ITERATIVE METHODS FOR NONLINEAR QUASI
COMPLEMENTARITY PROBLEMS

MUHAMMAD ASLAM NOOR

Mathematics Department,College of Science
King Saud University,Riyadh 11451,Saudi Arabia.

ABSTRACT.   In this paper, we consider and study an iterative algorithm
for finding the approximate solution of the nonlinear quasi complementa--
rity problem of finding $u \in K(u)$, such that

$$Tu \in K^*(u) \quad \text{and} \quad (u-m(u),Tu)=0$$

where m is a point-to-point mapping, T is a (nonlinear) continuous mapping
from a real Hilbert space H into itself and $K^*(u)$ is the polar cone of the
convex cone $K(u)$ in H. We also discuss the convergence criteria and se-
veral special cases, which can be obtained from our main results.

1. INTRODUCTION.

    The relationship between variational inequalities and complementarity
is well known. Variational inequalities have been extended and generalized
in various directions to study a wide class of difficulty problems arising
in different branches of mathematical and engineering sciences. Bensous-
san and Lions [1] introduced a general class of variational inequalities
known as quasi variational inequality, which has many applications in
fluid dynamics, imulse control and genral theory of equilibrium. In a
quasi variational inequality formulation, the convex set also depends on
the solution. Related to the variational inequality problem, there is
also a complementarity problem, which has many applications in operations
research, economics equilibrium and management sciences. In fact, it is
known that if the underlying set in both these problems is a convex cone,
then complementarity problem and variational inequality problem are equi-
valent. This equivalent has been used quite effectively in suggesting
unified and general algorithms for solving complementarity problems, see
Ahn [2] and Noor [3,4,5]..

    Motivated and inpired by the research activities in these fields, we
introduce a new class of complementarity problems, which is the natural

generalization of linear and nonlinear complementarity problems studied
extensively in recent years. After introducing the problem, we show
that its solution can be obtained from an iterative scheme. We also
discuss the convergence criteria for the approximate solution. Several
special cases, which can be obtained from our main results, are disucssed.
Our results are an extension and improvements of the previously known
results.

2. PRELIMINARIES AND BASIC RESULTS.

Let H be a real Hilbert space with its dual H', whose inner product
and norm are denoted by $(.,.)$ and $\|.\|$ respectively. We identify the
Hilbert space H with its dual space H'. We define by $\leq$ partil ordering
on H, which is consistant with the topological vector structure of H that
is the cone of all non-negative elements $\{v \in H : v \geq 0\}$ is a closed convex
cone. Let K be a closed convex set in H. If $T:K \to H$ is a continuous map-
ping , then we consider the problem of finding $u \in K$ such that

$$(Tu,v-u) \geq 0, \qquad \text{for all } v \in K. \qquad (1.1)$$

Inequality of type (1.1) is known as variational inequality. If the
convex set K also depends upon the solution, then variational inequality
(1.1) is called quasi variational inequality. To be more specific, the
quasi variational inequality problem is to find $u \in K(u)$ such that

$$(Tu,v-u) \geq 0, \qquad \text{for all } v \in K(u). \qquad (1.2)$$

In many important applications, K(u) has the following form

$$K(u) = m(u) + K, \qquad (1.3)$$

where m is a point-to-point mapping. In this case $K^*(u)=(m(u)+K)^* =$
$m(u) \cap K^*$. For applications of quasi variational inequalities, see
[6,7,8,9].

Related to quasi variational inequalities, we now introduce a new
class of complementarity problems, which will be called quasi complement-
arity problem of finding u such that

$$u-m(u) \geq 0, \quad Tu \geq 0, \quad (u-m(u),Tu) = 0 \qquad (1.4)$$

If T is a nonlinear transformation, then problem (1.4) is called the non-
linear quasi complementarity problem (NQCP). If T is an affine transfor-
mation of the form $u \to Mu+q$, where M is a nxn matrix, q is a n-vector, then
problem (1.4) is equivalent to finding u such that

$$u-m(u) \geq 0, \quad Mu+q \geq 0, \quad (u-m(u),Mu+q)=0, \qquad (1.5)$$

which is called linear quasi complementarity problem. Problem of type
(1.5) was introduced and studied by Dolcetta [10] , Mosco [7] and Pang
[11,12].

