

TIME-EVOLUTION OF A CAUSTIC

NASIT ARI and ARTHUR D. GORMAN
Department of Engineering Science
Lafayette College
Easton, Pennsylvania 18042 U.S.A.

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ABSTRACT. The Lagrange manifold formalism is adapted to study the time-evolution of caustics associated with high frequency wave propagation in media with both spatial and temporal inhomogeneities.

KEYWORDS AND PHRASES: Wave Propagation, Lagrange manifold, caustics, turning points.

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1. INTRODUCTION

The utility of the asymptotic series, or geometrical optics, approach developed by Keller [1] and his students [2] for studying wave-type linear partial differential equations is well known. For example, applied to a differential equation of the form

$$\nabla^2 \psi(\bar{r}, t) - f(\bar{r}, t) \frac{\partial^2 \psi(\bar{r}, t)}{\partial t^2} - \lambda^2 g(\bar{r}, t) \psi(\bar{r}, t) = 0 \quad (1.1)$$

where, for definiteness, $\psi(\bar{r}, t)$ is the wave function, \bar{r} refers to the spatial coordinates, t is the time and λ is a large parameter, a solution of the form

$$\psi(\bar{r}, t) = \exp [i\lambda S(\bar{r}, t)] A(\bar{r}, t, \lambda), \quad (1.2)$$

where

$$A(\bar{r}, t, \lambda) \doteq \sum_{k=0}^{\infty} A_k(\bar{r}, t) (i\lambda)^{-k}, \quad A_{-k} = 0 \quad (1.3)$$

is assumed. $S(\bar{r}, t)$ may be regarded as a phase and $A(\bar{r}, t, \lambda)$ as an amplitude. Then substituting Equation (1.2) into Equation (1.1) followed by a re-grouping in powers of $i\lambda$ leads to

$$\left\{ (i\lambda)^2 \left[(\nabla S)^2 - f(\bar{r}, t) \left(\frac{\partial S}{\partial t} \right)^2 + g(\bar{r}, t) \right] + \right. \\ (i\lambda) \left[\nabla^2 S - f(\bar{r}, t) \frac{\partial^2 S}{\partial t^2} + 2\nabla S \cdot \nabla - 2f(\bar{r}, t) \frac{\partial S}{\partial t} \frac{\partial}{\partial t} \right] \\ \left. (i\lambda)^0 \left[\nabla^2 - f(\bar{r}, t) \frac{\partial^2}{\partial t^2} \right] \right\}_{k=0} \sum A_k(\bar{r}, t) (i\lambda)^{-k} = 0. \quad (1.4)$$

Then by introducing the wave number and frequency

$$\bar{p} = \nabla S, \quad \omega = -\frac{\partial S}{\partial t}, \quad (1.5)$$

respectively, the coefficient of the $(i\lambda)^2$ term may be regarded as a Hamiltonian

$$H = \bar{p} \cdot \bar{p} - f(\bar{r}, t) \omega^2 + g(\bar{r}, t).$$

The standard approach for obtaining the phase involves the introduction of Hamilton's equations

$$\frac{d\bar{r}}{d\gamma} = \nabla_{\bar{p}} H \quad \frac{d\bar{p}}{d\gamma} = -\nabla_{\bar{r}} H \quad (1.6)$$

$$\frac{dt}{d\gamma} = -\frac{\partial H}{\partial \omega} \quad \frac{d\omega}{d\gamma} = \frac{\partial H}{\partial t} \quad (1.7)$$

which leads to the ray trajectories (map)

$$\bar{r} = \bar{r}(\gamma, \bar{\sigma}) \quad \bar{p} = \bar{p}(\gamma, \bar{\sigma}) \quad (1.8)$$

$$t = t(\gamma, \bar{\sigma}) \quad \omega = \omega(\gamma, \bar{\sigma}), \quad (1.9)$$

where γ is the ray-path parameter and $\bar{\sigma}$ a parameterized initial condition. But at those space-time points where the coordinate space map becomes singular i.e., on the caustic curve where

$$\det \begin{pmatrix} \frac{\partial \bar{r}}{\partial \bar{\mu}} \\ \frac{\partial \bar{p}}{\partial \bar{\mu}} \end{pmatrix} = 0 \quad (1.10)$$

with $\bar{\mu} = (\gamma, \bar{\sigma})$, the geometrical optics procedure cannot be applied directly.

Such difficulties at caustics can often be circumvented by using the Lagrange Manifold formalism of Maslov [3] and Arnold [4], which has recently been extended to determine a class of asymptotic solutions [5] for phenomena modelled by Equation (1.1). Here we present a variation of this extension which enables a modelling of the time evolution of the caustic. This algorithm also leads to determination of the field on the caustic; but because so much of this aspect of the procedure in [5] applies directly, for brevity we emphasize only those aspects pertinent to modelling the evolution of the caustic. For clarity, we consider the scalar wave equation given in Equation (1.1), although the analogous vector wave equation could also have been considered. An example is included to illustrate the procedure.

