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Research Article

Common Coupled Fixed Point Theorems for Contractive Mappings in Fuzzy Metric Spaces

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We prove a common fixed point theorem for mappings under ϕ-contractive conditions in fuzzy metric spaces. We also give an example to illustrate the theorem. The result is a genuine generalization of the corresponding result of S.Sedghi et al. (2010)

1. Introduction

Since Zadeh [1] introduced the concept of fuzzy sets, many authors have extensively developed the theory of fuzzy sets and applications. George and Veeramani [2, 3] gave the concept of fuzzy metric space and defined a Hausdorff topology on this fuzzy metric space which have very important applications in quantum particle physics particularly in connection with both string and E-infinity theory.

Bhaskar and Lakshmikantham [4], Lakshmikantham and Ćirić [5] discussed the mixed monotone mappings and gave some coupled fixed point theorems which can be used to discuss the existence and uniqueness of solution for a periodic boundary value problem. Sedghi et al. [6] gave a coupled fixed point theorem for contractions in fuzzy metric spaces, and Fang [7] gave some common fixed point theorems under ϕ-contractions for compatible and weakly compatible mappings in Menger probabilistic metric spaces. Many authors [8–23] have proved fixed point theorems in (intuitionistic) fuzzy metric spaces or probabilistic metric spaces.

In this paper, using similar proof as in [7], we give a new common fixed point theorem under weaker conditions than in [6] and give an example which shows that the result is a genuine generalization of the corresponding result in [6].

2. Preliminaries

First we give some definitions.

Definition 1 (see [2]). A binary operation $* : [0,1] \times [0,1] \to [0,1]$ is continuous t-norm if $*$ is satisfying the following conditions:

(1) $*$ is commutative and associative;

(2) $*$ is continuous;

(3) $a * 1 = a$ for all $a \in [0,1]$;

(4) $a * b \leq c * d$ whenever $a \leq c$ and $b \leq d$ for all $a, b, c, d \in [0,1]$.

Definition 2 (see [24]). Let $\sup_{0<t<1} \Delta(t,t) = 1$. A t-norm Δ is said to be of H-type if the family of functions $\{\Delta^m(t)\}_{m=1}^{\infty}$ is equicontinuous at $t = 1$, where

$$\Delta^1(t) = t\Delta t, \quad \Delta^{m+1}(t) = t\Delta(\Delta^m(t)), \quad m = 1, 2, \ldots, \ t \in [0,1]. \tag{2.1}$$

The t-norm $\Delta_M = \min$ is an example of t-norm of H-type, but there are some other t-norms Δ of H-type [24].

Obviously, Δ is a H-type t norm if and only if for any $\lambda \in (0,1)$, there exists $\delta(\lambda) \in (0,1)$ such that $\Delta^m(t) > 1 - \lambda$ for all $m \in \mathbb{N}$, when $t > 1 - \delta$.

Definition 3 (see [2]). A 3-tuple $(X, M, *)$ is said to be a fuzzy metric space if X is an arbitrary nonempty set, $*$ is a continuous t-norm, and M is a fuzzy set on $X^2 \times (0, +\infty)$ satisfying the following conditions, for each $x, y, z \in X$ and $t, s > 0$:

(FM-1) $M(x, y, t) > 0$;

(FM-2) $M(x, y, t) = 1$ if and only if $x = y$;

(FM-3) $M(x, y, t) = M(y, x, t)$;

(FM-4) $M(x, y, t) * M(y, z, s) \leq M(x, z, t + s)$;

(FM-5) $M(x, y, \cdot) : (0, \infty) \to [0,1]$ is continuous.

Let $(X, M, *)$ be a fuzzy metric space. For $t > 0$, the open ball $B(x, r, t)$ with a center $x \in X$ and a radius $0 < r < 1$ is defined by

$$B(x, r, t) = \{y \in X : M(x, y, t) > 1 - r\}. \tag{2.2}$$

A subset $A \subset X$ is called open if, for each $x \in A$, there exist $t > 0$ and $0 < r < 1$ such that $B(x, r, t) \subset A$. Let τ denote the family of all open subsets of X. Then τ is called the topology on X induced by the fuzzy metric M. This topology is Hausdorff and first countable.

Example 1. Let (X, d) be a metric space. Define t-norm $a * b = ab$ and for all $x, y \in X$ and $t > 0$, $M(x, y, t) = t/(t + d(x, y))$. Then $(X, M, *)$ is a fuzzy metric space. We call this fuzzy metric M induced by the metric d the standard fuzzy metric.

