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ON SOME PROPERTIES OF POLYNOMIAL RINGS

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ABSTRACT. For a commutative ring with unity R, it is proved that R is a PF - ring if and only if the annihilator, $\text{ann}_R(a)$, for each $a \in R$ is a pure ideal in R, Also it is proved that the polynimial ring, R[X], is a PF-ring if and only if R is a PF-ring. Finally, we prove that R is a PP-ring if and only if R[X] is a PP-ring.

KEY WORDS AND PHRASES. Polynomial Rings, Pure ideal, PF-ring, PP-ring, R-flatness, and idempotent elements.
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1. INTRODUCTION.

All our rings in this paper are commutative with unity. An ideal I of a ring R is called pure if for any $x \in I$, there exists $y \in I$ such that $xy = x$. A ring is called a PF-ring if every principal ideal aR is a flat R- module. A ring R is called a PP-ring if every principal ideal aR is a projective R-module. One can easily show that aR is projective if and only if the annihilator, $\text{ann}_R(a)$, is generated by an idempotent element, (see [1], [2]).

First, we state a proposition characterizing flat R-modules elementwise. This is a well known result in commutative ring theory, (see [3]).

PROPOSITION 1. An R-module M is a flat R-module if and only if for any pair of finite subsets $\{x_1, x_2, \ldots, x_n\}$ and $\{a_1, a_2, \ldots, a_n\}$ of M and R respectively, such that $\sum_{i=1}^{n} x_i a_i = 0$ there exists elements $z_1, \ldots, z_k \in M$ and $b_{ij} \in R$; $i = 1,2,\ldots,k$ such that $\sum_{i=1}^{n} b_j a_i = 0$, $j=1, 2, \ldots, k$, and $x_i = \sum_{j=1}^{k} z_j b_{ji}$, $i = 1, 2, \ldots, n$.

In the following theorem we establish that R is a PF-ring if and only if $\text{ann}_R(a)$ for each $a \in R$ is a pure ideal.

THEOREM 1. For any ring R, R is a PF-ring if and only if $\text{ann}_R(m)$ for each $m \in R$ is a pure ideal.

PROOF. Let $x_1, x_2, \ldots, x_n \in mR$ and $a_1, a_2, \ldots, a_n \in R$ with $\sum_{i=1}^{n} x_i a_i = 0$. Then there exists $m_1, m_2, \ldots, m_n \in R$ such that $x_i = m_i m$, $i = 1, 2, \ldots, n$. So $\sum_{i=1}^{n} m m_i a_i = 0$. Hence $m \in \text{ann}_R(\sum_{i=1}^{n} m_i a_i)$.

Since $\operatorname{ann}_R(\sum_{i=1}^{n} m_i a_i)$ is a pure ideal, there exists $b \in \operatorname{ann}_R(\sum_{i=1}^{n} m_i a_i)$ such that $bm = m$. Now take $m \in mR$ and $bm_1, bm_2, \ldots, bm_n \in R$. These elements satisfy $\sum_{i=1}^{n} bm_i a_i = 0$ and $bm_i m = m_i m = x_i$, $i = 1, 2, \ldots, n$. Therefore mR is a flat R-module.

Conversely, let $b \in \operatorname{ann}_R(m)$. Then $mb = 0$. Since bR is a flat R-module, there exists $c \in bR$ and $d \in R$ such that $dm = 0$ and $b = cd$. Now $c = c_1 b$, so $b = cd = bc_1 d$. Moreover $c_1 d \in \operatorname{ann}_R(m)$. Therefore $\operatorname{ann}_R(m)$ is a pure ideal.

LEMMA 1. Let $I_1, I_2, \ldots, I_n$ be a finite set of pure ideals of a ring R, then $J = \bigcap_{j=1}^{n} I_j$ is a pure ideal.

PROOF. Let $x \in J$. Then $x \in I_j$ for each j. Thus there exists $y_1 \in I_1$, $y_2 \in I_2, \ldots, y_n \in I_n$ with $xy_j = x$, $j = 1, 2, \ldots, n$. Then $y = y_1 y_2 \cdots y_n \in J$ and $xy = x$.

Let R be a reduced (without nonzero nilpotent elements) ring. Let $h(X) = h_0 + h_1 X + \ldots + h_n X^n \in R[X]$. Then $\operatorname{ann}_{R[X]}(h(X)) = N[X]$, where N is the annihilator of the ideal generated by $h_0, h_1, \ldots, h_n$, that is $N = \operatorname{ann}_R(h_0, h_1, \ldots, h_n) = \bigcap_{i=0}^{n} \operatorname{ann}_R(h_i)$. Moreover if $f(X) = a_0 + a_1 X + \ldots + a_m X^m \in \operatorname{ann}_{R[X]}(h(X))$ then $a_i h_j = 0$ for all $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, n$ (see [4]).

LEMMA 2. Let R be a PF-ring, then R is reduced.

PROOF. Let a be a nilpotent element in R, $a \neq 0$. Let n be the least positive integer greater than 1 such that $a^n = 0$. Hence $a \in \operatorname{ann}_R(a^{n-1})$. Since $\operatorname{ann}_R(a^{n-1})$ is pure, there exists $b \in \operatorname{ann}_R(a^{n-1})$ with $ab = a$. Now $o = ba^{n-1} = a^{n-1}$ since $ba = a$. Contradiction. Thus R is reduced.

THEOREM 2. The ring of polynomials, R[X], is a PF-ring if and only if R is a PF-ring.

