n\}$ satisfying $(x_n, y_n) \in M$ and

$$\begin{aligned} x_{n-1} \preceq x_n \in T(x_{n-1}, y_{n-1}), \quad n = 1, 2, \dots, \\ y_{n-1} \succeq y_n \in T(y_{n-1}, x_{n-1}), \quad n = 1, 2, \dots. \end{aligned} \tag{4.2}$$

So

$$x_0 \preceq x_1 \preceq \cdots \preceq x_n \preceq \cdots \preceq y_n \preceq \cdots \preceq y_2 \preceq y_1. \tag{4.3}$$

Hence,

$$\varphi(x_0) \geq \varphi(x_1) \geq \cdots \geq \varphi(x_n) \geq \cdots \geq \varphi(y_n) \geq \cdots \geq \varphi(y_1) \geq \varphi(y_0). \tag{4.4}$$

From this we get that $\varphi(x_n)$ and $\varphi(y_n)$ are convergent sequences. By the definition of "$\preceq$" as in the proof of Theorem 3.1, it is easy to prove that $\{x_n\}$ and $\{y_n\}$ are p-Cauchy sequences. Since X is p-Cauchy complete, let $\{x_n\}$ converge to x^* and $\{y_n\}$ converge to y^*. Since T is upper semicontinuous, $x^* \in T(x^*, y^*)$ and $y^* \in T(y^*, x^*)$. Hence (x^*, y^*) is a coupled fixed-point of T. □

Corollary 4.3. *Let (X, ϑ) be a Hausdorff uniform space, p is an E-distance on X, $\varphi : X \to \mathbb{R}$ be a function bounded below, and "$\preceq$" be the order in X introduced by φ. Let X be also a p-Cauchy complete space, $f : X \times X \to X$ be a mapping and $M = \{(x, y) \in X \times X : x \preceq y$ and $x \preceq f(x, y)$ and $f(x, y) \preceq y\}$. Suppose that;*

(i) *f is $\tau(\vartheta)$-continuous,*

(ii) *$M \neq \emptyset$,*

(iii) *for each $(x, y) \in M$, $x \preceq f(x, y)$ and $f(y, x) \preceq y$.*

Then f has a coupled fixed-point (x^*, y^*), that is, $x^* = f(x^*, y^*)$ and $y^* = f(y^*, x^*)$. And there exist two sequences $\{x_n\}$ and $\{y_n\}$ with $x_{n-1} \preceq x_n = f(x_{n-1}, y_{n-1})$, $y_{n-1} \succeq y_n = f(y_{n-1}, x_{n-1})$, $n = 1, 2, \dots$ such that $x_n \to x^*$ and $y_n \to y^*$.

Corollary 4.4. *Let (X, ϑ) be a Hausdorff uniform space, p is an E-distance on X,$\varphi : X \to \mathbb{R}$ be a function bounded below, and "$\preceq$" be the order in X introduced by φ. Let X be also a p-Cauchy complete space, $f : X \times X \to X$ be a mapping and $M = \{(x, y) \in X \times X : x \preceq y$ and $x \preceq f(x, y)$ and $f(x, y) \preceq y\}$. Suppose that;*

(i) *f is $\tau(\vartheta)$-continuous,*

(ii) *$M \neq \emptyset$,*

(iii) *f is mixed monotone, that is for each $x_1 \preceq x_2$ and $y_1 \succeq y_2$, $f(x_1, y_1) \preceq f(x_2, y_2)$.*

Then f has a coupled fixed-point (x^*, y^*). And there exist two sequences $\{x_n\}$ and $\{y_n\}$ with $x_{n-1} \preceq x_n = f(x_{n-1}, y_{n-1})$, $y_{n-1} \succeq y_n = f(y_{n-1}, x_{n-1})$, $n = 1, 2, \dots$ such that $x_n \to x^*$ and $y_n \to y^*$.