Definition 4 (see [2]). Let $(X, M, *)$ be a fuzzy metric space, then

(1) a sequence $\{x_n\}$ in X is said to be convergent to x (denoted by $\lim_{n\to\infty} x_n = x$) if

$$\lim_{n\to\infty} M(x_n, x, t) = 1, \tag{2.3}$$

for all $t > 0$;

(2) a sequence $\{x_n\}$ in X is said to be a Cauchy sequence if for any $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$, such that

$$M(x_n, x_m, t) > 1 - \varepsilon, \tag{2.4}$$

for all $t > 0$ and n, $m \geq n_0$;

(3) a fuzzy metric space $(X, M, *)$ is said to be complete if and only if every Cauchy sequence in X is convergent.

Remark 1 (see [25]). (1) For all $x, y \in X$, $M(x, y, \cdot)$ is nondecreasing.

(2) It is easy to prove that if $x_n \to x$, $y_n \to y$, $t_n \to t$, then

$$\lim_{n\to\infty} M(x_n, y_n, t_n) = M(x, y, t). \tag{2.5}$$

(3) In a fuzzy metric space $(X, M, *)$, whenever $M(x, y, t) > 1 - r$ for x, y in X, $t > 0$, $0 < r < 1$, we can find a t_0, $0 < t_0 < t$ such that $M(x, y, t_0) > 1 - r$.

(4) For any $r_1 > r_2$, we can find an r_3 such that $r_1 * r_3 \geq r_2$ and for any r_4 we can find a r_5 such that $r_5 * r_5 \geq r_4$ $(r_1, r_2, r_3, r_4, r_5 \in (0, 1))$.

Definition 5 (see [6]). Let $(X, M, *)$ be a fuzzy metric space. M is said to satisfy the n-property on $X^2 \times (0, \infty)$ if

$$\lim_{n\to\infty} \left[M(x, y, k^n t)\right]^{n^p} = 1, \tag{2.6}$$

whenever $x, y \in X$, $k > 1$ and $p > 0$.

Lemma 1. *Let $(X, M, *)$ be a fuzzy metric space and M satisfies the n-property; then*

$$\lim_{t\to+\infty} M(x, y, t) = 1, \quad \forall x, y \in X. \tag{2.7}$$

Proof. If not, since $M(x, y, \cdot)$ is nondecreasing and $0 \leq M(x, y, \cdot) \leq 1$, there exists $x_0, y_0 \in X$ such that $\lim_{t\to+\infty} M(x_0, y_0, t) = \lambda < 1$, then for $k > 1$, $k^n t \to +\infty$ when $n \to \infty$ as $t > 0$ and we get $\lim_{n\to\infty}[M(x_0, y_0, k^n t)]^{n^p} = 0$, which is a contraction. □

Remark 2. Condition (2.7) cannot guarantee the n-property. See the following example.

Example 2. Let (X, d) be an ordinary metric space, $a * b \leq ab$ for all $a, b \in [0,1]$, and ψ be defined as following:

$$\psi(t) = \begin{cases} \alpha\sqrt{t}, & 0 < t \leq 4, \\ 1 - \dfrac{1}{\ln t}, & t > 4, \end{cases} \tag{2.8}$$

where $\alpha = (1/2)(1 - 1/\ln 4)$. Then $\psi(t)$ is continuous and increasing in $(0, \infty)$, $\psi(t) \in (0,1)$ and $\lim_{t\to+\infty}\psi(t) = 1$. Let

$$M(x, y, t) = [\psi(t)]^{d(x,y)}, \quad \forall x, y \in X, t > 0, \tag{2.9}$$

then $(X, M, *)$ is a fuzzy metric space and

$$\lim_{t\to+\infty} M(x, y, t) = \lim_{t\to+\infty} [\psi(t)]^{d(x,y)} = 1, \quad \forall x, y \in X. \tag{2.10}$$

But for any $x \neq y$, $p = 1$, $k > 1$, $t > 0$,

$$\lim_{n\to\infty} [M(x, y, k^n t)]^{n^p} = \lim_{n\to\infty} [\psi(k^n t)]^{d(x,y)\cdot n^p} = \lim_{n\to\infty} \left[1 - \frac{1}{\ln(k^n t)}\right]^{n\cdot d(x,y)} = e^{-d(x,y)/\ln k} \neq 1. \tag{2.11}$$

Define $\Phi = \{\phi : R^+ \to R^+\}$, where $R^+ = [0, +\infty)$ and each $\phi \in \Phi$ satisfies the following conditions:

(ϕ-1) ϕ is nondecreasing;

(ϕ-2) ϕ is upper semicontinuous from the right;

(ϕ-3) $\sum_{n=0}^{\infty} \phi^n(t) < +\infty$ for all $t > 0$, where $\phi^{n+1}(t) = \phi(\phi^n(t))$, $n \in \mathbb{N}$.

