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Research Article

A General Iterative Approach to Variational Inequality Problems and Optimization Problems

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We introduce a new general iterative scheme for finding a common element of the set of solutions of variational inequality problem for an inverse-strongly monotone mapping and the set of fixed points of a nonexpansive mapping in a Hilbert space and then establish strong convergence of the sequence generated by the proposed iterative scheme to a common element of the above two sets under suitable control conditions, which is a solution of a certain optimization problem. Applications of the main result are also given.

1. Introduction

Let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and induced norm $\| \cdot \|$. Let C be a nonempty closed convex subset of H and $S : C \to C$ be self-mapping on C. We denote by $F(S)$ the set of fixed points of S and by P_C the metric projection of H onto C.

Let A be a nonlinear mapping of C into H. The variational inequality problem is to find a $u \in C$ such that

$$\langle v - u, Au \rangle \geq 0, \quad \forall v \in C. \tag{1.1}$$

We denote the set of solutions of the variational inequality problem (1.1) by $\mathrm{VI}(C, A)$. The variational inequality problem has been extensively studied in the literature; see [1–5] and the references therein.

Recently, in order to study the problem (1.1) coupled with the fixed point problem, many authors have introduced some iterative schemes for finding a common element of the set of the solutions of the problem (1.1) and the set of fixed points of nonexpansive mappings; see [6–9] and the references therein. In particular, in 2005, Iiduka and Takahashi [8] introduced an iterative scheme for finding a common point of the set of fixed points of a nonexapansive mapping S and the set of solutions of the problem (1.1) for an inverse-strong monotone mapping A: $x_1 \in C$ and

$$x_{n+1} = \alpha_n x + (1-\alpha_n) S P_C(x_n - \lambda_n A x_n), \quad n \geq 1, \tag{1.2}$$

where $\{\alpha_n\} \subset [0,1)$ and $\{\lambda_n\} \subset [0, 2\alpha]$. They proved that the sequence generated by (1.2) strongly converges strongly to $P_{F(S)\cap \mathrm{VI}(C,A)}x$. In 2010, Jung [10] provided the following new composite iterative scheme for the fixed point problem and the problem (1.1): $x_1 = x \in C$ and

$$\begin{aligned} y_n &= \alpha_n f(x_n) + (1-\alpha_n) S P_C(x_n - \lambda_n A x_n), \\ x_{n+1} &= (1-\beta_n) y_n + \beta_n S P_C(y_n - \lambda_n A y_n), \quad n \geq 1, \end{aligned} \tag{1.3}$$

where f is a contraction with constant $k \in (0,1), \{\alpha_n\}, \{\beta_n\} \in [0,1]$, and $\{\lambda_n\} \subset [0, 2\alpha]$. He proved that the sequence $\{x_n\}$ generated by (1.3) strongly converges strongly to a point in $F(S) \cap \mathrm{VI}(C, A)$, which is the unique solution of a certain variational inequality.

On the other hand, the following optimization problem has been studied extensively by many authors:

$$\min_{x\in\Omega} \frac{\mu}{2}\langle Bx, x\rangle + \frac{1}{2}\|x-u\|^2 - h(x), \tag{1.4}$$

where $\Omega = \bigcap_{n=1}^{\infty} C_n$, $C_1, C_2, \ldots$ are infinitely many closed convex subsets of H such that $\bigcap_{n=1}^{\infty} C_n \neq \emptyset$, $u \in H$, $\mu \geq 0$ is a real number, B is a strongly positive bounded linear operator on H (i.e., there is a constant $\overline{\gamma} > 0$ such that $\langle Bx, x\rangle \geq \overline{\gamma}\|x\|^2$, for all $x \in H$), and h is a potential function for γf (i.e., $h'(x) = \gamma f(x)$ for all $x \in H$). For this kind of optimization problems, see, for example, Deutsch and Yamada [11], Jung [10], and Xu [12, 13] when $\Omega = \bigcap_{i=1}^{N} C_i$ and $h(x) = \langle x, b\rangle$ for a given point b in H.

In 2007, related to a certain optimization problem, Marino and Xu [14] introduced the following general iterative scheme for the fixed point problem of a nonexpansive mapping:

$$x_{n+1} = \alpha_n \gamma f(x_n) + (I - \alpha_n B) S x_n, \quad n \geq 0, \tag{1.5}$$

where $\{\alpha_n\} \in (0,1)$ and $\gamma > 0$. They proved that the sequence $\{x_n\}$ generated by (1.5) converges strongly to the unique solution of the variational inequality

$$\left\langle (B - \gamma f)x^*, x - x^* \right\rangle \geq 0, \quad x \in F(S), \tag{1.6}$$

which is the optimality condition for the minimization problem

$$\min_{x\in F(S)} \frac{1}{2}\langle Bx, x\rangle - h(x), \tag{1.7}$$

where h is a potential function for γf. The result improved the corresponding results of Moudafi [15] and Xu [16].

In this paper, motivated by the above-mentioned results, we introduce a new general composite iterative scheme for finding a common point of the set of solutions of the variational inequality problem (1.1) for an inverse-strongly monotone mapping and the set of fixed points of a nonexapansive mapping and then prove that the sequence generated by the proposed iterative scheme converges strongly to a common point of the above two sets, which is a solution of a certain optimization problem. Applications of the main result are also discussed. Our results improve and complement the corresponding results of Chen et al. [6], Iiduka and Takahashi [8], Jung [10], and others.

2. Preliminaries and Lemmas

Let H be a real Hilbert space and let C be a nonempty closed convex subset of H. We write $x_n \rightharpoonup x$ to indicate that the sequence $\{x_n\}$ converges weakly to x. $x_n \to x$ implies that $\{x_n\}$ converges strongly to x.

First we recall that a mapping $f : C \to C$ is a *contraction* on C if there exists a constant $k \in (0,1)$ such that $\|f(x) - f(y)\| \leq k\|x - y\|$, $x,\ y \in C$. A mapping $T : C \to C$ is called *nonexpansive* if $\|Tx - Ty\| \leq \|x - y\|$, $x,\ y \in C$. We denote by $F(T)$ the set of fixed points of T.

For every point $x \in H$, there exists a unique nearest point in C, denoted by $P_C(x)$, such that

$$\|x - P_C(x)\| \leq \|x - y\| \tag{2.1}$$

for all $y \in C$. P_C is called the *metric projection* of H onto C. It is well known that P_C is nonexpansive and P_C satisfies

$$\langle x - y, P_C(x) - P_C(y)\rangle \geq \|P_C(x) - P_C(y)\|^2 \tag{2.2}$$

for every $x,\ y \in H$. Moreover, $P_C(x)$ is characterized by the properties:

$$\begin{gathered} \|x - y\|^2 \geq \|x - P_C(x)\|^2 + \|y - P_C(x)\|^2, \\ u = P_C(x) \Longleftrightarrow \langle x - u, u - y\rangle \geq 0, \quad \forall x \in H,\ y \in C. \end{gathered} \tag{2.3}$$

In the context of the variational inequality problem for a nonlinear mapping A, this implies that

$$u \in \mathrm{VI}(C, A) \Longleftrightarrow u = P_C(u - \lambda Au), \quad \text{for any } \lambda > 0. \tag{2.4}$$

It is also well known that H satisfies the *Opial condition*, that is, for any sequence $\{x_n\}$ with $x_n \rightharpoonup x$, the inequality

$$\liminf_{n\to\infty}\|x_n - x\| < \liminf_{n\to\infty}\|x_n - y\| \tag{2.5}$$

holds for every $y \in H$ with $y \neq x$.

