Internat. J. Math. & Math. Sci.
Vol. 10 No. 1 (1987) 51-55

ON n^{th} - ORDER DIFFERENTIAL OPERATORS WITH BOHR-NEUGEBAUER TYPE PROPERTY

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(Received July 2, 1985 and in revised form July 31, 1986)

ABSTRACT. Suppose B is a bounded linear operator in a Banach space. If the differential operator $\frac{d^n}{dt^n} - B$ has a Bohr-Neugebauer type property for Bochner almost periodic functions, then, for any Stepanov almost periodic continuous function $g(t)$ and any Stepanov-bounded solution of the differential equation $\frac{d^n}{dt^n} u(t) - Bu(t) = g(t)$, $u^{(n-1)}, \dots, u', u$ are all almost periodic.

KEY WORDS AND PHRASES. *Bounded linear operator, Bohr-Neugebauer property, Bochner (Stepanov or weakly) almost periodic function, completely continuous normal operator.*
1970 AMS SUBJECT CLASSIFICATION SCHEME. PRIMARY 34C25, 34G05; SECONDARY 43A60.

1. INTRODUCTION.

Suppose X is a Banach space and J is the interval $-\infty < t < \infty$. A function $f \in L^p_{loc}(J;X)$ with $1 \leq p < \infty$ is said to be Stepanov - bounded or S^p -bounded on J if

$$\|f\|_{S^p} = \sup_{t \in J} \left[\int_t^{t+1} \|f(s)\|^p ds \right]^{1/p} < \infty. \tag{1.1}$$

For the definitions of almost periodicity, weak almost periodicity and S^p-almost periodicity, we refer the reader to pp. 3, 39 and 77, Amerio-Prouse [1].

Suppose that B is a bounded linear operator having domain and range in X. We say that the differential operator $\frac{d^n}{dt^n} - B$ has Bohr-Neugebauer property if, for any almost periodic X-valued function $f(t)$ and any bounded (on J) solution of the equation

$$\frac{d^n}{dt^n} u(t) - Bu(t) = f(t) \qquad \text{on } J, \tag{1.2}$$

$u^{(n-1)}, \dots, u', u$ are all almost periodic.

Our main result is as follows.

THEOREM 1. For a bounded linear operator B with domain D(B) and range R(B) in a Banach space X, let the differential operator $\frac{d^n}{dt^n} - B$ be such that, for any almost periodic X-valued function $f(t)$ and any S^p-bounded solution $u: J \to D(B)$ of the equation (1.2), $u^{(n-1)}, \dots, u', u$ are all S^1-almost periodic. If $p > 1$, then, for any S^p-bounded solution $u: J \to D(B)$ of the equation

$$\frac{d^n}{dt^n} u(t) - Bu(t) = g(t) \qquad \text{on } J, \tag{1.3}$$

$u^{(n-1)},\dots,u',u$ are all almost periodic.

REMARK 1. Theorem 1 is a generalization of a result of Zaidman [6].

2. PROOF OF THEOREM 1.

By (1.3), we have the representation

$$u^{(n-1)}(t) = u^{(n-1)}(0) + \int_0^t Bu(s)ds + \int_0^t g(s)ds \quad \text{on } J. \tag{2.1}$$

If $0 < t_2 - t_1 < 1$ and $p^{-1} + q^{-1} = 1$, then, by the Hölder's inequality,

$$\left\| \int_{t_1}^{t_2} Bu(s)\, ds \right\| \leq \|B\| . \int_{t_1}^{t_2} \|u(s)\|\, ds$$

$$\leq \|B\| . \left[\int_{t_1}^{t_2} \|u(s)\|^p\, ds \right]^{p^{-1}} . (t_2 - t_1)^{q^{-1}}$$

$$\leq \|B\| . \left[\int_{t_1}^{t_1+1} \|u(s)\|^p \right]^{p^{-1}} . (t_2 - t_1)^{q^{-1}}$$

$$\leq \|B\| . \|u\|_{S^p} . (t_2 - t_1)^{q^{-1}} . \tag{2.2}$$

Hence $\int_0^t Bu(s)\, ds$ is uniformly continuous on J. Further, by Theorem 8, p. 79, Amerio-Prouse [1], $\int_0^t g(s)\, ds$ is uniformly continuous on J. Consequently, $u^{(n-1)}$ is uniformly continuous on J.