It is easy to prove that, if $\phi \in \Phi$, then $\phi(t) < t$ for all $t > 0$.

Lemma 2 (see [7]). *Let $(X, M, *)$ be a fuzzy metric space, where $*$ is a continuous t-norm of H-type. If there exists $\phi \in \Phi$ such that if*

$$M(x, y, \phi(t)) \geq M(x, y, t), \tag{2.12}$$

for all $t > 0$, then $x = y$.

Definition 6 (see [5]). An element $(x, y) \in X \times X$ is called a coupled fixed point of the mapping $F : X \times X \to X$ if

$$F(x, y) = x, \qquad F(y, x) = y. \tag{2.13}$$

Definition 7 (see [5]). An element $(x, y) \in X \times X$ is called a coupled coincidence point of the mappings $F : X \times X \to X$ and $g : X \to X$ if

$$F(x, y) = g(x), \qquad F(y, x) = g(y). \tag{2.14}$$

Definition 8 (see [7]). An element $(x, y) \in X \times X$ is called a common coupled fixed point of the mappings $F : X \times X \to X$ and $g : X \to X$ if

$$x = F(x, y) = g(x), \qquad y = F(y, x) = g(y). \tag{2.15}$$

Definition 9 (see [7]). An element $x \in X$ is called a common fixed point of the mappings $F : X \times X \to X$ and $g : X \to X$ if

$$x = g(x) = F(x, x). \tag{2.16}$$

Definition 10 (see [7]). The mappings $F : X \times X \to X$ and $g : X \to X$ are said to be compatible if

$$\begin{aligned} &\lim_{n\to\infty} M(gF(x_n, y_n), F(g(x_n), g(y_n)), t) = 1, \\ &\lim_{n\to\infty} M(gF(y_n, x_n), F(g(y_n), g(x_n)), t) = 1, \end{aligned} \tag{2.17}$$

for all $t > 0$ whenever $\{x_n\}$ and $\{y_n\}$ are sequences in X, such that

$$\lim_{n\to\infty} F(x_n, y_n) = \lim_{n\to\infty} \mathrm{g}(x_n) = x, \qquad \lim_{n\to\infty} F(y_n, x_n) = \lim_{n\to\infty} g(y_n) = y, \tag{2.18}$$

for all $x, y \in X$ are satisfied.

Definition 11 (see [7]). The mappings $F : X \times X \to X$ and $g : X \to X$ are called commutative if

$$g(F(x, y)) = F(gx, gy), \tag{2.19}$$

for all $x, y \in X$.

Remark 3. It is easy to prove that, if F and g are commutative, then they are compatible.

3. Main Results

For convenience, we denote

$$[M(x, y, \mathrm{t})]^n = \underbrace{M(x, y, t) * M(x, y, t) * \cdots * M(x, y, t)}_{n}, \tag{3.1}$$

for all $n \in \mathbb{N}$.

Theorem 1. *Let* $(X, M, *)$ *be a complete FM-space, where* $*$ *is a continuous* t*-norm of H-type satisfying* (2.7). *Let* $F : X \times X \to X$ *and* $g : X \to X$ *be two mappings and there exists* $\phi \in \Phi$ *such that*

$$M(F(x,y), F(u,v), \phi(t)) \geq M(g(x), g(u), t) * M(g(y), g(v), t), \tag{3.2}$$

for all $x, y, u, v \in X$, $t > 0$.

Suppose that $F(X \times X) \subseteq g(X)$, *and* g *is continuous,* F *and* g *are compatible. Then there exist* $x, y \in X$ *such that* $x = g(x) = F(x, x)$, *that is,* F *and* g *have a unique common fixed point in* X.

Proof. Let $x_0, y_0 \in X$ be two arbitrary points in X. Since $F(X \times X) \subseteq g(X)$, we can choose $x_1, y_1 \in X$ such that $g(x_1) = F(x_0, y_0)$ and $g(y_1) = F(y_0, x_0)$. Continuing in this way we can construct two sequences $\{x_n\}$ and $\{y_n\}$ in X such that

$$g(x_{n+1}) = F(x_n, y_n), \quad g(y_{n+1}) = F(y_n, x_n), \quad \forall n \geq 0. \tag{3.3}$$

The proof is divided into 4 steps.

Step 1. Prove that $\{gx_n\}$ and $\{gy_n\}$ are Cauchy sequences.

Since $*$ is a t-norm of H-type, for any $\lambda > 0$, there exists a $\mu > 0$ such that

$$\underbrace{(1-\mu) * (1-\mu) * \cdots * (1-\mu)}_{k} \geq 1 - \lambda, \tag{3.4}$$

for all $k \in \mathbb{N}$.