A mapping A of C into H is called *inverse-strongly monotone* if there exists a positive real number α such that

$$\langle x-y, Ax-Ay\rangle \geq \alpha\|Ax-Ay\|^2 \tag{2.6}$$

for all $x,\ y\in C$; see [4, 7, 17]. For such a case, A is called α-inverse-strongly monotone. We know that if $A=I-T$, where T is a nonexpansive mapping of C into itself and I is the identity mapping of H, then A is $1/2$-inverse-strongly monotone and $\mathrm{VI}(C,A)=F(T)$. A mapping A of C into H is called *strongly monotone* if there exists a positive real number η such that

$$\langle x-y, Ax-Ay\rangle \geq \eta\|x-y\|^2 \tag{2.7}$$

for all $x,\ y\in C$. In such a case, we say A is η-strongly monotone. If A is η-strongly monotone and κ-*Lipschitz continuous*, that is, $\|Ax-Ay\|\leq\kappa\|x-y\|$ for all $x,\ y\in C$, then A is η/κ^2-inverse-strongly monotone. If A is an α-inverse-strongly monotone mapping of C into H, then it is obvious that A is $1/\alpha$-Lipschitz continuous. We also have that for all $x,\ y\in C$ and $\lambda>0$,

$$\begin{aligned}\|(I-\lambda A)x-(I-\lambda A)y\|^2 &= \|(x-y)-\lambda(Ax-Ay)\|^2\\ &= \|x-y\|^2-2\lambda\langle x-y, Ax-Ay\rangle+\lambda^2\|Ax-Ay\|^2\\ &\leq \|x-y\|^2+\lambda(\lambda-2\alpha)\|Ax-Ay\|^2.\end{aligned} \tag{2.8}$$

So, if $\lambda\leq 2\alpha$, then $I-\lambda A$ is a nonexpansive mapping of C into H. The following result for the existence of solutions of the variational inequality problem for inverse strongly-monotone mappings was given in Takahashi and Toyoda [9].

Proposition 2.1. *Let C be a bounded closed convex subset of a real Hilbert space and let A be an α-inverse-strongly monotone mapping of C into H. Then,* $\mathrm{VI}(C,A)$ *is nonempty.*

A set-valued mapping $T:H\to 2^H$ is called *monotone* if for all $x,\ y\in H$, $f\in Tx$, and $g\in Ty$ imply $\langle x-y, f-g\rangle\geq 0$. A monotone mapping $T:H\to 2^H$ is *maximal* if the graph $G(T)$ of T is not properly contained in the graph of any other monotone mapping. It is known that a monotone mapping T is maximal if and only if for $(x,f)\in H\times H$, $\langle x-y, f-g\rangle\geq 0$ for every $(y,g)\in G(T)$ implies $f\in Tx$. Let A be an inverse-strongly monotone mapping of C into H and let N_Cv be the *normal cone* to C at v, that is, $N_Cv=\{w\in H:\langle v-u, w\rangle\geq 0,\ \text{for all } u\in C\}$, and define

$$Tv=\begin{cases}Av+N_Cv, & v\in C,\\ \emptyset, & v\notin C.\end{cases} \tag{2.9}$$

Then T is maximal monotone and $0\in Tv$ if and only if $v\in\mathrm{VI}(C,A)$; see [18, 19].

We need the following lemmas for the proof of our main results.

Lemma 2.2. *In a real Hilbert space H, there holds the following inequality:*

$$\|x + y\|^2 \leq \|x\|^2 + 2\langle y, x + y\rangle, \tag{2.10}$$

for all $x, y \in H$.

Lemma 2.3 (Xu [12]). *Let $\{s_n\}$ be a sequence of nonnegative real numbers satisfying*

$$s_{n+1} \leq (1 - \lambda_n)s_n + \beta_n + \gamma_n, \quad n \geq 1, \tag{2.11}$$

where $\{\lambda_n\}$ and $\{\beta_n\}$ satisfy the following conditions:

(i) $\{\lambda_n\} \subset [0,1]$ *and* $\sum_{n=1}^{\infty} \lambda_n = \infty$ *or, equivalently,* $\prod_{n=1}^{\infty}(1 - \lambda_n) = 0$;

(ii) $\limsup_{n\to\infty}(\beta_n/\lambda_n) \leq 0$ *or* $\sum_{n=1}^{\infty} |\beta_n| < \infty$;

(iii) $\gamma_n \geq 0$ $(n \geq 1)$, $\sum_{n=1}^{\infty} \gamma_n < \infty$.

Then $\lim_{n\to\infty} s_n = 0$.

Lemma 2.4 (Marino and Xu [14]). *Assume that A is a strongly positive linear bounded operator on a Hilbert space H with constant $\overline{\gamma} > 0$ and $0 < \rho \leq \|B\|^{-1}$. Then $\|I - \rho B\| \leq 1 - \rho\overline{\gamma}$.*

The following lemma can be found in [20, 21] (see also Lemma 2.2 in [22]).

Lemma 2.5. *Let C be a nonempty closed convex subset of a real Hilbert space H, and let $g : C \to \mathbb{R} \cup \{\infty\}$ be a proper lower semicontinunous differentiable convex function. If x^* is a solution to the minimization problem*

$$g(x^*) = \inf_{x\in C} g(x), \tag{2.12}$$

then

$$\langle g'(x), x - x^*\rangle \geq 0, \quad x \in C. \tag{2.13}$$

In particular, if x^* solves the optimization problem

$$\min_{x\in C} \frac{\mu}{2}\langle Bx, x\rangle + \frac{1}{2}\|x - u\|^2 - h(x), \tag{2.14}$$

then

$$\langle u + (\gamma f - (I + \mu B))x^*, x - x^*\rangle \leq 0, \quad x \in C, \tag{2.15}$$

where h is a potential function for γf.

3. Main Results

In this section, we present a new general composite iterative scheme for inverse-strongly monotone mappings and a nonexpansive mapping.