Theorem 4.5. *Let (X, ϑ) be a Hausdorff uniform space, p is an E-distance on X, $\varphi : X \to \mathbb{R}$ be a continuous function, and "$\preceq$" be the order in X introduced by φ. Let X be also a p-Cauchy complete space, $T : X \times X \to 2^X$ be a multivalued mapping, $[x, +\infty) = \{y \in X : x \preceq y\}$, $(-\infty, y] = \{x \in X : x \preceq y\}$, and $M = \{(x, y) \in X \times X : x \preceq y, T(x, y) \cap [x, +\infty) \neq \emptyset$ and $T(y, x) \cap (-\infty, y] \neq \emptyset\}$. Suppose that;*

(i) *T is mixed monotone, that is, for $x_1 \preceq y_1$, $x_2 \succeq y_2$ and $u \in T(x_1, y_1)$, $v \in T(y_1, x_1)$, there exist $w \in T(x_2, y_2)$, $z \in T(y_2, x_2)$ such that $u \preceq w$, $v \succeq z$,*

(ii) *$M \neq \emptyset$,*

(iii) *$T(x, y)$ is compact for each $(x, y) \in X \times X$.*

Then T has a coupled fixed-point.

Proof. By (ii), there exists $(x_0, y_0) \in M$ with $x_0 \preceq y_0$, $T(x_0, y_0) \cap [x_0, +\infty) \neq \emptyset$ and $T(y_0, x_0) \cap (-\infty, y_0] \neq \emptyset$. Let $C = \{(x, y) : x_0 \preceq x, y \preceq y_0, T(x, y) \cap [x, +\infty) \neq \emptyset$ and $T(y, x) \cap (-\infty, y] \neq \emptyset\}$. Then $(x_0, y_0) \in C$. Define the order relation "$\preceq$" in C by

$$(x_1, y_1) \preceq (x_2, y_2) \Longleftrightarrow x_1 \preceq x_2, y_2 \preceq y_1. \tag{4.5}$$

It is easy to prove that $(C, \preceq)$ becomes an ordered space.

We will prove that C has a maximum element. Let $\{x_v, y_v\}_{v \in \Lambda}$ be a totally ordered subset in C, where Λ is a directed set. For $v, \mu \in \Lambda$ and $v \leq \mu$, one has $(x_v, y_v) \preceq (x_\mu, y_\mu)$. So $x_v \preceq x_\mu$ and $y_\mu \preceq y_v$, which implies that

$$\begin{aligned} &\varphi(x_0) \geq \varphi(x_v) \geq \varphi(x_\mu) \geq \varphi(y_0), \\ &\varphi(y_0) \leq \varphi(y_\mu) \leq \varphi(y_v) \leq \varphi(x_0) \end{aligned} \tag{4.6}$$

for $v \leq \mu$.

Since $\{\varphi(x_v)\}$ and $\{\varphi(y_v)\}$ are convergence nets in $\mathbb{R}$. From

$$p(x_v, x_\mu) \leq \varphi(x_v) - \varphi(x_\mu), \qquad p(y_\mu, y_v) \leq \varphi(y_\mu) - \varphi(y_v), \tag{4.7}$$

we get that $\{x_v\}$ and $\{y_v\}$ are p-Cauchy nets in X. By the p-Cauchy completeness of X, let x_v convergence to x^* and y_v convergence to y^* in X. For given $\mu \in \Lambda$,

$$\begin{aligned} p(x_\mu, x^*) &= \lim_v p(x_\mu, x_v) \leq \lim_v (\varphi(x_\mu) - \varphi(x_v)) = \varphi(x_\mu) - \varphi(x^*), \\ p(y_\mu, y^*) &= \lim_v p(y_\mu, y_v) \leq \lim_v (\varphi(y_v) - \varphi(y_\mu)) = \varphi(y_v) - \varphi(y^*). \end{aligned} \tag{4.8}$$

So $x_0 \preceq x_\mu \preceq x^*$ and $y_\mu \succeq y^* \succeq y_0$ for all $\mu \in \Lambda$.