Now consider a sequence $\{\rho_k(t)\}_{k=1}^{\infty}$ of non-negative continuous functions on J such that

$$\rho_k(t) = 0 \quad \text{for } |t| \geq k^{-1}, \ \int_{-k^{-1}}^{k^{-1}} \rho_k(t)dt = 1. \tag{2.3}$$

The convolution between u and ρ_k is defined by

$$(u * \rho_k)(t) = \int_J u(t-s)\rho_k(s)ds = \int_J u(s)\rho_k(t-s)ds. \tag{2.4}$$

From (1.3), it follows that

$$\frac{d^n}{dt^n}(u * \rho_k)(t) - B(u * \rho_k)(t) = (g * \rho_k)(t) \text{ on } J. \tag{2.5}$$

Again by Hölder's inequality,

$$\|(u*\rho_k)\ (t)\| = \| \int_{-1}^{1} u(t - s)\rho_k(s)\ ds\|$$

$$\leq \left[\int_{-1}^{1} \|u(t - s)\|^p\, ds \right]^{p^{-1}} . \left[\int_{-1}^{1} \left[\rho_k(s)\right]^q\, ds \right]^{q^{-1}}$$

$$= c_{\rho_k} \left[\int_{t-1}^{t+1} \|u(\sigma)\|^p\, d\sigma \right]^{p^{-1}}$$

$$\leq 2\, c_{\rho_k} \|u\|_{S^p} \text{ for all } t \in J \text{ and } k = 1,2,\dots . \tag{2.6}$$

Similarly, the S^1-almost periodicity of g(t) implies the almost periodicity of $(g*\rho_k)$ (t) for all k = 1,2,... .

Consequently, it follows from our assumption on the operator $\frac{d^n}{dt^n} - B$ that $(u * \rho_k)^{(n-1)}(t),\dots,(u * \rho_k)'(t),(u * \rho_k)(t)$ are all S^1-almost periodic from J to X for all $k \geq 1$.

Further, since $u^{(n-1)}$ (t) is uniformly continuous on J, given $\varepsilon > 0$, there exists $\delta > o$ such that

$$\|u^{(n-1)}(t_1) - u^{(n-1)}(t_2)\| \leqq \varepsilon \text{ for } |t_1 - t_2| \leqq \delta. \tag{2.7}$$

Consequently, we have, for $|t_1 - t_2| \leqq \delta$,

$$\|(u^{(n-1)} * \rho_k)(t_1) - (u^{(n-1)} * \rho_k)(t_2)\|$$

$$\leqq \int_{-k^{-1}}^{k^{-1}} \|u^{(n-1)}(t_1 - s) - u^{(n-1)}(t_2 - s)\| \rho_k(s)\, ds$$

$$\leqq \varepsilon \int_{-k^{-1}}^{k^{-1}} \rho_k(s)\, ds = \varepsilon, \text{ by (2.3).} \tag{2.8}$$

Hence $(u * \rho_k)^{(n-1)}(t) = (u^{(n-1)} * \rho_k)(t)$ is uniformly continuous on J. So, by Theorem 7, p. 78, Amerio-Prouse [1], $(u^{(n-1)} * \rho_k)(t)$ is almost periodic.

