Internat. J. Math. & Math Sci.
VOL. 11 NO. 1 (1988) 43-46

RATIONAL CHOICE FUNCTION DERIVED FROM A FUZZY PREFERENCE

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(Received September 7, 1984 and in revised form April 22, 1987)

ABSTRACT. We shall prove that every fuzzy rational choice function is fuzzy regular (see Richter [6, p. 36]), count the total number of the fuzzy rational choice functions on a set of four elements and consider a semigroup of all fuzzy rational choice functions on a set.

KEY WORDS AND PHRASES. Fuzzy relation - fuzzy binary relation - fuzzy preference - choice function - fuzzy rational choice function - fuzzy transitive - fuzzy regular - semigroup. 1985 AMS CLASSIFICATION NUMBER 03E72

1. INTRODUCTION.

We have introduced a rational choice function derived from a fuzzy preference (see [2], [3], [4]). We shall establish two theorems (Theorems 1 and 2) which are motivated from the following theorems:

THEOREM 4 (Richter [6]). There exists a total rational choice which is not transitive rational.

THEOREM 6 (Richter [6]). There exists a rational choice which is not total rational.

We find that the number of all fuzzy rational choice functions on a set X = {a, b, c, d} of four elements is equal to 57751 (see [2]. We shall consider a semigroup. We note that in [4] there is a beautiful counting formula of the total number of all final choice functions on a finite set.

2. DEFINITIONS AND THEOREMS.

Let X be a finite set with more than two elements. For definitions of a choice function on X and a fuzzy binary relation (R, r) on X, we refer to [2] and [3].

DEFINITION 1 [2, p. 38]. Let (R, r) be a fuzzy relation X and let $a \in X$. Define $R(a) = \{x \in X: aRx \text{ and } r(a,x) \neq 0\}$ and $R_t(a) = \{x \in R(a): r(a,x) \geqq \frac{1}{t}\}$ for $\frac{1}{t} \in (0,1)]$. We define a function h_R as follows: Let $a \in A \subseteq X$. Then $a \in h_R(A)$ iff $A \subseteq R_A(a)$. We add that $h_R(\emptyset) = \emptyset$, the empty set. Note that h_R is in general, not a choice function. Let h be a choice function on X. If there exists a fuzzy relation (R, r) on X such that $h_R = h$, then we shall say that h is fuzzy rational and (R, r) rationalizes h.

NOTATION 1. We denote by F(X) the set of all fuzzy binary relations on X. We define $\Sigma = 2^X$ and $C(X, \Sigma)$ denotes the set of all choice functions h on X. Let $(R, r) \varepsilon F(X)$. We use $(x, y) \varepsilon R$ and x R y when $r(x,y) \neq 0$. Let $h \varepsilon C(X, \Sigma)$ be a choice function on X. Define $F(h) = \{(R, r) \varepsilon F(X): (R,r)$ rationalizes $h\}$.

DEFINITION 2. h is said to be fuzzy transitive (total, reflexive) if there exists (R, r) in $\varepsilon F(h)$ such that (R, r) is transitive (total, reflexive). (R,r) F(X) is regular if (R,r) is reflexive, total and transitive. h is fuzzy regular if there exists $(R, r) \varepsilon F(h)$ such that (R, r) is regular.

We shall prove the following theorem.

THEOREM 1. Every fuzzy rational choice function is fuzzy transitive.

PROOF. Let h be a fuzzy rational choice function on X. Then F(h) is non-empty and let $(R, r) \varepsilon F(h)$. Then $h= h_R$. Suppose that (R, r) is not transitive. Define $\{r\} = \{r(x,y) \neq 0: x,y \varepsilon X\}$ for (R, r). We can find a positivie number $t_o = \frac{1}{n+k}$ such that $t_o \notin \{r\}$, where k is a positive integer. We define a fuzzy relation (S, s) as follows: If $r(x,y) \neq 0$, then we put $s(x,y)=r(x,y)$, and if $r(x,y)=0$ then we put $s(x,y)=t_o$. It is clear that (S,s) is a transitive fuzzy relation on X. We show that $h_R=h_S$. To show this, we assume that $h_R \neq h_S$. Then there exists a non-empty set A such that $B = h_R(A) \neq h_S(A)=C$. We can assume that $c \varepsilon C$ and $a \notin B$. Then $(a,x) \varepsilon S$ for all $x \varepsilon A$, $s(a,x) \geqq \frac{1}{|A|} > \frac{1}{n+k} = t_o$, and hence $s(a,x) \neq t_o$. In view of {r} and $t_o \notin \{r\}$, it is clear that $s(a,x)=r(a,x)$ for all $x \varepsilon A$, and hence $a \varepsilon B$. This contradicts $a \notin B$. A similar proof for $b \varepsilon B$ and $b \notin C$ brings a contradiction. Therefore B=C and $h_R=h_S=h$. This proves Theorem 1.