2. FORMALISM

We assume that near caustics Equation (1.1) has an asymptotic solution of the form

$$\Psi(\bar{r}, t) = \int A(\bar{r}, \bar{p}, t, \lambda) \exp[i\lambda(\bar{r} \cdot \bar{p} - S(\bar{p}, t))] d\bar{p} = O(\lambda^{-\infty}). \quad (2.1)$$

The amplitude $A(\bar{r}, \bar{p}, t, \lambda)$ and its derivatives are assumed bounded and $\bar{r} \cdot \bar{p} - S(\bar{p}, t)$ is regarded as a phase, i.e.,

$$\phi(\bar{r}, \bar{p}, t) = \bar{r} \cdot \bar{p} - S(\bar{p}, t).$$

Carrying the differentiation in Equation (1.1) across the integral in Equation (2.1) leads to

$$i d\bar{p} \exp(i\lambda\phi) \left\{ (i\lambda)^2 \left[\bar{p} \cdot \bar{p} - f(\bar{r}, t)\omega^2 + g(\bar{r}, t) \right] + i\lambda \left[2\bar{p} \cdot \nabla_{\bar{r}} A + 2\omega f(\bar{r}, t) \frac{\partial A}{\partial t} \right] + (i\lambda)^0 \left[\nabla_{\bar{r}}^2 A - f(\bar{r}, t) \frac{\partial^2 A}{\partial t^2} \right] \right\} = 0 (\lambda^{-\infty}) \tag{2.2}$$

Where, analogous to Equations (1.5), the wavevector ($\bar{p} = \nabla\phi$) and frequency ($\omega = -\frac{\partial\phi}{\partial t}$) have been introduced. The coefficient of $(i\lambda)^2$ term is seen to be Maslov's Hamiltonian

$$H = \bar{p} \cdot \bar{p} - f(\bar{r}, t)\omega^2 + g(\bar{r}, t). \tag{2.3}$$

The field at any space-time point (\bar{r}, t) proceeds from the stationary phase ($\nabla_{\bar{p}}\phi=0$) evaluation of the integral in Equation (1.2), which turns the Hamiltonian into an eikonal equation

$$\bar{p} \cdot \bar{p} - f(\bar{r}, t)\omega^2 + g(\bar{r}, t) = 0 \tag{2.4}$$

and determines the time-parameterized Lagrange Manifold

$$\bar{r} = \nabla_{\bar{p}} S(\bar{p}, t). \tag{2.5}$$

In the Lagrange Manifold formalism caustic points are determined from the phase [5], or equivalently from $S(\bar{p}, t)$. To obtain this $S(\bar{p}, t)$, we first find the trajectories (through Hamilton's equations)

$$\bar{r} = \bar{r}(\gamma, \bar{\sigma}) \quad \bar{p} = \bar{p}(\gamma, \bar{\sigma}) \tag{1.8}$$

$$t = t(\gamma, \bar{\sigma}) \quad \omega = \omega(\gamma, \bar{\sigma}) \tag{1.9}$$

then invert the wavevector and time transformations

$$\bar{p} = \bar{p}(\gamma, \bar{\sigma}) \quad t = t(\gamma, \bar{\sigma}) \tag{2.6}$$

to obtain

$$\gamma = \gamma(\bar{p}, t) \quad \bar{\sigma} = \bar{\sigma}(\bar{p}, t). \tag{2.7}$$

Substitution into the coordinate space map determines the Lagrange Manifold explicitly

$$\bar{r} = \bar{r}(\gamma(\bar{p}, t), \bar{\sigma}(\bar{p}, t)) = \nabla_{\bar{p}} S(\bar{p}, t),$$

where the time appears as a parameter. Then an integration along the trajectories

$$S(\bar{p}, t) = \int_{\bar{p}_0}^{\bar{p}} \bar{r} \cdot d\bar{p} \tag{2.8}$$

leads to the phase

$$\phi(\bar{r}, \bar{p}, t) = \bar{r} \cdot \bar{p} - S(\bar{p}, t). \tag{2.9}$$

The caustic points, equivalent to those specified in Equation (1.10), are those at which

$$\det \left\{ \frac{\partial^2 \Phi}{\partial p_i \partial p_j} \right\} = \det \left\{ \frac{\partial^2 S}{\partial p_i \partial p_j} \right\} = 0 \quad ; (i, j = 1, 2, 3) \quad (2.10)$$

At a given time this condition leads to sets of triplets (\bar{p}) , which upon substitution into Equation (2.5) determines the caustic in coordinate space. The time evolution of the caustic proceeds by considering Equation (2.10) for several values of time. Corresponding to each such time is a set of triplets (\bar{p}) , which upon substitution into the Lagrange manifold yield the time evolution of the caustic in coordinate space.

The determination of the field on the caustic requires the development of a transport equation for the amplitudes. As this development so parallels that referred to above [5], we do not include it here.