Theorem 3.1. *Let C be a nonempty closed convex subset of a real Hilbert space H such that $C \pm C \subset C$. Let A be an α-inverse-strongly monotone mapping of C into H and S a nonexpansive mapping of C into itself such that $F(S) \cap \mathrm{VI}(C, A) \neq \emptyset$. Let $u \in C$ and let B be a strongly positive bounded linear operator on C with constant $\overline{\gamma} \in (0,1)$ and f a contraction of C into itself with constant $k \in (0,1)$. Assume that $\mu > 0$ and $0 < \gamma < (1+\mu)\overline{\gamma}/k$. Let $\{x_n\}$ be a sequence generated by*

$$\begin{aligned} x_1 &= x \in C, \\ y_n &= \alpha_n\big(u + \gamma f(x_n)\big) + \big(I - \alpha_n(I + \mu B)\big) S P_C(x_n - \lambda_n A x_n), \\ x_{n+1} &= (1-\beta_n) y_n + \beta_n S P_C(y_n - \lambda_n A y_n), \quad n \geq 1, \end{aligned} \tag{IS}$$

where $\{\lambda_n\} \subset [0, 2\alpha]$, $\{\alpha_n\} \subset [0,1)$, and $\{\beta_n\} \subset [0,1]$. Let $\{\alpha_n\}$, $\{\lambda_n\}$, and $\{\beta_n\}$ satisfy the following conditions:

(i) $\alpha_n \to 0$ $(n \to \infty)$; $\sum_{n=1}^{\infty} \alpha_n = \infty$;

(ii) $\beta_n \subset [0, a)$ *for all* $n \geq 0$ *and for some* $a \in (0,1)$;

(iii) $\lambda_n \in [c, d]$ *for some* c, d *with* $0 < c < d < 2\alpha$;

(iv) $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$, $\sum_{n=1}^{\infty} |\beta_{n+1} - \beta_n| < \infty$, $\sum_{n=1}^{\infty} |\lambda_{n+1} - \lambda_n| < \infty$.

Then $\{x_n\}$ converges strongly to $q \in F(S) \cap \mathrm{VI}(C, A)$, which is a solution of the optimization problem

$$\min_{x \in F(S) \cap \mathrm{VI}(C,A)} \frac{\mu}{2}\langle Bx, x\rangle + \frac{1}{2}\|x - u\|^2 - h(x), \tag{OP1}$$

where h is a potential function for γf.

Proof. We note that from the control condition (i), we may assume, without loss of generality, that $\alpha_n \leq (1 + \mu\|B\|)^{-1}$. Recall that if B is bounded linear self-adjoint operator on H, then

$$\|B\| = \sup\{|\langle Bu, u\rangle| : u \in H,\ \|u\| = 1\}. \tag{3.1}$$

Observe that

$$\begin{aligned} \big\langle \big(I - \alpha_n(I + \mu B)\big)u, u\big\rangle &= 1 - \alpha_n - \alpha_n \mu \langle Bu, u\rangle \\ &\geq 1 - \alpha_n - \alpha_n \mu \|B\| \\ &\geq 0, \end{aligned} \tag{3.2}$$

which is to say that $I - \alpha_n(I + \mu B)$ is positive. It follows that

$$\begin{aligned}\|I - \alpha_n(I + \mu B)\| &= \sup\{\langle (I - \alpha_n(I + \mu B))u, u\rangle : u \in H,\ \|u\| = 1\}\\ &= \sup\{1 - \alpha_n - \alpha_n\mu\langle Bu, u\rangle : u \in H,\ \|u\| = 1\}\\ &\le 1 - \alpha_n(1 + \mu\overline{\gamma})\\ &< 1 - \alpha_n(1 + \mu)\overline{\gamma}.\end{aligned} \tag{3.3}$$

Now we divide the proof into several steps.

Step 1. We show that $\{x_n\}$ is bounded. To this end, let $z_n = P_C(x_n - \lambda_n A x_n)$ and $w_n = P_C(y_n - \lambda_n A y_n)$ for every $n \ge 1$. Let $p \in F(S) \cap \mathrm{VI}(C, A)$. Since $I - \lambda_n A$ is nonexpansive and $p = P_C(p - \lambda_n A p)$ from (2.4), we have

$$\begin{aligned}\|z_n - p\| &\le \|(x_n - \lambda_n A x_n) - (p - \lambda_n A p)\|\\ &\le \|x_n - p\|.\end{aligned} \tag{3.4}$$

Similarly, we have

$$\|w_n - p\| \le \|y_n - p\|. \tag{3.5}$$

Now, set $\overline{B} = (I + \mu B)$. Let $p \in F(S) \cap \mathrm{VI}(C, A)$. Then, from (IS) and (3.4), we obtain

$$\begin{aligned}\|y_n - p\| &= \left\|\alpha_n u + \alpha_n\left(\gamma f(x_n) - \overline{B}p\right) + \left(I - \alpha_n\overline{B}\right)(S z_n - p)\right\|\\ &\le (1 - (1 + \mu)\overline{\gamma}\alpha_n)\|z_n - p\| + \alpha_n\|u\|\\ &\quad + \alpha_n\gamma\|f(x_n) - f(p)\| + \alpha_n\left\|\gamma f(p) - \overline{B}p\right\|\\ &\le (1 - (1 + \mu)\overline{\gamma}\alpha_n)\|z_n - p\| + \alpha_n\|u\|\\ &\quad + \alpha_n\gamma k\|x_n - p\| + \alpha_n\left\|\gamma f(p) - \overline{B}p\right\|\\ &= (1 - ((1 + \mu)\overline{\gamma} - \gamma k)\alpha_n)\|x_{n+1} - p\|\\ &\quad + ((1 + \mu)\overline{\gamma} - \gamma k)\alpha_n\frac{\left\|\gamma f(p) - \overline{B}p\right\| + \|u\|}{(1 + \mu)\overline{\gamma} - \gamma k}.\end{aligned} \tag{3.6}$$

From (3.5) and (3.6), it follows that

$$\begin{aligned}\|x_{n+1}-p\| &= \|(1-\beta_n)(y_n-p)+\beta_n(Sw_n-p)\| \\ &\le (1-\beta_n)\|y_n-p\|+\beta_n\|w_n-p\| \\ &\le (1-\beta_n)\|y_n-p\|+\beta_n\|y_n-p\| \\ &= \|y_n-p\| \\ &\le \max\left\{\|x_n-p\|, \frac{\left\|\gamma f(p)-\overline{B}p\right\|+\|u\|}{(1+\mu)\overline{\gamma}-\gamma k}\right\}.\end{aligned} \tag{3.7}$$

By induction, it follows from (3.7) that

$$\|x_n-p\| \le \max\left\{\|x_1-p\|, \frac{\left\|\gamma f(p)-\overline{B}p\right\|+\|u\|}{(1+\mu)\overline{\gamma}-\gamma k}\right\} \quad n\ge 1. \tag{3.8}$$

Therefore, $\{x_n\}$ is bounded. So $\{y_n\}$, $\{z_n\}$, $\{w_n\}$, $\{f(x_n)\}$, $\{Ax_n\}$, $\{Ay_n\}$, and $\{\overline{B}Sz_n\}$ are bounded. Moreover, since $\|Sz_n-p\|\le\|x_n-p\|$ and $\|Sw_n-p\|\le\|y_n-p\|$, $\{Sz_n\}$ and $\{Sw_n\}$ are also bounded. And by the condition (i), we have

$$\begin{aligned}\|y_n-Sz_n\| &= \alpha_n\|(u+\gamma f(x_n))-(I+\mu B)Sz_n\| \\ &= \alpha_n\left\|(u+\gamma f(x_n))-\overline{B}Sz_n\right\| \longrightarrow 0 \quad (\text{as } n\longrightarrow\infty).\end{aligned} \tag{3.9}$$