Since $M(x, y, \cdot)$ is continuous and $\lim_{t \to +\infty} M(x, y, t) = 1$ for all $x, y \in X$, there exists $t_0 > 0$ such that

$$M(gx_0, gx_1, t_0) \geq 1 - \mu, \qquad M(gy_0, gy_1, t_0) \geq 1 - \mu. \tag{3.5}$$

On the other hand, since $\phi \in \Phi$, by condition (ϕ-3) we have $\sum_{n=1}^{\infty} \phi^n(t_0) < \infty$. Then for any $t > 0$, there exists $n_0 \in \mathbb{N}$ such that

$$t > \sum_{k=n_0}^{\infty} \phi^k(t_0). \tag{3.6}$$

From condition (3.2), we have

$$\begin{aligned} M(gx_1, gx_2, \phi(t_0)) &= M(F(x_0, y_0), F(x_1, y_1), \phi(t_0)) \\ &\geq M(gx_0, gx_1, t_0) * M(gy_0, gy_1, t_0), \\ M(gy_1, gy_2, \phi(t_0)) &= M(F(y_0, x_0), F(y_1, x_1), \phi(t_0)) \\ &\geq M(gy_0, gy_1, t_0) * M(gx_0, gx_1, t_0). \end{aligned} \tag{3.7}$$

Similarly, we can also get

$$
\begin{aligned}
M\left(gx_2, gx_3, \phi^2(t_0)\right) &= M\left(F(x_1, y_1), F(x_2, y_2), \phi^2(t_0)\right) \\
&\geq M(gx_1, gx_2, \phi(t_0)) * M(gy_1, gy_2, \phi(t_0)) \\
&\geq [M(gx_0, gx_1, t_0)]^2 * [M(gy_0, gy_1, t_0)]^2, \\
M\left(gy_2, gy_3, \phi^2(t_0)\right) &= M\left(F(y_1, x_1), F(y_2, x_2), \phi^2(t_0)\right) \\
&\geq [M(gy_0, gy_1, t_0)]^2 * [M(gx_0, gx_1, t_0)]^2.
\end{aligned}
\tag{3.8}
$$

Continuing in the same way we can get

$$
\begin{aligned}
M(gx_n, gx_{n+1}, \phi^n(t_0)) &\geq [M(gx_0, gx_1, t_0)]^{2^{n-1}} * [M(gy_0, gy_1, t_0)]^{2^{n-1}}, \\
M(gy_n, gy_{n+1}, \phi^n(t_0)) &\geq [M(gy_0, gy_1, t_0)]^{2^{n-1}} * [M(gx_0, gx_1, t_0)]^{2^{n-1}}.
\end{aligned}
\tag{3.9}
$$

So, from (3.5) and (3.6), for $m > n \geq n_0$, we have

$$
\begin{aligned}
&M(gx_n, gx_m, t) \\
&\quad\geq M\left(gx_n, gx_m, \sum_{k=n_0}^{\infty} \phi^k(t_0)\right) \\
&\quad\geq M\left(gx_n, gx_m, \sum_{k=n}^{m-1} \phi^k(t_0)\right) \\
&\quad\geq M(gx_n, gx_{n+1}, \phi^n(t_0)) * M\left(gx_{n+1}, gx_{n+2}, \phi^{n+1}(t_0)\right) * \cdots * M\left(gx_{m-1}, gx_m, \phi^{m-1}(t_0)\right) \\
&\quad\geq [M(gy_0, gy_1, t_0)]^{2^{n-1}} * [M(gx_0, gx_1, t_0)]^{2^{n-1}} * [M(gy_0, gy_1, t_0)]^{2^n} \\
&\qquad * [M(gx_0, gx_1, t_0)]^{2^n} * \cdots * [M(gy_0, gy_1, t_0)]^{2^{m-2}} * [M(gx_0, gx_1, t_0)]^{2^{m-2}} \\
&\quad= [M(gy_0, gy_1, t_0)]^{2^{(m-n)(m+n-3)}} * [M(gx_0, gx_1, t_0)]^{2^{(m-n)(m+n-3)}} \\
&\quad\geq \underbrace{(1-\mu) * (1-\mu) * \cdots * (1-\mu)}_{2^{2(m-n)(m+n-3)}} \geq 1 - \lambda,
\end{aligned}
\tag{3.10}
$$

which implies that

$$
M(gx_n, gx_m, t) > 1 - \lambda, \tag{3.11}
$$

for all $m, n \in \mathbb{N}$ with $m > n \geq n_0$ and $t > 0$. So $\{g(x_n)\}$ is a Cauchy sequence.

Similarly, we can get that $\{g(y_n)\}$ is also a Cauchy sequence.

Step 2. Prove that g and F have a coupled coincidence point.