For $\mu \in \Lambda$, by the condition (i), for each $u_\mu \in T(x_\mu, y_\mu)$ with $x_\mu \preceq u_\mu$ and $v_\mu \in T(y_\mu, x_\mu)$ with $v_\mu \preceq y_\mu$, there exist $w_\mu \in T(x^*, y^*)$ and $z_\mu \in T(y^*, x^*)$ such that $u_\mu \preceq w_\mu$ and $v_\mu \succeq z_\mu$. By the compactness of $T(x^*, y^*)$ and $T(y^*, x^*)$, there exist convergence subnets $\{w_{\mu^l}\}$ of $\{w_\mu\}$

and $\{z_{\mu^|}\}$ of $\{z_\mu\}$. Suppose that $\{w_{\mu^|}\}$ converges to $w \in T(x^*, y^*)$ and $\{z_{\mu^|}\}$ converges to $z \in T(y^*, x^*)$. Take $\Lambda^|$, such that $\mu^| \geq \Lambda^|$ implies $u_\mu \preceq v_\mu \preceq v_{\mu^|}$. We have

$$\begin{aligned} p(u_\mu, w) &= \lim_{\mu^|} p\left(u_\mu, u_{\mu^|}\right) \leq \lim_{\mu^|}\left(\varphi(u_\mu) - \varphi\left(u_{\mu^|}\right)\right) = \varphi(u_\mu) - \varphi(w), \\ p(z, v_\mu) &= \lim_{\mu^|} p\left(v_{\mu^|}, v_\mu\right) \leq \lim_{\mu^|}\left(\varphi\left(v_{\mu^|}\right) - \varphi(v_\mu)\right) = \varphi(z) - \varphi(v_\mu). \end{aligned} \tag{4.9}$$

So $x_\mu \preceq u_\mu \preceq w$ and $z \preceq v_\mu \preceq y_\mu$ for all μ. And

$$\begin{aligned} p(x^*, w) &= \lim_{\mu^|} p\left(x_{\mu^|}, u_{\mu^|}\right) \leq \lim_{\mu^|}\left(\varphi\left(x_{\mu^|}\right) - \varphi\left(u_{\mu^|}\right)\right) = \varphi(x^*) - \varphi(w), \\ p(z, y^*) &= \lim_{\mu^|} p\left(v_{\mu^|}, y_{\mu^|}\right) \leq \lim_{\mu^|}\left(\varphi\left(v_{\mu^|}\right) - \varphi\left(y_{\mu^|}\right)\right) = \varphi(z) - \varphi(y^*). \end{aligned} \tag{4.10}$$

So $x^* \preceq w$ and $z \preceq y^*$, this gives that $(x^*, y^*) \in C$. Hence we have proven that $\{x_\mu, y_\mu\}_{\mu \in \Lambda}$ has an upper bound in C.

By Zorn's lemma, there exists a maximum element $(\overline{x}, \overline{y})$ in C. By the definition of C, there exist $\overline{u} \in T(\overline{x}, \overline{y})$, $\overline{v} \in T(\overline{y}, \overline{x})$, such that $x_0 \preceq \overline{u}, \overline{v} \preceq y_0$ and $\overline{x} \preceq \overline{u}, \overline{v} \preceq \overline{y}$. By the condition (i) there exist $\overline{w} \in T(\overline{u}, \overline{v})$, $\overline{z} \in T(\overline{v}, \overline{u})$ such that $x_0 \preceq \overline{u} \preceq \overline{w}$ and $\overline{z} \preceq \overline{v} \preceq y_0$. Hence $(\overline{u}, \overline{v}) \in C$ and $(\overline{x}, \overline{y}) \preceq (\overline{u}, \overline{v})$. Since $(\overline{x}, \overline{y})$ is maximum element in C, it follows that $(\overline{x}, \overline{y}) = (\overline{u}, \overline{v})$, and it follows that $\overline{x} = \overline{u} \in T(\overline{x}, \overline{u})$ and $\overline{y} = \overline{v} \in T(\overline{y}, \overline{x})$. So $(\overline{x}, \overline{y})$ is a coupled fixed-point of T. □

Corollary 4.6. *Let (X, ϑ) be a Hausdorff uniform space, p is an E-distance on X, $\varphi : X \to \mathbb{R}$ be a continuous function, and "$\preceq$" be the order in X introduced by φ. Let X be also a p-Cauchy complete space and $f : X \times X \to X$ be a mapping. Suppose that;*

(i) *f is mixed monotone, that is for $x_1 \preceq y_1$, $x_2 \succeq y_2$ and $f(x_1, y_1) \preceq f(y_2, x_2)$,*

(ii) *there exist $x_0, y_0 \in X$ such that $x_0 \preceq f(x_0, y_0)$ and $f(y_0, x_0) \preceq y_0$.*

Then f has a coupled fixed-point.

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