Step 2. We show that $\lim_{n\to\infty}\|x_{n+1}-x_n\|=0$ and $\lim_{n\to\infty}\|y_{n+1}-y_n\|=0$. Indeed, since $I-\lambda_nA$ and P_C are nonexpansive and $z_n=P_C(x_n-\lambda_nAx_n)$, we have

$$\begin{aligned}\|z_n-z_{n-1}\| &\le \|(x_n-\lambda_nAx_n)-(x_{n-1}-\lambda_{n-1}Ax_{n-1})\| \\ &\le \|x_n-x_{n-1}\|+|\lambda_n-\lambda_{n-1}|\|Ax_{n-1}\|.\end{aligned} \tag{3.10}$$

Similarly, we get

$$\|w_n-w_{n-1}\| \le \|y_n-y_{n-1}\|+|\lambda_n-\lambda_{n-1}|\|Ay_{n-1}\|. \tag{3.11}$$

Simple calculations show that

$$\begin{aligned}y_n-y_{n-1} &= \alpha_n(u+\gamma f(x_n))+\left(I-\alpha_n\overline{B}\right)Sz_n-\alpha_{n-1}(u+\gamma f(x_{n-1}))-\left(I-\alpha_{n-1}\overline{B}\right)Sz_{n-1} \\ &= (\alpha_n-\alpha_{n-1})\left(u+\gamma f(x_{n-1})-\overline{B}Sz_{n-1}\right)+\alpha_n\gamma(f(x_n)-f(x_{n-1})) \\ &\quad +\left(I-\alpha_n\overline{B}\right)(Sz_n-Sz_{n-1}).\end{aligned} \tag{3.12}$$

So, we obtain

$$\begin{aligned} \left\|y_n - y_{n-1}\right\| &\le |\alpha_n - \alpha_{n-1}|\left(\|u\| + \gamma\left\|f(x_{n-1})\right\| + \left\|\overline{B}\right\|\|Sz_{n-1}\|\right) \\ &\quad + \alpha_n\gamma k\|x_n - x_{n-1}\| + (1 - (1+\mu)\overline{\gamma}\alpha_n)\|z_n - z_{n-1}\| \\ &\le |\alpha_n - \alpha_{n-1}|\left(\|u\| + \gamma\left\|f(x_{n-1})\right\| + \left\|\overline{B}\right\|\|Sz_{n-1}\|\right) \\ &\quad + \alpha_n\gamma k\|x_n - x_{n-1}\| + (1 - (1+\mu)\overline{\gamma}\alpha_n)\|x_n - x_{n-1}\| \\ &\quad + |\lambda_n - \lambda_{n-1}|\|Ax_{n-1}\|. \end{aligned} \tag{3.13}$$

Also observe that

$$\begin{aligned} x_{n+1} - x_n = (1-\beta_n)(y_n - y_{n-1}) + (\beta_n - \beta_{n-1})(Sw_{n-1} - y_{n-1}) \\ + \beta_n(Sw_n - Sw_{n-1}). \end{aligned} \tag{3.14}$$

By (3.11), (3.13), and (3.14), we have

$$\begin{aligned} \|x_{n+1} - x_n\| &\le (1-\beta_n)\left\|y_n - y_{n-1}\right\| + \left|\beta_n - \beta_{n-1}\right|\left(\|Sx_{n-1}\| + \left\|y_{n-1}\right\|\right) \\ &\quad + \beta_n\|w_n - w_{n-1}\| \\ &\le (1-\beta_n)\left\|y_n - y_{n-1}\right\| + \beta_n\left\|y_n - y_{n-1}\right\| + \beta_n|\lambda_n - \lambda_{n-1}|\left\|Ay_{n-1}\right\| \\ &\quad + \left|\beta_n - \beta_{n-1}\right|\left(\|Sw_{n-1}\| + \left\|y_{n-1}\right\|\right) \\ &\le \left\|y_n - y_{n-1}\right\| + |\lambda_n - \lambda_{n-1}|\left\|Ay_{n-1}\right\| + \left|\beta_n - \beta_{n-1}\right|\left(\|Sw_{n-1}\| + \left\|y_{n-1}\right\|\right) \\ &\le (1 - ((1+\mu)\overline{\gamma} - \gamma k)\alpha_n)\|x_n - x_{n-1}\| \\ &\quad + |\alpha_n - \alpha_{n-1}|\left(\|u\| + \gamma\left\|f(x_{n-1})\right\| + \left\|\overline{B}\right\|\|Sz_{n-1}\|\right) \\ &\quad + |\lambda_n - \lambda_{n-1}|\left(\left\|Ay_{n-1}\right\| + \|Ax_{n-1}\|\right) + \left|\beta_n - \beta_{n-1}\right|\left(\|Sw_{n-1}\| + \left\|y_{n-1}\right\|\right) \\ &\le (1 - ((1+\mu)\overline{\gamma} - \gamma k)\alpha_n)\|x_n - x_{n-1}\| \\ &\quad + M_1|\alpha_n - \alpha_{n-1}| + M_2|\lambda_n - \lambda_{n-1}| + M_3\left|\beta_n - \beta_{n-1}\right|, \end{aligned} \tag{3.15}$$

where $M_1 = \sup\{\|u\| + \gamma\|f(x_n)\| + \|\overline{B}\|\|T_n z_n\| : n \ge 1\}$, $M_2 = \sup\{\|Ay_n\| + \|Ax_n\| : n \ge 1\}$, and $M_3 = \sup\{\|Sw_n\| + \|y_n\| : n \ge 1\}$. From the conditions (i) and (iv), it is easy to see that

$$\begin{aligned} &\lim_{n\to\infty}((1+\mu)\overline{\gamma} - \gamma k)\alpha_n = 0, \qquad \sum_{n=1}^{\infty}((1+\mu)\overline{\gamma} - \gamma k)\alpha_n = \infty, \\ &\sum_{n=2}^{\infty}\left(M_1|\alpha_n - \alpha_{n-1}| + M_2|\lambda_n - \lambda_{n-1}| + M_3\left|\beta_n - \beta_{n-1}\right|\right) < \infty. \end{aligned} \tag{3.16}$$

Applying Lemma 2.3 to (3.15), we obtain

$$\lim_{n\to\infty}\|x_{n+1}-x_n\|=0. \tag{3.17}$$

Moreover, by (3.10) and (3.13), we also have

$$\lim_{n\to\infty}\|z_{n+1}-z_n\|=0, \qquad \lim_{n\to\infty}\|y_{n+1}-y_n\|=0. \tag{3.18}$$