Since X complete, there exist $x, y \in X$ such that

$$\lim_{n\to\infty} F(x_n, y_n) = \lim_{n\to\infty} g(x_n) = x, \qquad \lim_{n\to\infty} F(y_n, x_n) = \lim_{n\to\infty} g(y_n) = y. \tag{3.12}$$

Since F and g are compatible, we have by (3.12),

$$\begin{aligned} &\lim_{n\to\infty} M(gF(x_n, y_n), F(g(x_n), g(y_n)), t) = 1, \\ &\lim_{n\to\infty} M(gF(y_n, x_n), F(g(y_n), g(x_n)), t) = 1. \end{aligned} \tag{3.13}$$

for all $t > 0$. Next we prove that $g(x) = F(x, y)$ and $g(y) = F(y, x)$.

For all $t > 0$, by condition (3.2), we have

$$\begin{aligned} &M(gx, F(x, y), \phi(t)) \\ &\quad \geq M(ggx_{n+1}, F(x, y), \phi(k_1 t)) * M(gx, ggx_{n+1}, \phi(t) - \phi(k_1 t)) \\ &\quad = M(gF(x_n, y_n), F(x, y), \phi(k_1 t)) * M(gx, ggx_{n+1}, \phi(t) - \phi(k_1 t)) \\ &\quad \geq M(gF(x_n, y_n), F(gx_n, gy_n), \phi(k_1 t) - \phi(k_2 t)) \\ &\qquad * M(F(gx_n, gy_n), F(x, y), \phi(k_2 t)) * M(gx, ggx_{n+1}, \phi(t) - \phi(k_1 t)) \\ &\quad \geq M(gF(x_n, y_n), F(gx_n, gy_n), \phi(k_1 t) - \phi(k_2 t)) \\ &\qquad * M(ggx_n, gx, k_2 t) * M(ggy_n, gy, k_2 t) * M(gx, ggx_{n+1}, \phi(t) - \phi(k_1 t)), \end{aligned} \tag{3.14}$$

for all $0 < k_2 < k_1 < 1$. Let $n \to \infty$, since g and F are compatible, with the continuity of g, we get

$$M(gx, F(x, y), \phi(t)) \geq 1, \tag{3.15}$$

which implies that $gx = F(x, y)$. Similarly, we can get $gy = F(y, x)$.

Step 3. Prove that $gx = y$ and $gy = x$.

Since $*$ is a t-norm of H-type, for any $\lambda > 0$, there exists an $\mu > 0$ such that

$$\underbrace{(1-\mu) * (1-\mu) * \cdots * (1-\mu)}_{k} \geq 1 - \lambda, \tag{3.16}$$

for all $k \in \mathbb{N}$.

Since $M(x, y, \cdot)$ is continuous and $\lim_{t\to+\infty} M(x, y, t) = 1$ for all $x, y \in X$, there exists $t_0 > 0$ such that $M(gx, y, t_0) \geq 1 - \mu$ and $M(gy, x, t_0) \geq 1 - \mu$.

On the other hand, since $\phi \in \Phi$, by condition (ϕ-3) we have $\sum_{n=1}^{\infty} \phi^n(t_0) < \infty$. Then for any $t > 0$, there exists $n_0 \in \mathbb{N}$ such that $t > \sum_{k=n_0}^{\infty} \phi^k(t_0)$. Since

$$\begin{aligned} M(gx, gy_{n+1}, \phi(t_0)) &= M(F(x,y), F(y_n, x_n), \phi(t_0)) \\ &\geq M(gx, gy_n, t_0) * M(gy, gx_n, t_0), \end{aligned} \tag{3.17}$$

letting $n \to \infty$, we get

$$M(gx, y, \phi(t_0)) \geq M(gx, y, t_0) * M(gy, x, t_0). \tag{3.18}$$

Similarly, we can get

$$M(gy, x, \phi(t_0)) \geq M(gx, y, t_0) * M(gy, x, t_0). \tag{3.19}$$

From (3.18) and (3.19) we have

$$M(gx, y, \phi(t_0)) * M(gy, x, \phi(t_0)) \geq [M(gx, y, t_0)]^2 * [M(gy, x, t_0)]^2. \tag{3.20}$$

By this way, we can get for all $n \in \mathbb{N}$,

$$\begin{aligned} M(gx, y, \phi^n(t_0)) * M(gy, x, \phi^n(t_0)) &\geq \left[M\left(gx, y, \phi^{n-1}(t_0)\right)\right]^2 * \left[M\left(gy, x, \phi^{n-1}(t_0)\right)\right]^2 \\ &\geq [M(gx, y, t_0)]^{2^n} * [M(gy, x, t_0)]^{2^n}. \end{aligned} \tag{3.21}$$

