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A MONOTONE PATH IN AN EDGE-ORDERED GRAPH

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ABSTRACT. An edge-ordered graph is an ordered pair (G,f), where G is a graph and f is a bijective function, $f:E(G) \to \{1,2,\ldots,|E(G)|\}$. A monotone path of length k in (G,f) is a simple path $P_{k+1}:v_1v_2\ldots v_{k+1}$ in G such that either $f(\{v_i,v_{i+1}\}) < f(\{v_{i+1},v_{i+2}\})$ or $f(\{v_i,v_{i+1}\}) > f(\{v_{i+1},v_i\})$ for $i = 1,2,\ldots,k-1$.

It is proved that a graph G has the property that (G,f) contains a monotone path of length three for every f iff G contains as a subgraph, an odd cycle of length at least five or one of six listed graphs.

KEY WORDS AND PHRASES. Edge-ordered graph, monotone path.

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1. INTRODUCTION.

Graphs in this paper are finite, loopless and have no multiple edges. We denote by $G = G(V,E)$ a graph with $E(G)$ as its edge-set of cardinality $e(G)$ and $V(G)$ as its vertex-set. Let K_n, P_n, C_n be the complete graph, the path and the cycle, on n vertices, respectively. The vertex-chromatic number of G is denoted by $\chi(G)$, and $d(v)$ is the degree of a vertex $v \in V(G)$. By $H \subset G$ we mean that H is a subgraph of G and $H \not\subset G$ is the negation of this fact.

Definitions and Notation

1. An edge-ordered graph is an ordered pair (G,f), where G is a graph and f is a bijective function, $f:E(G) \to \{1,2,3,\ldots,e(G)\}$.

2. A monotone path of length k, $k \geq 3$ in (G,f), denoted by MP_{k+1}, is a simple path $P_{k+1}:v_1v_2\ldots v_{k+1}$ in G such that either

$$f(\{v_i,v_{i+1}\}) < f(\{v_{i+1},v_{i+2}\})$$

or

$$f(\{v_i,v_{i+1}\}) > f(\{v_{i+1},v_{i+2}\}) \quad \text{for} \quad i = 1,2,\ldots,k-1.$$

3. We denote by $G \to MP_k$ the fact that (G,f) contains an MP_k for every function f, and let

$$A_k = \{G | G \to MP_k\}, \qquad k \geq 3$$

The following Theorem 1.1 is well known, see [1], [2], [3], for a proof and generalizations:

THEOREM 1.1. For every positive integer k, there is a minimal integer $g(k)$, such that $K_n \in A_k$ for every $n \geq g(k)$.

The main result of this paper is:

THEOREM 1.2. A graph G belongs to A_4 iff G contains either C_{2n+1}, $n \geq 2$, or one of the following graphs:

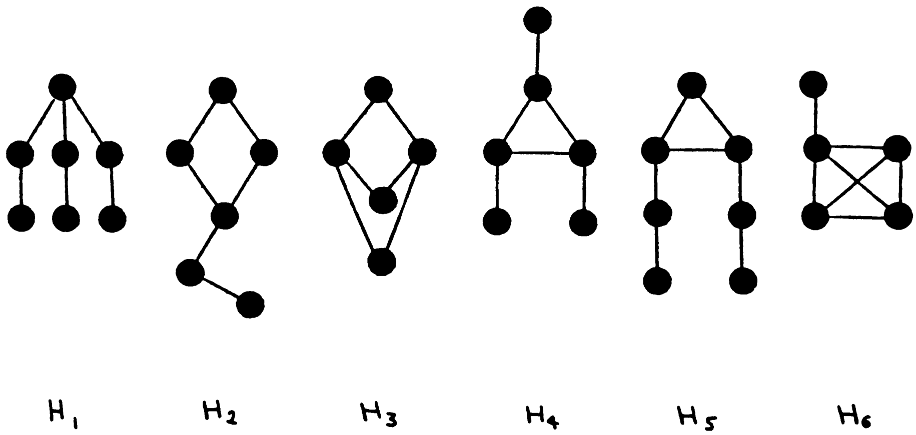


Fig. 1

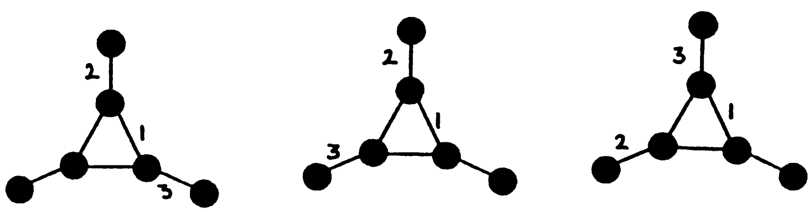
REMARK. Notice that a graph G belongs to A_3 iff G contains a path P_3.

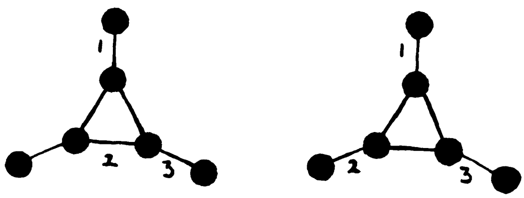
2. PROOFS

The following lemmas are essential for the proof of Theorem 1.2.

LEMMA 2.1. The graphs H_1, H_2, H_3, H_4, H_5, H_6, and C_{2n+1} where $n \geq 2$ belong to A_4.

PROOF. The proof is a straightforward verification for each of the graphs. We prove that $H_4 \in A_4$. The proof of the remaining cases is similar. Assume that there is an f such that no MP_4 occurs in (H_4,f). It turns out that up to isomorphism, the integers 1,2,3 can be assigned to the edges of H_4 in the following 5 ways:





Now, one can see that in each case it is impossible to complete the labeling of the edges such that (H_4,f) does not contain an MP_4.

The following definition is needed for the next lemma.

DEFINITION. Let $a,b,c_1,c_2,\ldots,c_{m+1},a_1,\ldots,a_{2n}$ be non-negative integers where $m \geqq 0$ and $n \geqq 2$. The graph $L_1(m,a,b,c_1,c_2,\ldots,c_{m+1})$, $L_2(a,b)$, $L_3(a,b)$, and $R_{2n}(a_1,a_2,\ldots a_{2n})$ are defined in Fig. 2.

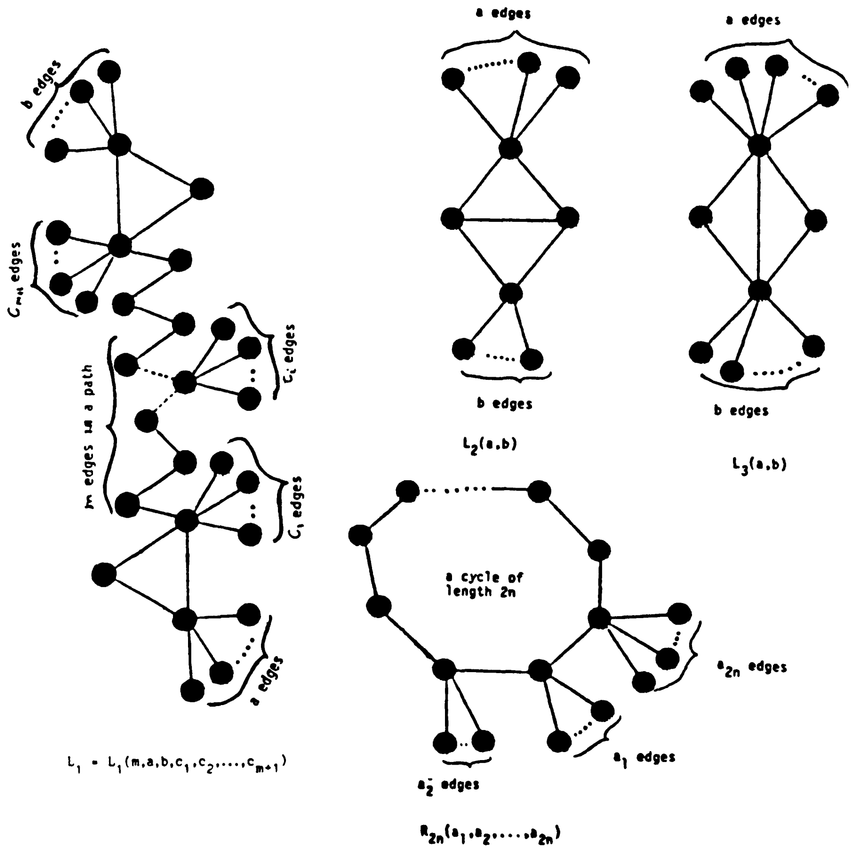
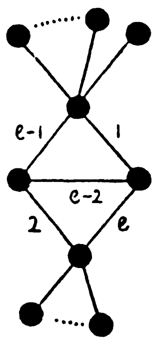