Step 3. We show that $\lim_{n\to\infty}\|x_n-y_n\|=0$ and $\lim_{n\to\infty}\|x_n-Sz_n\|=0$. Indeed,

$$\begin{aligned}\|x_{n+1}-y_n\| &= \beta_n\|Sw_n-y_n\| \\ &\le \beta_n(\|Sw_n-Sz_n\|+\|Sz_n-y_n\|) \\ &\le a(\|w_n-z_n\|+\|Sz_n-y_n\|) \\ &\le a(\|y_n-x_n\|+\|Sz_n-y_n\|) \\ &\le a(\|y_n-x_{n+1}\|+\|x_{n+1}-x_n\|+\|Sz_n-y_n\|)\end{aligned} \tag{3.19}$$

which implies that

$$\|x_{n+1}-y_n\|\le\frac{a}{1-a}(\|x_{n+1}-x_n\|+\|Sz_n-y_n\|). \tag{3.20}$$

Obviously, by (3.9) and Step 2, we have $\|x_{n+1}-y_n\|\to 0$ as $n\to\infty$. This implies that

$$\|x_n-y_n\|\le\|x_n-x_{n+1}\|+\|x_{n+1}-y_n\|\longrightarrow 0 \quad \text{as } n\longrightarrow\infty. \tag{3.21}$$

By (3.9) and (3.21), we also have

$$\|x_n-Sz_n\|\le\|x_n-y_n\|+\|y_n-Sz_n\|\longrightarrow 0 \quad \text{as } n\longrightarrow\infty. \tag{3.22}$$

Step 4. We show that $\lim_{n\to\infty}\|x_n - z_n\| = 0$ and $\lim_{n\to\infty}\|y_n - z_n\| = 0$. To this end, let $pF(S) \cap \mathrm{VI}(C, A)$. Since $z_n = P_C(x_n - \lambda_n A x_n)$ and $p = P_C(p - \lambda_n p)$, we have

$$
\begin{aligned}
\|y_n - p\|^2 &= \left\|\alpha_n\left(u + \gamma f(x_n) - \overline{B}p\right) + \left(I - \alpha_n \overline{B}\right)(Sz_n - p)\right\|^2 \\
&\le \left(\alpha_n\left\|u + \gamma f(x_n) - \overline{B}p\right\| + \left\|I - \alpha_n \overline{B}\right\|\|Sz_n - p\|\right)^2 \\
&\le \alpha_n\left\|u + \gamma f(x_n) - \overline{B}p\right\|^2 + (1 - \alpha_n(1+\mu)\overline{\gamma})\|z_n - p\|^2 \\
&\quad + 2\alpha_n(1 - \alpha_n(1+\mu)\overline{\gamma})\left\|u + \gamma f(x_n) - \overline{B}p\right\|\|z_n - p\| \\
&\le \alpha_n\left\|u + \gamma f(x_n) - \overline{B}p\right\|^2 \\
&\quad + (1 - \alpha_n(1+\mu)\overline{\gamma})\left[\|x_n - p\|^2 + \lambda_n(\lambda_n - 2\alpha)\|Ax_n - Ap\|^2\right] \\
&\quad + 2\alpha_n(1 - \alpha_n(1+\mu)\overline{\gamma})\left\|\gamma u + f(x_n) - \overline{B}p\right\|\|z_n - p\| \\
&\le \alpha_n\left\|u + \gamma f(x_n) - \overline{B}p\right\|^2 + \|x_n - p\|^2 \\
&\quad + (1 - \alpha_n(1+\mu)\overline{\gamma})c(d - 2\alpha)\|Ax_n - Ap\|^2 \\
&\quad + 2\alpha_n\left\|u + \gamma f(x_n) - \overline{B}p\right\|\|z_n - p\|.
\end{aligned}
\tag{3.23}
$$

So we obtain

$$
\begin{aligned}
&-(1 - \alpha_n(1+\mu)\overline{\gamma})c(d - 2\alpha)\|Ax_n - Ap\|^2 \\
&\le \alpha_n\left\|\gamma u + f(x_n) - \overline{B}p\right\|^2 + (\|x_n - p\| + \|y_n - p\|)(\|x_n - p\| - \|y_n - p\|) \\
&\quad + 2\alpha_n\left\|\gamma u + f(x_n) - \overline{B}p\right\|\|z_n - p\| \\
&\le \alpha_n\left\|\gamma u + f(x_n) - \overline{B}p\right\|^2 + (\|x_n - p\| + \|y_n - p\|)\|x_n - y_n\| \\
&\quad + 2\alpha_n\left\|\gamma u + f(x_n) - \overline{B}p\right\|\|z_n - p\|.
\end{aligned}
\tag{3.24}
$$

Since $\alpha_n \to 0$ from the condition (i) and $\|x_n - y_n\| \to 0$ from Step 3, we have $\|Ax_n - Ap\| \to 0$ $(n \to \infty)$. Moreover, from (2.4) we obtain

$$\begin{aligned}
\|z_n - p\|^2 &= \|P_C(x_n - \lambda_n Ax_n) - P_C(p - \lambda_n Ap)\|^2 \\
&\le \langle x_n - \lambda_n Ax_n - (p - \lambda_n Ap), z_n - p\rangle \\
&= \frac{1}{2}\Big\{\|(x_n - \lambda_n Ax_n) - (p - \lambda_n Ap)\|^2 + \|z_n - p\|^2 \\
&\quad -\|(x_n - \lambda_n Ax_n) - (p - \lambda_n Ap) - (z_n - p)\|^2\Big\} \\
&\le \frac{1}{2}\Big\{\|x_n - p\|^2 + \|z_n - p\|^2 - \|x_n - z_n\|^2 \\
&\quad +2\lambda_n\langle x_n - z_n, Ax_n - Ap\rangle - \lambda_n^2\|Ax_n - Ap\|^2\Big\},
\end{aligned} \tag{3.25}$$

and so

$$\|z_n - p\|^2 \le \|x_n - p\|^2 - \|x_n - z_n\|^2 + 2\lambda_n\langle x_n - z_n, Ax_n - Ap\rangle - \lambda_n^2\|Ax_n - Ap\|^2. \tag{3.26}$$

Thus

$$\begin{aligned}
\|y_n - p\|^2 &\le \alpha_n\left\|u + \gamma f(x_n) - \overline{B}p\right\|^2 + (1 - \alpha_n(1+\mu)\overline{\gamma})\|z_n - p\|^2 \\
&\quad + 2\alpha_n(1 - \alpha_n(1+\mu)\overline{\gamma})\left\|\gamma u + f(x_n) - \overline{B}p\right\|\|z_n - p\| \\
&\le \alpha_n\left\|u + \gamma f(x_n) - \overline{B}p\right\|^2 + \|x_n - p\|^2 - (1 - \alpha_n(1+\mu)\overline{\gamma})\|x_n - z_n\|^2 \\
&\quad + 2(1 - \alpha_n(1+\mu)\overline{\gamma})\lambda_n\langle x_n - z_n, Ax_n - Ap\rangle \\
&\quad - (1 - \alpha_n(1+\mu)\overline{\gamma})\lambda_n^2\|Ax_n - Ap\|^2 \\
&\quad + 2\alpha_n\left\|u + \gamma f(x_n) - \overline{B}\right\|\|z_n - p\|.
\end{aligned} \tag{3.27}$$