Then, we have

$$\begin{aligned} M(gx, y, t) * M(gy, x, t) &\geq M\left(gx, y, \sum_{k=n_0}^{\infty} \phi^k(t_0)\right) * M\left(gy, x, \sum_{k=n_0}^{\infty} \phi^k(t_0)\right) \\ &\geq M(gx, y, \phi^{n_0}(t_0)) * M(gy, x, \phi^{n_0}(t_0)) \\ &\geq [M(gx, y, t_0)]^{2^{n_0}} * [M(gy, x, t_0)]^{2^{n_0}} \\ &\geq \underbrace{(1-\mu) * (1-\mu) * \cdots * (1-\mu)}_{2^{2n_0}} \geq 1 - \lambda. \end{aligned} \tag{3.22}$$

So for any $\lambda > 0$ we have

$$M(gx, y, t) * M(gy, x, t) \geq 1 - \lambda, \tag{3.23}$$

for all $t > 0$. We can get that $gx = y$ and $gy = x$.

Step 4. Prove that $x = y$.

Since $*$ is a t-norm of H-type, for any $\lambda > 0$, there exists an $\mu > 0$ such that

$$\underbrace{(1-\mu) * (1-\mu) * \cdots * (1-\mu)}_{k} \geq 1 - \lambda, \tag{3.24}$$

for all $k \in \mathbb{N}$.

Since $M(x, y, \cdot)$ is continuous and $\lim_{t \to +\infty} M(x, y, t) = 1$, there exists $t_0 > 0$ such that $M(x, y, t_0) \geq 1 - \mu$.

On the other hand, since $\phi \in \Phi$, by condition (ϕ-3) we have $\sum_{n=1}^{\infty} \phi^n(t_0) < \infty$. Then for any $t > 0$, there exists $n_0 \in \mathbb{N}$ such that $t > \sum_{k=n_0}^{\infty} \phi^k(t_0)$.

Since for $t_0 > 0$,

$$\begin{aligned} M(gx_{n+1}, gy_{n+1}, \phi(t_0)) &= M(F(x_n, y_n), F(y_n, x_n), \phi(t_0)) \\ &\geq M(gx_n, gy_n, t_0) * M(gy_n, gx_n, t_0). \end{aligned} \tag{3.25}$$

Letting $n \to \infty$ yields

$$M(x, y, \phi(t_0)) \geq M(x, y, t_0) * M(y, x, t_0). \tag{3.26}$$

Thus we have

$$\begin{aligned} M(x, y, t) &\geq M\left(x, y, \sum_{k=n_0}^{\infty} \phi^k(t_0)\right) \\ &\geq M(x, y, \phi^{n_0}(t_0)) \\ &\geq [M(x, y, t_0)]^{2^{n_0}} * [M(y, x, t_0)]^{2^{n_0}} \\ &\geq \underbrace{(1-\mu) * (1-\mu) * \cdots * (1-\mu)}_{2^{2n_0}} \geq 1 - \lambda, \end{aligned} \tag{3.27}$$

which implies that $x = y$.

Thus we have proved that F and g have a unique common fixed point in X.

This completes the proof of the Theorem 1. □

Taking $g = I$ (the identity mapping) in Theorem 1, we get the following consequence.

Corollary 1. *Let $(X, M, *)$ be a complete FM-space, where $*$ is a continuous t-norm of H-type satisfying* (2.7). *Let $F : X \times X \to X$ and there exists $\phi \in \Phi$ such that*

$$M(F(x, y), F(u, v), \phi(t)) \geq M(x, u, t) * M(y, v, t), \tag{3.28}$$

for all $x, y, u, v \in X$, $t > 0$.

Then there exist $x \in X$ such that $x = F(x, x)$, that is, F admits a unique fixed point in X.

Let $\phi(t) = kt$, where $0 < k < 1$, the following by Lemma 1, we get the following.

Corollary 2 (see [6]). *Let $a * b \geq ab$ for all $a, b \in [0, 1]$ and $(X, M, *)$ be a complete fuzzy metric space such that M has n-property. Let $F : X \times X \to X$ and $g : X \to X$ be two functions such that*

$$M(F(x,y), F(u,v), kt) \geq M(gx, gu, t) * M(gy, gv, t), \tag{3.29}$$

for all $x, y, u, v \in X$, where $0 < k < 1$, $F(X \times X) \subset g(X)$ and g is continuous and commutes with F. Then there exists a unique $x \in X$ such that $x = g(x) = F(x, x)$.

Next we give an example to demonstrate Theorem 1.

Example 3. Let $X = [-2, 2]$, $a * b = ab$ for all $a, b \in [0, 1]$. φ is defined as (2.8). Let

$$M(x, y, t) = [\varphi(t)]^{|x-y|}, \tag{3.30}$$

for all $x, y \in [0, 1]$. Then $(X, M, *)$ is a complete FM-space.