THEOREM 2. Every fuzzy rational choice function h on X is fuzzy total.

PROOF. Let h be a fuzzy rational choice function on X. Then there exists (R, r) such that $h_R=h$. For $x, y \varepsilon X$ and $x \neq y$, it is clear that $h_R\{x,y\} \subseteq \{x,y\}$. Thus we have that either $r(x,y) \geqq \frac{1}{2}$ or $r(y,x) \geqq \frac{1}{2}$. Therefore (R,r) is total. This proves Theorem 2.

COROLLARY 1. Every fuzzy rational choice function is regular. The proof follows from Theorems 1 and 2.

3. A SEMIGROUP.

We begin with the following definition.

DEFINITION 3. Let $(R, r) \in F(X)$ be a fuzzy relation. (R, r) is completely total if $r(a,b) \neq 0$ and $r(b,a) \neq 0$ for all $a,b \varepsilon X$. A choice function h is fuzzy completely total if there exists $(R,r) \varepsilon F(X)$ such that $h_R=h$ and (R, r) is completely total. h is fuzzy completely regular if there exists (R, r) such that $h=h_R$ is fuzzy regular and fuzzy completely total.

We have considered a semigroup in [2] and [4]. We denote by CR(X) the set of all completely regular fuzzy rational choice functions on X. By Theorem 4-(i)[2], we have that $h_P h_Q \subseteq h_{P \cup Q}$, $h_P, h_Q \varepsilon CR(X)$. Thus we have the following theorem.

THEOREM 3. CR(X) forms a semigroup under the binary operation defined by $h_P h_Q = h_{P \cup Q}$, $h_P, h_Q \varepsilon CR(X)$.

We note that if $h \varepsilon CR(X)$, then there exists (P,p) such that $h=h_P$ and (P, p) is regular and completely total.

PROOF. It is clear that the binary operation is associative. It is also clear that $P \cup Q = R$ (or(R, r)) is regular and completely total. Letting $P \cup Q =$

R, $h_R(A) \subseteq A$ is a part of the definition of h_R (see Definition 1). We prove that $h_R(A)$ is non-empty when A is non-empty. We assume that $A \neq \emptyset$ and $|A| = m$. Since $h_P(A) \neq \emptyset$, there exists $a \in h_P(A)$ and hence $p(a,x) \geqq \frac{1}{m}$, for all $x \in A$. From $r(a,x)=\max\{p(a,x), q(a,x)\}$ it follows that $r(a,x) \geqq \frac{1}{m}$ for all $x \in A$. This shows that $a \in h_R(A)$. This proves Theorem 3.

The following example is to show that $h_P(h_Q)$, the composite set function, is not a fuzzy rational choice even though h_P and h_Q are both fuzzy rational choices on X.

EXAMPLE 1. Let $X=\{a, b, c\}$. Let $(R, r)=\{r(a,a)=r(b,b)=r(c,c)=1$, $r(a,b)=r(a,c)=r(b,c)= \frac{1}{2}$, $r(b,a)=r(c,a)=r(c,b) = \frac{1}{4}\}$ and $(Q,q)=\{q(a,a)=q(b,b)=q(c,c)=1$, $q(b,a)=q(c,a)=q(c,b)= \frac{1}{2}$, $q(b,c)= \frac{1}{3}$, $q(a,b)=q(a,c)= \frac{1}{5}\}$. Then we can prove that there is not a fuzzy relation (P, p) such that $h_P=h_R(h_Q)$.

We list the following theorem.

THEOREM 4. Let (r, r) be a fuzzy relation on X. A necessary and sufficient condition for h_R to be a choice function on X is that for every non-empty subset A of X there exists at least one member a in A such that $r(a,x) \geqq \frac{1}{|A|}$ for all $x \in A$.

PROOF. We suppose that the condition holds for (R,r). Let $A \neq \emptyset$ and assume that there is a in A such that $r(a,x) \geqq \frac{1}{|A|}$ for all $x \in A$. Then $A \subseteq R_{|A|}(a)$ and $a \in h_R(A)$. $h_R(A) \subseteq A$ is a part of the definition of h_R. Thus h_R is a choice function on X. Suppose h_R is a choice on X. Then for each $A \neq \emptyset$ there is a in A such that $a \in h_r(A)$ from which we obtain that $r(a,x) \geqq \frac{1}{|A|}$. This proves Theorem 4.

4. THE NUMBER OF ALL FUZZY RATIONAL CHOICES ON {a,b,c,d}. Let X be a set of n elements. We denote the number of all fuzzy rational choice functions on X by $h_{F(X)}(n)$. In [2] we showed that $h_{F(X)}(3) = 93$. In this section we announce that $h_{F(X)}(4) = 57751$. WE shall prove this in a separate paper. A justification of $h_{F(X)}(4) = 57751$ needs several pages.

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