Then, we have

$$\begin{aligned}
&(1-\alpha_n(1+\mu)\overline{\gamma})\|x_n-z_n\|^2\\
&\quad\le \alpha_n\left\|u+\gamma f(x_n)-\overline{B}p\right\|^2+(\|x_n-p\|+\|y_n-p\|)(\|x_n-p\|-\|y_n-p\|)\\
&\qquad+2(1-\alpha_n(1+\mu)\overline{\gamma})\lambda_n\langle x_n-z_n,Ax_n-Ap\rangle-(1-\alpha_n(1+\mu)\overline{\gamma})\lambda_n^2\|Ax_n-Ap\|^2\\
&\qquad+2\alpha_n\left\|u+\gamma f(x_n)-\overline{B}p\right\|\|z_n-p\|\\
&\quad\le \alpha_n\left\|u+\gamma f(x_n)-\overline{B}p\right\|^2+(\|x_n-p\|+\|y_n-p\|)\|x_n-y_n\|\\
&\qquad+2(1-\alpha_n(1+\mu)\overline{\gamma})\lambda_n\langle x_n-z_n,Ax_n-Ap\rangle-(1-\alpha_n(1+\mu)\overline{\gamma})\lambda_n^2\|Ax_n-Ap\|^2\\
&\qquad+2\alpha_n\left\|u+\gamma f(x_n)-\overline{B}p\right\|\|z_n-p\|.
\end{aligned}\tag{3.28}$$

Since $\alpha_n\to 0$, $\|x_n-y_n\|\to 0$ and $\|Ax_n-Au\|\to 0$, we get $\|x_n-z_n\|\to 0$. Also by (3.21)

$$\|y_n-z_n\|\le\|y_n-x_n\|+\|x_n-z_n\|\longrightarrow 0\quad(n\longrightarrow\infty).\tag{3.29}$$

Step 5. We show that $\lim_{n\to\infty}\|Sz_n-z_n\|=0$. In fact, since

$$\begin{aligned}
\|Sz_n-z_n\|&\le\|Sz_n-y_n\|+\|y_n-z_n\|\\
&=\alpha_n\left\|u+\gamma f(x_n)-\overline{B}Sz_n\right\|+\|y_n-z_n\|,
\end{aligned}\tag{3.30}$$

from (3.9) and (3.29) , we have $\lim_{n\to\infty}\|Sz_n-z_n\|=0$.

Step 6. We show that

$$\limsup_{n\to\infty}\langle u+(\gamma f-(I+\mu B))q,y_n-q\rangle=\limsup_{n\to\infty}\left\langle u+\left(\gamma f-\overline{B}\right)q,y_n-q\right\rangle\le 0,\tag{3.31}$$

where q is a solution of the optimization problem (OP1). First we prove that

$$\limsup_{n\to\infty}\left\langle u+\left(\gamma f-\overline{B}\right)q,Sz_n-q\right\rangle\le 0.\tag{3.32}$$

Since $\{z_n\}$ is bounded, we can choose a subsequence $\{z_{n_i}\}$ of $\{z_n\}$ such that

$$\limsup_{n\to\infty}\left\langle u+\left(\gamma f-\overline{B}\right)q, Sz_n-q\right\rangle=\lim_{i\to\infty}\left\langle u+\left(\gamma f-\overline{B}\right)q, Sz_{n_i}-q\right\rangle. \tag{3.33}$$

Without loss of generality, we may assume that $\{z_{n_i}\}$ converges weakly to $z\in C$.

Now we will show that $z\in F(S)\cap \mathrm{VI}(C,A)$. First we show that $z\in F(S)$. Assume that $z\notin F(S)$. Since $z_{n_i}\rightharpoonup z$ and $Sz\neq z$, by the Opial condition and Step 5, we obtain

$$\begin{aligned}
\liminf_{i\to\infty}\|z_{n_i}-z\| &< \liminf_{i\to\infty}\|z_{n_i}-Sz\| \\
&\leq \liminf_{i\to\infty}(\|z_{n_i}-Sz_{n_i}\|+\|Sz_{n_i}-Sz\|) \\
&= \liminf_{i\to\infty}\|Sz_{n_i}-Sz\| \\
&\leq \liminf_{i\to\infty}\|z_{n_i}-z\|,
\end{aligned} \tag{3.34}$$

which is a contradiction. Thus we have $z\in F(S)$.

Next, let us show that $z\in \mathrm{VI}(C,A)$. Let

$$Tv=\begin{cases} Av+N_Cv, & v\in C,\\ \emptyset, & v\notin C.\end{cases} \tag{3.35}$$

Then T is maximal monotone. Let $(v,w)\in G(T)$. Since $w-Av\in N_Cv$ and $z_n\in C$, we have

$$\langle v-z_n, w-Av\rangle\geq 0. \tag{3.36}$$

On the other hand, from $z_n=P_C(x_n-\lambda_nAx_n)$, we have $\langle v-z_n, z_n-(x_n-\lambda_nAz_n)\rangle\geq 0$ and hence

$$\langle v-z_n, \frac{z_n-x_n}{\lambda_n}+Ax_n\rangle\geq 0. \tag{3.37}$$

Therefore, we have

$$
\begin{aligned}
\langle v - z_{n_i}, w \rangle &\geq \langle v - z_{n_i}, Av \rangle \\
&\geq \langle v - z_{n_i}, Av \rangle - \left\langle v - z_{n_i}, \frac{z_{n_i} - x_{n_i}}{\lambda_{n_i}} + Ax_{n_i} \right\rangle \\
&= \left\langle v - z_{n_i}, Av - Ax_{n_i} - \frac{z_{n_i} - x_{n_i}}{\lambda_{n_i}} \right\rangle \\
&= \langle v - z_{n_i}, Av - Az_{n_i} \rangle + \langle v - z_{n_i}, Az_{n_i} - Ax_{n_i} \rangle - \left\langle v - z_{n_i}, \frac{z_{n_i} - x_{n_i}}{\lambda_{n_i}} \right\rangle \\
&\geq \langle v - z_{n_i}, Az_{n_i} - Ax_{n_i} \rangle - \left\langle v - z_{n_i}, \frac{z_{n_i} - x_{n_i}}{\lambda_{n_i}} \right\rangle.
\end{aligned}
\tag{3.38}
$$

Since $\|z_n - x_n\| \to 0$ in Step 4 and A is α-inverse-strongly monotone, we have $\langle v - z, w \rangle \geq 0$ as $i \to \infty$. Since T is maximal monotone, we have $z \in T^{-1}0$ and hence $z \in \mathrm{VI}(C, A)$.