In particular, if the mapping m is zero, then problems (1.4) and
(1.5) are equivalent to the following complementarity problems;

$$u \geq 0, \quad Tu \geq 0, \quad (u, Tu) = 0 \tag{1.6}$$

and

$$u \geq 0, \quad Mu+q \geq 0, \quad (u, Mu+q) = 0, \tag{1.7}$$

which have been extensively studied by Cottle [13,14], Karamardian [15], Mangasarian [16] and Ahn [17] in recent years.

Let $K^* = \{u \varepsilon H, (u,v) \geq 0 \text{ for all } v \varepsilon K\}$, be the polar cone of the convex cone K in H. If the convex cone also depends upon the solution itself, then we introduce the generalized quasi complementarity problem of finding $u \varepsilon K(u)$ such that

$$Tu \varepsilon K^*(u) \quad \text{and} \quad (u-m(u), Tu) = 0, \tag{1.8}$$

It is clear that the nonlinear quasi complementarity problem (1.4) is a special case of the generalized nonlinear quasi complementarity problme (1.8). Furthermore, if the point-to-point mapping m is zero, then problem (1.8) is equivalent to finding $u \varepsilon K$ such that

$$Tu \varepsilon K^* \quad \text{and} \quad (u, Tu) = 0,$$

which is known as the generalized nonlinear complementarity problem considered and studied by Karamardian [15], Cottle [14], Noor [3] and many others.

3. MAIN RESULTS.

We need the following results, the first is a generalization of a result of Karamardian [15] and Pang [11].

LEMMA 3.1. If K is the positive cone in H, then $u \varepsilon K(u)$, defined by (1.3), is a solution of (1.8) if and only if u satisfies the quasi variational inequality (1.2).

PROOF. The proof is similar to that of lemma 3.1 in [18,11].

THEOREM 3.1 [19,20]. For $K(u)$ given by (1.3), $u \varepsilon K(u)$ is a solution of quasi variational inequality (1.2) if and only if u satisfies the following relation

$$u = m(u) + P_K[u - \rho(Tu) - m(u)] \tag{3.1}$$

for some $\rho > 0$, where m is an arbitrary point-to-point mapping and P_K is the projection of H into K, the positive cone in H. For the description of P_K, see [8].

From lemma 3.1 and theorem 3.1, we conclude that the solution to problem (1.8) may be obtained by computing the fixed point of the function defined by (3.1). In fact, theorem 3.1 enables us to find the approximate solution u_n by the following iterative scheme:

$$u_{n+1} = m(u_n) + P_K[u_n - \rho T(u_n) - m(u_n)], \quad n = 0,1,2,\ldots \tag{3.2}$$

We also need the following concepts.

DEFINITION 3.1. A mapping $T: H \to H$ is said to be

i. *Strongly Monotone*, if there exists a constant $\alpha > 0$ such that

$$\langle Tu-Tv, u-v \rangle \geq \alpha \| u-v \|^2, \qquad \text{for all } u, v \varepsilon H.$$

ii. *Lipschitz Continuous*, if there exists a constant $\beta > 0$ such that

$$\| Tu - Tv \| < \beta \| u - v \| , \quad \text{for all } u, v \in H.$$

In particular, it follows that $\alpha \leq \beta$.

In the next theorem, we study the conditions under which the approximate solution obtained from (3.2) converges to the exact solution u of quasi complementarity problem (QCP, 1.8).