PROOF. Let $f(X) = a_0 + a_1 X + \ldots + a_m X^m \in \operatorname{ann}_{R[X]}(h(X))$ where $h(X) = h_0 + h_1 X + \ldots + h_n X^n$.

Since R[X] has no nonzero nilpotent elements,

$$a_i \in J = \bigcap_{j=o}^{n} \operatorname{ann}_R(h_j)\ , \quad i = o, 1, 2, \ldots\ldots, m\ .$$

By Lemma 1, J is pure. Hence there exist $b_1, b_2, \ldots, b_m \in J$ such that $a_i b_i = a_i$, $i = 1, 2, \ldots, m$. Now our aim is to find $c \in J$ such that $c\, f(X) = f(X)$. We construct this element inductively.

First, $a_0 b_0 = a_0$. Consider

$$\begin{aligned} &(a_0 + a_1 X)(b_0 + b_1 - b_1 b_0) \\ &= a_0 b_0 + a_0 b_1 - a_0 b_0 b_1 + a_1 b_0 X + a_1 b_1 X - a_1 b_0 b_1 X \\ &= a_0 + a_0 b_1 - a_0 b_1 + a_1 b_0 X + a_1 X - a_1 b_0 X \\ &= a_0 + a_1 X. \end{aligned}$$

Let $c_1 = b_0 + b_1 - b_1 b_0$, then

$$\begin{aligned} &(a_0 + a_1 X + a_2 X^2)(c_1 + b_2 - c_1 b_2) \\ &= (a_0 + a_1 X) c_1 + b_2 (1 - c_1)(a_0 + a_1 X) + a_2 c_1 X^2 + a_2 b_2 X^2 - a_2 b_2 c_1 X^2 \end{aligned}$$

$$= a_0 + a_1X + a_2c_1X^2 + a_2b_2X^2 - a_2c_1X^2$$
$$= a_0 + a_1X + a_2X^2$$

Similarly, $c_2 = c_1 + b_2 - c_1b_2, \ldots$

$c_m = c_{m-1} + b_m - c_{m-1}b_m$ and

$(a_0 + a_1X + \ldots + a_iX^i)\ c_i = a_0 + a_1X + \ldots + a_iX^i$

$i = 0, 1, 2, \ldots, m$. Moreover $c_0, c_1, \ldots, c_m \in J$.

Thus there exist $c = c_m \in J$ with $cf(X) = f(X)$.

Conversely, assume R[X] is a PF-ring. Let $a \in R$ and $b \in \operatorname{ann}_R(a)$. Then $b \in \operatorname{ann}_{R[X]}(a)$. Since R is a PF-ring there exists

$$g(X) = c_0 + c_1X + \ldots + c_kX^k \in \operatorname{ann}_{R[X]}(a)$$

with $b\,g(X) = b$. Hence $bc_0 = b$ and $c_0a = 0$.

Consequently, R is a PF-ring.

THEOREM 3. R is a PP-ring if and only if R[X] is a PP-ring.

PROOF. It is enough to show that $\operatorname{ann}_{R[X]}(f(X))$ is generated by an idempotent element in R[X], where $f(X) = a_0 + a_1X + \ldots + a_nX^n$. Since R is reduced, $\operatorname{ann}_{R[X]}(f(X)) = N[X]$ where N is the annihilator of the ideal generated by $a_0, a_1, \ldots, a_n$.

$$N = \operatorname{ann}_R(a_0, a_1, \ldots, a_n)$$
$$= \bigcap_{i=0}^{n} \operatorname{ann}_R(a_i)$$
$$= \bigcap_{i=0}^{\infty} e_iR,\ e_i^2 = e_i \text{ because R is a PP-ring.}$$
$$= (e_1e_2\ldots e_n)R$$
$$= eR, \text{ where } e = e_1e_2\ldots e_n$$

Hence $\operatorname{ann}_{R[X]}(f(X)) = eR[X]$, $e^2 = e$

Conversely, let R[X] be a PP-ring, let $a \in R$, then consider $\operatorname{ann}_R(a)$. Since R[X] is a PP-ring, $\operatorname{ann}_{R[X]}(a) = g(X)R[X]$, where $g(X)^2 = g(X)$. If $g(X) = b_0 + b_1X + \ldots + b_mX^m$, then $b_0^2 = b_0$. We claim $\operatorname{ann}_R(a) = b_0R$. Let $b \in \operatorname{ann}_R(a)$, then $ba = 0$. So $b \in g(X)R[X]$. Thus $b = (b_0 + b_1X + \ldots + b_mX^m)(c_0 + c_1X + \ldots + c_tX^t)$. Therefore $b = b_0c_0$, that is $b \in b_0R$.

For the other way around, let $b \in b_0R$. Then $b = b_0c_0$ for some $c_0 \in R$. Since $b_0a = 0$. That is $b \in \operatorname{ann}_R(a)$. Thus $\operatorname{ann}_R(a) = b_0R$.

REFERENCES

1. EVANS, M. On commutative PP-rings, Pac. Jour. Math., 41(1972) 687-697.

2. VASCONCELOS, W. On finitely generated flat modules, Trans. Amer. Math. Soc., 138 (1969) 505-512.

3. LARSON, M. and MCCARTHY, P. Multiplicative Ideal Theory, Vol 43 in pure and applied Mathematics, Academic Press, New York and London (1971).

4. BREWER, J. Power series over commutative rings, Lecture notes in pure and applied Mathematics 64, Marcel Dekker, New York and Basel (1981).