Therefore, $z \in F(S) \cap \mathrm{VI}(C, A)$. Now from Lemma 2.5 and Step 5, we obtain

$$
\begin{aligned}
\limsup_{n\to\infty} \left\langle u + \left(\gamma f - \overline{B}\right)q, Sz_n - q \right\rangle &= \lim_{i\to\infty} \left\langle u + \left(\gamma f - \overline{B}\right)q, Sz_{n_i} - q \right\rangle \\
&= \lim_{i\to\infty} \left\langle u + \left(\gamma f - \overline{B}\right)q, z_{n_i} - q \right\rangle \\
&= \left\langle u + \left(\gamma f - \overline{B}\right)q, z - q \right\rangle \\
&\leq 0.
\end{aligned}
\tag{3.39}
$$

By (3.9) and (3.39), we conclude that

$$
\begin{aligned}
&\limsup_{n\to\infty} \left\langle u + \left(\gamma f - \overline{B}\right)q, y_n - q \right\rangle \\
&\quad\leq \limsup_{n\to\infty} \left\langle u + \left(\gamma f - \overline{B}\right)q, y_n - Sz_n \right\rangle + \limsup_{n\to\infty} \left\langle u + \left(\gamma f - \overline{B}\right)q, Sz_n - q \right\rangle \\
&\quad\leq \limsup_{n\to\infty} \left\| u + \left(\gamma f - \overline{B}\right)q \right\| \|y_n - Sz_n\| + \limsup_{n\to\infty} \left\langle u + \left(\gamma f - \overline{B}\right)q, Sz_n - q \right\rangle \\
&\quad\leq 0.
\end{aligned}
\tag{3.40}
$$

Step 7. We show that $\lim_{n\to\infty}\|x_n - q\| = 0$ and $\lim_{n\to\infty}\|u_n - q\| = 0$, where q is a solution of the optimization problem (OP1). Indeed from (IS) and Lemma 2.2, we have

$$
\begin{aligned}
\|x_{n+1} - q\|^2 &\le \|y_n - q\|^2 \\
&= \left\|\alpha_n\left(u + \gamma f(x_n) - \overline{B}q\right) + \left(I - \alpha_n\overline{B}\right)(Sz_n - q)\right\| \\
&\le \left\|\left(I - \alpha_n\overline{B}\right)(Sz_n - q)\right\|^2 + 2\alpha_n\left\langle u + \gamma f(x_n) - \overline{B}q, y_n - q\right\rangle \\
&\le (1 - (1+\mu)\overline{\gamma}\alpha_n)^2\|z_n - q\|^2 + 2\alpha_n\gamma\langle f(x_n) - f(q), y_n - q\rangle \\
&\quad + 2\alpha_n\left\langle u + \gamma f(q) - \overline{B}q, y_n - q\right\rangle \\
&\le (1 - (1+\mu)\overline{\gamma}\alpha_n)^2\|x_n - q\|^2 + 2\alpha_n\gamma k\|x_n - q\|\|y_n - q\| \\
&\quad + 2\alpha_n\left\langle u + \left(\gamma f - \overline{B}\right)q, y_n - q\right\rangle \\
&\le (1 - (1+\mu)\overline{\gamma}\alpha_n)^2\|x_n - q\|^2 + 2\alpha_n\gamma k\|x_n - q\|(\|y_n - x_n\| + \|x_n - q\|) \\
&\quad + 2\alpha_n\left\langle u + \left(\gamma f - \overline{B}\right)q, y_n - q\right\rangle \\
&= (1 - 2((1+\mu)\overline{\gamma} - \gamma k)\alpha_n)\|x_n - q\|^2 \\
&\quad + \alpha_n^2((1+\mu)\overline{\gamma})^2\|x_n - q\|^2 + 2\alpha_n\gamma k\|x_n - q\|\|y_n - x_n\| \\
&\quad + 2\alpha_n\left\langle u + \left(\gamma f - \overline{B}\right)q, y_n - q\right\rangle,
\end{aligned}
\tag{3.41}
$$

that is,

$$
\begin{aligned}
\|x_{n+1} - q\|^2 &\le (1 - 2((1+\mu)\overline{\gamma} - \gamma k)\alpha_n)\|x_n - q\|^2 \\
&\quad + \alpha_n^2((1+\mu)\overline{\gamma})^2M_4^2 + 2\alpha_n\gamma k\|y_n - x_n\|M_4 \\
&\quad + 2\alpha_n\left\langle u + \left(\gamma f - \overline{B}\right)q, y_n - q\right\rangle \\
&= (1 - \overline{\alpha_n})\|x_n - q\|^2 + \overline{\beta_n},
\end{aligned}
\tag{3.42}
$$

where $M_4 = \sup\{\|x_n - q\| : n \ge 1\}$, $\overline{\alpha_n} = 2((1+\mu)\overline{\gamma} - \gamma k)\alpha_n$, and

$$
\overline{\beta_n} = \alpha_n\left[\alpha_n(1+\mu\overline{\gamma})^2M_4^2 + 2\gamma k\|y_n - x_n\|M_4 + 2\left\langle u + \left(\gamma f - \overline{B}\right)q, y_n - q\right\rangle\right].
\tag{3.43}
$$

From (i), $\|y_n - x_n\| \to 0$ in Steps 3, and 6, it is easily seen that $\overline{\alpha_n} \to 0$, $\sum_{n=1}^{\infty}\overline{\alpha_n} = \infty$, and $\limsup_{n\to\infty}(\overline{\beta_n}/\overline{\alpha_n}) \le 0$. Hence, by Lemma 2.3, we conclude $x_n \to q$ as $n \to \infty$. This completes the proof. □

As a direct consequence of Theorem 3.1, we have the following results.

Corollary 3.2. *Let $H, C, S, B, f, u, \gamma, \overline{\gamma}, k$, and μ be as in Theorem 3.1. Let $\{x_n\}$ be a sequence generated by*

$$x_1 = x \in C,$$
$$y_n = \alpha_n(u + \gamma f(x_n)) + (I - \alpha_n(I + \mu B))Sx_n, \tag{3.44}$$
$$x_{n+1} = (1 - \beta_n)y_n + \beta_n S y_n, \quad n \geq 1,$$

where $\{\alpha_n\}$ and $\{\beta_n\} \subset [0,1]$. Let $\{\alpha_n\}$ and $\{\beta_n\}$ satisfy the conditions (i), (ii), and (iv) in Theorem 3.1. Then $\{x_n\}$ converges strongly to $q \in F(S)$, which is a solution of the optimization problem

$$\min_{x \in F(S)} \frac{\mu}{2}\langle Bx, x\rangle + \frac{1}{2}\|x - u\|^2 - h(x), \tag{OP2}$$

where h is a potential function for γf.