Fig. 2

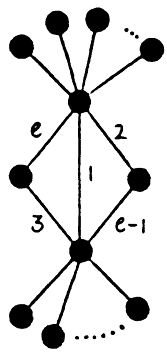
LEMMA 2.2. (i). For all non-negative integers $a,b,c_1,c_2,\ldots c_{m+1},a_1,\ldots,a_{2n}$ where $m \geq 0$ and $n \geq 2$, the graphs L_1, $L_2(a,b)$, $L_3(a,b)$, and $R_{2n}(a_1,a_2,\ldots,a_{2n})$ do not belong to A_4.

(ii). The complete graph K_4 does not belong to A_4.

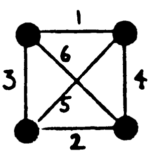
PROOF. We set e for $e(G)$. For the proof of (i), a partial labeling of the edges of the graphs in question is presented in Fig. 3. The labeling of the remaining edges is arbitrary. An MP_4 will not occur. A labeling of $E(K_4)$ is also presented in Fig. 3.

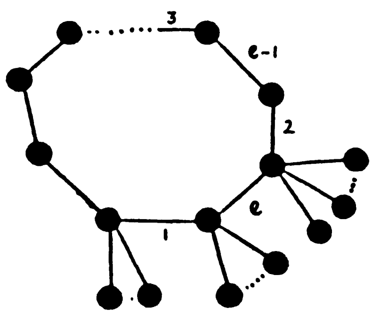


$L_2(a,b)$



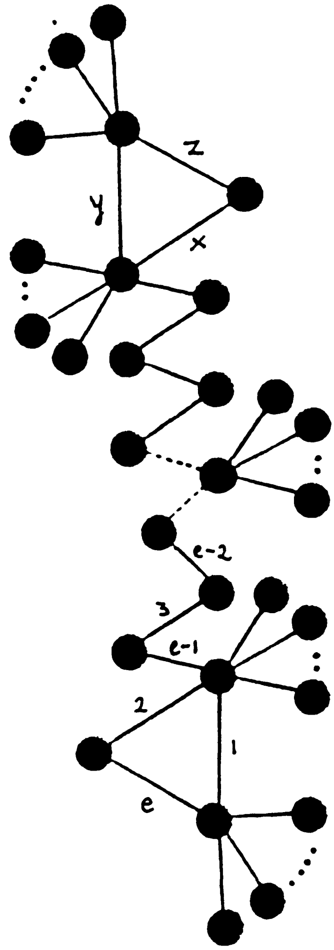
$L_3(a,b)$





$R_{2n}(a_1,a_2,\ldots,a_{2n})$

Fig. 3a



$$x = \begin{cases} e - 1 - \frac{m}{2}, & m \text{ even} \\ \frac{m+1}{2} + 2, & m \text{ odd} \end{cases}$$