Furthermore, by the uniform continuity of $u^{(n-1)}(t)$ on J, the sequence of convolutions $(u^{(n-1)} * \rho_k)(t)$ converges to $u^{(n-1)}(t)$ as $k \to \infty$, uniformly on J. Hence $u^{(n-1)}(t)$ is almost periodic from J to X, and so is bounded on J. Therefore $u^{(n-2)}(t)$ is uniformly continuous on J. Consequently, $(u^{(n-2)} * \rho_k)(t)$ is almost periodic and $(u^{(n-2)} * \rho_k)(t) \to u^{(n-2)}(t)$ as $k \to \infty$, uniformly on J. Hence $u^{(n-2)}(t)$ is almost periodic.

Thus we conclude successively that $u^{(n-1)},\ldots,u',u$ are all almost periodic from J to X, which completes the proof of the theorem.

REMARK 2. The conclusion of Theorem 1 remains valid for any S^1-bounded and uniformly continuous solution of the equation (1.3).

PROOF. By the Lemma of Rao [5], such a solution is bounded on J. Consequently, by the representation (2.1), $u^{(n-1)}$ is uniformly continuous on J.

REMARK 3. If $B = 0$, then Theorem 1 holds for $p \geqq 1$.

3. NOTES.

(i) Suppose X is a separable Hilbert space, and consider the differential equation

$$\frac{d^n}{dt^n} u(t) - Bu(t) = f(t) \quad \text{on } J, \tag{3.1}$$

where $f : J \to X$ is an almost periodic function, and $B : X \to X$ is a completely continuous normal operator. Then, if u is a bounded solution of (3.1), $u^{(n)}$ is almost periodic (as shown in the proof of Theorem 1 of Cooke [3]). Therefore, by the Corollary to Lemma 5 of Cooke [3], $u^{(n-1)},\ldots,u',u$ are all almost periodic. That is, the operator $\frac{d^n}{dt^n} - B$ has Bohr-Neugebauer property.

Now assume that u is an S^p-bounded solution $(1 < p < \infty)$ of the equation (3.1). If we replace g by f in the proof of our Theorem 1, then, by the Bohr-Neugebauer property of the operator $\frac{d^n}{dt^n} - B$, it follows that $u^{(n-1)},\ldots,u',u$ are all almost periodic. Hence the operator $\frac{d^n}{dt^n} - B$ satisfies the assumption of Theorem 1 for $p > 1$.

(ii) Finally, suppose X is a reflexive space and $B = 0$. Given an almost periodic X-valued function $f(t)$, assume $u(t)$ is a bounded solution of the differential equation

$$\frac{d^n}{dt^n} u(t) = f(t) \quad \text{on } J. \tag{3.2}$$

Then it follows from Lemma 2 of Cooke [3] that $u^{(n-1)},\dots,u'$ are all bounded on J. Hence we conclude successively that $u^{(n-1)},\dots,u',u$ are all almost periodic (see Amerio-Prouse [1], p. 55 and Authors' Remark on p. 82).
Therefore the operator $\frac{d^n}{dt^n}$ has Bohr-Neugebauer property.

Now, given an S^1-almost periodic continuous X-valued function $g(t)$, suppose $u(t)$ is an S^p-bounded solution $(1 \leq p < \infty)$ of the differential equation

$$\frac{d^n}{dt^n} u(t) = g(t) \qquad \text{on } J. \tag{3.3}$$

From (3.3), it follows that

$$\frac{d^n}{dt^n}(u * \rho_k)(t) = (g * \rho_k)(t) \qquad \text{on } J, \tag{3.4}$$

where $\{\rho_k(t)\}_{k=1}^{\infty}$ is the sequence defined in the proof of our Theorem 1. Then $(u * \rho_k)(t)$ is bounded on J and $(g * \rho_k)(t)$ is almost periodic from J to X. So, by the Bohr-Neugebauer property of the operator $\frac{d^n}{dt^n}$, $(u * \rho_k)^{(n-1)}(t),\dots,(u * \rho_k)'(t)$, $(u * \rho_k)(t)$ are all almost periodic.