Corollary 3.3. *Let $H, C, A, B, f, u, \gamma, \overline{\gamma}, k$, and μ be as in Theorem 3.1. Let $\{x_n\}$ be a sequence generated by*

$$x_1 = x \in C,$$
$$y_n = \alpha_n(u + \gamma f(x_n)) + (I - \alpha_n(I + \mu B))P_C(x_n - \lambda_n A x_n), \tag{3.45}$$
$$x_{n+1} = (1 - \beta_n)y_n + \beta_n P_C(y_n - \lambda_n A y_n), \quad n \geq 1,$$

where $\{\lambda_n\} \subset [0, 2\alpha]$, $\{\alpha_n\} \subset [0,1)$, and $\{\beta_n\} \subset [0,1]$. Let $\{\alpha_n\}$, $\{\lambda_n\}$ and $\{\beta_n\}$ satisfy the conditions (i), (ii), (iii), and (iv) in Theorem 3.1. Then $\{x_n\}$ converges strongly to $q \in \mathrm{VI}(C, A)$, which is a solution of the optimization problem

$$\min_{x \in VI(C,A)} \frac{\mu}{2}\langle Bx, x\rangle + \frac{1}{2}\|x - u\|^2 - h(x), \tag{OP3}$$

where h is a potential function for γf.

Remark 3.4. (1) Theorem 3.1 and Corollary 3.3 improve and develop the corresponding results in Chen et al. [6], Iiduka and Takahashi [8], and Jung [10].

(2) Even though $\beta_n = 0$ for $n \geq 1$, the iterative scheme (3.44) in Corollary 3.2 is a new one for fixed point problem of a nonexpansive mapping.

4. Applications

In this section, as in [6, 8, 10], we prove two theorems by using Theorem 3.1. First of all, we recall the following definition.

A mapping $T : C \to C$ is called *strictly pseudocontractive* if there exists α with $0 \leq \alpha < 1$ such that

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + \alpha\|(I - T)x - (I - T)y\|^2 \tag{4.1}$$

for every $x,\ y \in C$. If $k = 0$, then T is nonexpansive. Put $A = I - T$, where $T : C \to C$ is a strictly pseudo-contractive mapping with constant α. Then A is $(1 - \alpha)/2$-inverse-strongly monotone; see [2]. Actually, we have, for all $x,\ y \in C$,

$$\|(I - A)x - (I - A)y\|^2 \leq \|x - y\|^2 + \alpha\|Ax - Ay\|^2. \tag{4.2}$$

On the other hand, since H is a real Hilbert space, we have

$$\|(I - A)x - (I - A)y\|^2 = \|x - y\|^2 + \|Ax - Ay\|^2 - 2\langle x - y, Ax - Ay\rangle. \tag{4.3}$$

Hence we have

$$\langle x - y, Ax - Ay\rangle \geq \frac{1 - \alpha}{2}\|Ax - Ay\|^2. \tag{4.4}$$

Using Theorem 3.1, we found a strong convergence theorem for finding a common fixed point of a nonexpansive mapping and a strictly pseudo-contractive mapping.

Theorem 4.1. *Let $H, C, S, B, f, u, \gamma, \overline{\gamma}, k$, and μ be as in Theorem 3.1. Let T be an α-strictly pseudo-contractive mapping of C into itself such that $F(S) \cap F(T) \neq \emptyset$. Let $\{x_n\}$ be a sequence generated by*

$$\begin{aligned} x_1 &= x \in C, \\ y_n &= \alpha_n(u + \gamma f(x_n)) + (I - \alpha_n(I + \mu B))S((1 - \lambda_n)x_n + \lambda_n T x_n), \\ x_{n+1} &= (1 - \beta_n)y_n + \beta_n S((1 - \lambda_n)y_n + \lambda_n T y_n), \quad n \geq 1, \end{aligned} \tag{4.5}$$

where $\{\lambda_n\} \subset [0, 1 - \alpha]$, $\{\alpha_n\} \subset [0, 1)$, and $\{\beta_n\} \subset [0, 1]$. Let $\{\alpha_n\}$, $\{\lambda_n\}$, and $\{\beta_n\}$ satisfy the conditions (i), (ii), (iii), and (iv) in Theorem 3.1. Then $\{x_n\}$ converges strongly to $q \in F(S) \cap F(T)$, which is a solution of the optimization problem

$$\min_{x \in F(S) \cap F(T)} \frac{\mu}{2}\langle Bx, x\rangle + \frac{1}{2}\|x - u\|^2 - h(x), \tag{OP4}$$

where h is a potential function for γf.

Proof. Put $A = I - T$. Then A is $(1 - \alpha)/2$-inverse-strongly monotone. We have $F(T) = \mathrm{VI}(C, A)$ and $P_C(x_n - \lambda_n A x_n) = (1 - \lambda_n)x_n + \lambda_n T x_n$. Thus, the desired result follows from Theorem 3.1. □

Using Theorem 3.1, we also obtain the following result.

Theorem 4.2. *Let H be a real Hilbert space. Let A be an α-inverse-strongly monotone mapping of Hinto H and S a nonexpansive mapping of H into itself such that $F(S) \cap A^{-1}0 \neq \emptyset$. Let $u \in H$, and let B be a strongly positive bounded linear operator on H with constant $\overline{\gamma} > 0$ and $f : H \to H$ a contraction with constant $k \in (0,1)$. Assume that $\mu > 0$ and $0 < \gamma < (1+\mu)\overline{\gamma}/k$. Let $\{x_n\}$ be a sequence generated by*

$$\begin{gathered} x_1 = x \in H, \\ y_n = \alpha_n\big(u + \gamma f(x_n)\big) + \big(I - \alpha_n(I + \mu B)\big)S(x_n - \lambda_n A x_n), \\ x_{n+1} = (1-\beta_n)y_n + \beta_n S(y_n - \lambda_n A y_n), \quad n \geq 1, \end{gathered} \tag{4.6}$$

where $\{\lambda_n\} \subset [0,2\alpha]$, $\{\alpha_n\} \subset [0,1)$, and $\{\beta_n\} \subset [0,1]$. Let $\{\alpha_n\}$, $\{\lambda_n\}$, and $\{\beta_n\}$ satisfy the conditions (i), (ii), (iii), and (iv) in Theorem 3.1. Then $\{x_n\}$ converges strongly to $q \in F(S) \cap A^{-1}0$, which is a solution of the optimization problem

$$\min_{x \in F(S) \cap A^{-1}0} \frac{\mu}{2}\langle Bx, x\rangle + \frac{1}{2}\|x - u\|^2 - h(x), \tag{OP5}$$

where h is a potential function for γf.

Proof. We have $A^{-1}0 = \mathrm{VI}(H, A)$. So, putting $P_H = I$, by Theorem 3.1, we obtain the desired result. □

Remark 4.3. (1) Theorems 4.1 and 4.2 complement and develop the corresponding results in Chen et al. [6] and Jung [10].

(2) In all our results, we can replace the condition $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$ on the control parameter $\{\alpha_n\}$ by the condition $\alpha_n \in (0,1]$ for $n \geq 1$, $\lim_{n\to\infty} \alpha_n/\alpha_{n+1} = 1$ [12, 13] or by the perturbed control condition $|\alpha_{n+1} - \alpha_n| < o(\alpha_{n+1}) + \sigma_n, \sum_{n=1}^{\infty} \sigma_n < \infty$ [23].

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