Let $\phi(t) = t/2$, $g(x) = x$ and $F : X \times X \to X$ be defined as

$$F(x, y) = \frac{x^2}{8} + \frac{y^2}{8} - 2, \quad \forall x, y \in X. \tag{3.31}$$

Then F satisfies all the condition of Theorem 1, and there exists a point $x = 2 - 2\sqrt{3}$ which is the unique common fixed point of g and F.

In fact, it is easy to see that $F(X \times X) = [-2, -1]$,

$$M(F(x,y), F(u,v), \phi(t)) = [\varphi(\phi(t))]^{|x^2-u^2+y^2-v^2|/8}, \tag{3.32}$$

For all $t > 0$ and $x, y \in [-2, 2]$. (3.28) is equivalent to

$$\left[\varphi\left(\frac{t}{2}\right)\right]^{|x^2-u^2+y^2-v^2|/8} \geq [\varphi(t)]^{|x-u|} \cdot [\varphi(t)]^{|y-v|}. \tag{3.33}$$

Since $\varphi(t) \in (0, 1)$, we can get

$$\left[\varphi\left(\frac{t}{2}\right)\right]^{|x^2-u^2+y^2-v^2|/8} \geq \left[\varphi\left(\frac{t}{2}\right)\right]^{|x-u|/2} \cdot \left[\varphi\left(\frac{t}{2}\right)\right]^{|y-v|/2}. \tag{3.34}$$

From (3.33), we only need to verify the following:

$$\left[\varphi\left(\frac{t}{2}\right)\right]^{|x-u|/2} \geq [\varphi(t)]^{|x-u|}, \tag{3.35}$$

that is,

$$\psi\left(\frac{t}{2}\right) \geq \left[\psi(t)\right]^2, \quad \forall t > 0. \tag{3.36}$$

We consider the following cases.

Case 1 ($0 < t \leq 4$). Then (3.36) is equivalent to

$$\alpha\sqrt{\frac{t}{2}} \geq \left(\alpha\sqrt{t}\right)^2, \tag{3.37}$$

it is easy to verified.

Case 2 ($t \geq 8$). Then (3.36) is equivalent to

$$1 - \frac{1}{\ln t/2} \geq \left(1 - \frac{1}{\ln t}\right)^2, \tag{3.38}$$

which is

$$2\ln t \cdot \ln\frac{t}{2} \geq \ln^2 t + \ln\frac{t}{2}, \tag{3.39}$$

since

$$\ln^2 t + \ln^2\frac{t}{2} - 2\ln t \cdot \ln\frac{t}{2} + \ln\frac{t}{2} - \ln^2\frac{t}{2} \leq 0, \tag{3.40}$$

that is

$$\ln^2 2 + \ln\frac{t}{2} - \ln^2\frac{t}{2} \leq 0, \tag{3.41}$$

holds for all $t \geq 8$. So (3.36) holds for $t \geq 8$.

Case 3 ($4 < t < 8$). Then (3.36) is equivalent to

$$\alpha\sqrt{\frac{t}{2}} \geq \left(1 - \frac{1}{\ln t}\right)^2. \tag{3.42}$$

Let $t = e^x$, we only need to verify

$$\frac{\alpha}{\sqrt{2}}e^{x/2} - \left(1 - \frac{1}{x}\right)^2 \geq 0, \tag{3.43}$$

for all x that $2\ln 2 < x < 3\ln 2$. We can verify it holds.

Thus it is verified that the functions F, g, ϕ satisfy all the conditions of Theorem 1; $x = 2 - 2\sqrt{3}$ is the common fixed point of F and g in X.

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References

[1] L. A. Zadeh, "Fuzzy sets," *Information and Computation*, vol. 8, pp. 338–353, 1965.

[2] A. George and P. Veeramani, "On some results in fuzzy metric spaces," *Fuzzy Sets and Systems*, vol. 64, no. 3, pp. 395–399, 1994.

[3] A. George and P. Veeramani, "On some results of analysis for fuzzy metric spaces," *Fuzzy Sets and Systems*, vol. 90, no. 3, pp. 365–368, 1997.

[4] T. G. Bhaskar and V. Lakshmikantham, "Fixed point theorems in partially ordered metric spaces and applications," *Nonlinear Analysis. Theory, Methods & Applications*, vol. 65, no. 7, pp. 1379–1393, 2006.

[5] V. Lakshmikantham and L. Ćirić, "Coupled fixed point theorems for nonlinear contractions in partially ordered metric spaces," *Nonlinear Analysis. Theory, Methods & Applications*, vol. 70, no. 12, pp. 4341–4349, 2009.