$$y = \begin{cases} e - 2 - \frac{m}{2}, & m \text{ even} \\ \frac{c+1}{2} + 3, & m \text{ odd} \end{cases}$$

$$z = \begin{cases} \frac{m}{2} + 2 & , m \text{ even} \\ e - 1 - \frac{m+1}{2} & , m \text{ odd} \end{cases}$$

$$L_1 = L_1(m,a,b,c_1,c_2,\dots,c_{m+1})$$

Fig. 3b

PROOF OF THEOREM 1.2. Clearly, every graph G that contains C_{2n+1}, $n \geq 2$, or an H_i, $i = 1,\dots,6$ belongs to A_4. To prove the opposite containment let $G \in A_4$. We may assume that G is connected and contains a P_4, hence $\chi(G) \geq 2$. We consider two cases: $\chi(G) = 2$ and $\chi(G) \geq 3$.

CASE 1. Let $\chi(G) = 2$. If G is a tree, let $P_t: x_1 x_2 \dots\dots x_t$ be its longest path. If $t = 4$, then G is double star yielding $G \notin A_4$, a contradiction. Hence, $t \geq 5$. Note that the maximality of P_t implies that there is no vertex-disjoint path to P_t, say P_n, where $n \geq 3$, with initial vertex x_2 or x_{t-1}. If for a certain i, $3 \leq i \leq t-2$ there is a vertex-disjoint path to P_t, say P_m, where $m \geq 3$, whose initial vertex is x_i, then $H_1 \subset G$, and we are through. Otherwise, G can be embedded in a graph $R_{2n}(a_1,a_2,\ldots,a_{2n})$ for a certain n and non-negative integers $a_1,a_2,\ldots,a_{2n}$ and in view of Lemma 2.2, $G \not\subset A_4$, a contradiction. Thus we may assume that G is not a tree.

Let C_{2t} be the shortest cycle in G. Assume first $t = 2$, i.e., C_{2t} is a 4-cycle. One can see that if $H_2 \not\subset G$ and $H_3 \not\subset G$ then $G = R_4(a_1,a_2,a_3,a_4)$ for some non-negative integers a_1,a_2,a_3,a_4 and hence by Lemma 2.2, $G \not\subset A_4$, a contradiction. Thus we may assume that $t \geq 3$. Similarly in view of the minimality of C_{2t} it follows that if $H_1 \not\subset G$ then $G = R_{2t}(a_1,a_2,\ldots,a_{2t})$ for some non-negative integers $a_1,a_2,\ldots,a_{2t}$ implying that $G \not\subset A_4$, a contradiction. Hence, the proof of Case 1 is completed.

CASE 2. Let $\chi(G) \geq 3$. Hence G contains an odd cycle C_{2n+1}. If $n \geq 2$ then we are through. So we may assume that G contains only triangles. Let C_3 be any triangle in G with a vertex-set $\{x,y,z\}$. Consider two cases:

(i) Let $d(x), d(y), d(z) \geq 3$. It follows that either $H_4 \subset G$ and we are through, or $K_4 \subset G$ or $L_2(0,1) \subset G$. By Lemma 2.2, $G \neq K_4$, hence $K_4 \subset G$ implies that $H_6 \subset G$. Again Lemma 2.2, $G \neq L_2(a,b)$ for all non-negative integers a and b. Hence $L_2(0,1) \subset G$ implies that one of the graphs H_2, H_4, or H_6 is contained in G. This completes the proof of case (i).

(ii) Assume that at least one of the vertices x,y,z is of degree 2. By Lemma 2.2, G is not a subgraph of L_1 or $L_2(0,b)$ or $L_3(a,b)$ for any non-negative integers a, b, and c; hence G must contain one of the graphs H_1, H_2, H_3, or H_5. This completes the proof of case (ii) and of the theorem.

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