THEOREM 3.2. Let the mapping T be strongly monotone and Lipschitz continuous. If the arbitrary mapping m is Lipschitz continuous, u_n and u are solutions satisfying (3.1) and (1.8), then u_n converges to u strongly in H, for

$$\left| \rho - \frac{\alpha}{\beta^2} \right| < \frac{\sqrt{\alpha^2 - 4\beta^2 (\gamma - \gamma^2)}}{\beta^2} , \quad \alpha \geq 2\beta \sqrt{\gamma - \gamma^2} \quad \text{and} \quad \gamma < 1/2,$$

where γ is the Lipschitz constant of m.

PROOF. The method of proof is similar to the one given in [5] . From lemma 3.1 and theorem 3.1 , we observe that the solution u of (1.8) can be characterized by the relation (3.1). Hence from (3.1) and (3.2), we have

$$\| u_{n+1} - u \| \leq \| m(u_n) - m(u) + P_K[u_n - \rho(Tu_n) - m(u_n)] - P_K[u - \rho(Tu) - m(u)] \|$$

$$\leq \| m(u_n) - m(u) \| + \| u_n - u - \rho(Tu_n - Tu) - m(u_n) + m(u) \| ,$$

since P_K is non-expansive, see [21,6].

$$\leq 2\gamma \| u_n - u \| + \| u_n - u - \rho(Tu_n - Tu) \|$$

Now by the strongly monotonicity and Lipschitz continuity of T, we obtain

$$\| u_n - u - \rho(Tu_n - Tu) \|^2 \leq (1 - 2\rho\alpha + \beta^2\rho^2) \| u_n - u \|^2.$$

Thus using the above inequality, we have

$$\| u_{n+1} - u \| \leq [\sqrt{(1 - 2\alpha\rho + \beta^2\rho^2)} + 2\gamma] \| u_n - u \|$$

$$\leq \theta \| u_n - u \| ,$$

where

$$\theta = [\sqrt{(1 - 2\rho\alpha + \beta^2\rho^2)} + 2\gamma] < 1 \quad \text{for} \left| \rho - \frac{\alpha}{\beta^2} \right| < \frac{\sqrt{\alpha^2 - 4\beta^2(\gamma - \gamma^2)}}{\beta^2} ,$$

$$\gamma < \frac{1}{2} \text{ and } \alpha \geq 2\beta\sqrt{\gamma - \gamma^2}$$

Since $\theta < 1$, so the fixed point problem (3.1) has a unique solution u and consequently the Picard iterate u_{n+1} converges to u; see [22] , which is the required result.

It is clear that the use of algorithms as constructive methods for proving the existence of solution brings theory and computation closer together. Theorem 3.2 enables us to find the approximate solution of (1.8) by the following iterative scheme:

i. Given $u_0 \in H$.

ii. $u_{n+1} = m(u_n) + P_K[u_n - \rho(Tu_n) - m(u_n)]$, $n = 0, 1, 2, 3, \ldots$.

where P_K is the projection of H into K, m is a point-to-point mapping and $\rho > 0$ is required to satisfy the following conditions:

$$\left| \rho - \frac{\alpha}{\beta^2} \right| < \frac{\sqrt{\alpha^2 - 4\beta^2(\gamma - \gamma^2)}}{\beta^2} , \gamma < \frac{1}{2} \text{ and } \alpha \geq 2\beta\sqrt{\gamma - \gamma^2} .$$

REMARK 3.1.   If the point-to-point mapping is zero, then it implies that the Lipschitz constant γ is zero. Thus our algorithm is exactly the same as considered by Fang [23] and Gana [24] . Furthermore, theorem 3.1 reduces to a result of Noor and Noor [21,25] for the corresponding variational inequalities. When T is of the form Tu=Mu+q, one can easily prove that T is strongly monotone if and only if M is positive definite, see [15] for proof. Obviously T is also Lipschitz continuous. For the linear quasi complementarity problem, see [18] where one can find a general algorithm for finding the approximate solution along with the necessary convergence criteria.

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