By (3.3), it follows from Theorem 8, p. 79, Amerio-Prouse [1] that $u^{(n-1)}(t)$ is uniformly continuous on J. Consequently, we conclude successively that $u^{(n-1)}(t),\dots,u'(t),u(t)$ are all almost periodic. Hence the operator $\frac{d^n}{dt^n}$ satisfies the assumption of Theorem 1 for $p \geq 1$.

4. CONSEQUENCES OF THEOREM 1.

Let $L(X,X)$ be the Banach space of all bounded linear operators on X into itself, with the uniform operator topology. As consequences of our Theorem 1, we demonstrate the following results.

THEOREM 2. In a reflexive space X, suppose $f : J \to X$ is an S^p-almost periodic continuous function $(1 \leq p < \infty)$, and $B : J \to L(X,X)$ is almost periodic with respect to the norm of $L(X,X)$. If $u : J \to X$ is any S^p-almost periodic solution of the differential equation

$$\frac{d^n}{dt^n} u(t) = B(t)u(t) + f(t) \qquad \text{on } J, \tag{4.1}$$

then $u^{(n-1)},\dots,u',u$ are all almost periodic from J to X.

PROOF. Since $B(t)$ is almost periodic from J to $L(X,X)$, and $u(t)$ is S^p-almost periodic from J to X, we can show that $B(t)u(t)$ is S^p-almost periodic from J to X (see Rao [4]). Hence $B(t)u(t) + f(t)$ is S^p-almost periodic from J to X. If we write

$$v(t) = B(t)u(t) + f(t) \qquad \text{on } J, \tag{4.2}$$

then (4.1) becomes

$$\frac{d^n}{dt^n} u(t) = v(t) \qquad \text{on } J. \tag{4.3}$$

By our Note (ii), the operator $\frac{d^n}{dt^n}$ satisfies the assumption of our Theorem 1 for $p \geq 1$. Since u is S^p-almost periodic, it is S^p-bounded on J. So, by Theorem 1, $u^{(n-1)},\dots,u',u$ are all almost periodic.

THEOREM 3. In a reflexive space X, suppose $f : J \to X$ is an S^p-almost periodic continuous function $(1 \leq p < \infty)$, and $B : X \to X$ is a completely continuous linear operator. If $u : J \to X$ is a weakly almost periodic (strong) solution of the differential equation

$$\frac{d^n}{dt^n} u(t) = Bu(t) + f(t) \qquad \text{on } J, \tag{4.4}$$

then $u^{(n-1)}, \ldots, u', u$ are all almost periodic.

PROOF. Since B is a bounded linear operator, Bu is also weakly almost periodic. Further, B being a completely continuous operator, the range of Bu is relatively compact. Hence, by Theorem 10, p. 45, Amerio-Prouse [1], Bu is almost periodic. Consequently, Bu + f is S^p-almost periodic. Now, if we write

$$w(t) = Bu(t) + f(t) \qquad \text{on } J, \tag{4.5}$$

then (4.4) becomes

$$\frac{d^n}{dt^n} u(t) = w(t) \qquad \text{on } J. \tag{4.6}$$

Since u is weakly almost periodic, it is bounded on J. Therefore, by Theorem 1, $u^{(n-1)}, \ldots, u', u$ are all almost periodic.

REMARK 4. Suppose X is a Hilbert space and $B \in L(X,X)$ with $B \geq 0$. Consider the differential equation

$$\frac{d^2}{dt^2} u(t) - Bu(t) = f(t) \qquad \text{on } J, \tag{4.7}$$

where $f : J \to X$ is an almost periodic function. Then any bounded solution $u : J \to X$ of the equation (4.7) is almost periodic (see Zaidman [7]). By (4.7), $u'(t)$ is uniformly continuous on J. Hence, by Theorem 6, p. 6, Amerio-Prouse [1], $u'(t)$ is almost periodic. Therefore the operator $\frac{d^2}{dt^2} - B$ has Bohr-Neugebauer property, and so satisfies the assumption of Theorem 1 for $p > 1$.

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