[6] S. Sedghi, I. Altun, and N. Shobe, "Coupled fixed point theorems for contractions in fuzzy metric spaces," *Nonlinear Analysis. Theory, Methods & Applications*, vol. 72, no. 3-4, pp. 1298–1304, 2010.

[7] J.-X. Fang, "Common fixed point theorems of compatible and weakly compatible maps in Menger spaces," *Nonlinear Analysis. Theory, Methods & Applications*, vol. 71, no. 5-6, pp. 1833–1843, 2009.

[8] L. B. Ćirić, D. Miheţ, and R. Saadati, "Monotone generalized contractions in partially ordered probabilistic metric spaces," *Topology and its Applications*, vol. 156, no. 17, pp. 2838–2844, 2009.

[9] D. O'Regan and R. Saadati, "Nonlinear contraction theorems in probabilistic spaces," *Applied Mathematics and Computation*, vol. 195, no. 1, pp. 86–93, 2008.

[10] S. Jain, S. Jain, and L. Bahadur Jain, "Compatibility of type (P) in modified intuitionistic fuzzy metric space," *Journal of Nonlinear Science and its Applications*, vol. 3, no. 2, pp. 96–109, 2010.

[11] L. B. Ćirić, S. N. Ješić, and J. S. Ume, "The existence theorems for fixed and periodic points of nonexpansive mappings in intuitionistic fuzzy metric spaces," *Chaos, Solitons and Fractals*, vol. 37, no. 3, pp. 781–791, 2008.

[12] L. Ćirić and V. Lakshmikantham, "Coupled random fixed point theorems for nonlinear contractions in partially ordered metric spaces," *Stochastic Analysis and Applications*, vol. 27, no. 6, pp. 1246–1259, 2009.

[13] L. Ćirić, N. Cakić, M. Rajović, and J. S. Ume, "Monotone generalized nonlinear contractions in partially ordered metric spaces," *Fixed Point Theory and Applications*, vol. 2008, Article ID 131294, 11 pages, 2008.

[14] A. Aliouche, F. Merghadi, and A. Djoudi, "A related fixed point theorem in two fuzzy metric spaces," *Journal of Nonlinear Science and its Applications*, vol. 2, no. 1, pp. 19–24, 2009.

[15] L. Ćirić, "Common fixed point theorems for a family of non-self mappings in convex metric spaces," *Nonlinear Analysis. Theory, Methods & Applications*, vol. 71, no. 5-6, pp. 1662–1669, 2009.

[16] K. P. R. Rao, A. Aliouche, and G. R. Babu, "Related fixed point theorems in fuzzy metric spaces," *Journal of Nonlinear Science and its Applications*, vol. 1, no. 3, pp. 194–202, 2008.

[17] L. Ćirić and N. Cakić, "On common fixed point theorems for non-self hybrid mappings in convex metric spaces," *Applied Mathematics and Computation*, vol. 208, no. 1, pp. 90–97, 2009.

[18] L. Ćirić, "Some new results for Banach contractions and Edelstein contractive mappings on fuzzy metric spaces," *Chaos, Solitons and Fractals*, vol. 42, no. 1, pp. 146–154, 2009.

[19] S. Shakeri, L. J. B. Ćirić, and R. Saadati, "Common fixed point theorem in partially ordered L-fuzzy metric spaces," *Fixed Point Theory and Applications*, vol. 2010, Article ID 125082, 13 pages, 2010.

[20] L. Ćirić, B. Samet, and C. Vetro, "Common fixed point theorems for families of occasionally weakly compatible mappings," *Mathematical and Computer Modelling*, vol. 53, no. 5-6, pp. 631–636, 2011.

[21] L. Ćirić, M. Abbas, R. Saadati, and N. Hussain, "Common fixed points of almost generalized contractive mappings in ordered metric spaces," *Applied Mathematics and Computation*, vol. 217, no. 12, pp. 5784–5789, 2011.

[22] L. Ćirić, M. Abbas, B. Damjanović, and R. Saadati, "Common fuzzy fixed point theorems in ordered metric spaces," *Mathematical and Computer Modelling*, vol. 53, no. 9-10, pp. 1737–1741, 2011.

[23] T. Kamran and N. Cakić, "Hybrid tangential property and coincidence point theorems," *Fixed Point Theory*, vol. 9, no. 2, pp. 487–496, 2008.

[24] O. Hadžić and E. Pap, *Fixed Point Theory in Probabilistic Metric Spaces*, vol. 536 of *Mathematics and its Applications*, Kluwer Academic, Dordrecht, The Netherlands, 2001.

[25] M. Grabiec, "Fixed points in fuzzy metric spaces," *Fuzzy Sets and Systems*, vol. 27, no. 3, pp. 385–389, 1988.