3. EXAMPLE

As an example, we consider wave propagation in a medium with $f(\bar{r}, t) = 1$, $g(\bar{r}, t) = bx + at - k^2$, with a, b, k constants. Consequently, the wave equation we consider is

$$\nabla^2 \psi - \frac{\partial^2 \psi}{\partial t^2} - \lambda^2 (bx + at - k^2) \psi = 0. \quad (3.1)$$

Let the initial-boundary condition be that at $r=0$, a point source at the origin $\bar{r}=(0,0)$ begins radiating at $t=t_0$, with initial frequency Ω and wave vector $\bar{p}=(p_0 \cos \theta, p_0 \sin \theta)$. We assume an asymptotic solution of the form

$$\psi(\bar{r}, t) = \int A(\bar{r}, \bar{p}, t, \lambda) \exp(i\lambda(\bar{r} \cdot \bar{p} - S(\bar{p}, t))) d\bar{p} = O(\lambda^{-\infty}) \quad (3.2)$$

Then proceeding through the algorithm we obtain Maslov's Hamiltonian (Equation (3.3)) and eikonal (Equation (3.4)), respectively

$$H = \bar{p} \cdot \bar{p} - \omega^2 + bx + at - k^2 \quad (3.3)$$

$$\bar{p} \cdot \bar{p} - \omega^2 + bx + at - k^2 = 0. \quad (3.4)$$

Next Hamilton's Equations (1.6) and (1.7), together with the initial conditions at $r=0$, are solved to obtain the maps

$$x = b\gamma^2 + 2p_0\gamma \cos \theta \quad p_x = -b\gamma + p_0 \cos \theta \quad (3.5)$$

$$y = 2p_0\gamma \sin \theta \quad p_y = p_0 \sin \theta \quad (3.6)$$

$$t = a\gamma^2 + 2\gamma\Omega + t_0 \quad \omega = a\gamma + \Omega. \quad (3.7)$$

While Hamilton's Equations relate the canonical variables, the selection of the initial specific condition $t=t_0$ at $r=0$ introduces an additional coupling between t_0 and Ω from the eikonal equation

$$p_0^2 - \Omega^2 - k^2 + at_0 = 0. \quad (3.8)$$

It is this coupling which allows the inclusion of time t as a parameter in the Lagrange Manifold. Specifically, elimination of t_0 between the time coordinate in Equation (3.7) and Equation (3.8), allows the arc length γ to be parameterized in time t . Then, inversion of the map $\bar{\mu} \rightarrow \bar{p}$ includes the time as a parameter, which is subsequently introduced into the Lagrange manifold. To illustrate, for clarity of exposition, we first use the simplicity of the example to eliminate θ between

Equations (3.5) and (3.6), obtaining

$$x = by^2 + 2\gamma p_x \tag{3.9}$$

$$y = 2\gamma p_y \tag{3.10}$$

Then from the space and time coordinates in Equations (3.5)-(3.7) and Equation (3.8), we determine

$$r = r(p_x, p_y, t) = \frac{(bp_x - a\Omega) + [(a\Omega - bp_x)^2 - (a^2 - b^2)(\Omega^2 + k^2 - p_x^2 - p_y^2 - at)]^{1/2}}{[a^2 - b^2]} \tag{3.11}$$

Finally, substituting into Equations (3.9) and (3.10), followed by integrations with respect to p_x and p_y respectively leads to the phase

$$\begin{aligned} \Phi(\bar{r}, \bar{p}, t) = & \bar{r} \cdot \bar{p} - \frac{2a}{(b^2 - a^2)^2} \left[ab \left(\Omega^2 p_x + \frac{p_x^3}{3} \right) - \frac{(b^2 + a^2) \Omega p_x^2}{2} \right] + \\ & + \frac{2}{3(b^2 - a^2)^2} \left[(a\Omega - bp_x)^2 - (a^2 - b^2)(\Omega^2 + k^2 - p_x^2 - p_y^2 - at) \right]^{3/2} \\ & - \frac{1}{b^2 - a^2} \left[b(\Omega^2 + k^2 - p_y^2 - at)p_x - \frac{bp_x^3}{3} + a\Omega p_y^2 + b^2 \Omega t \right]. \end{aligned} \tag{3.12}$$

(We note only the minus sign in Equation (3.11) leads to physically realizable caustics.)

As a specific example to illustrate the calculational aspects of the algorithm, let us choose $a = -1$, $b = 6.655 (= 45/7 \cos 15^\circ)$, $k^2 = 23$, $\theta = 15^\circ$, $t_0 = 1$, $\Omega = 5$, $|\bar{p}_0| = 7$ and the initial amplitude at the emitter $A_0 = 1$. For this case the Lagrange Manifold and phase are, respectively,

$$\begin{aligned} x = & 7.556 + .154t + .242p_x - .147p_x^2 - .154p_y^2 - (0.046p_x + 1.537) \cdot \\ & \cdot (1.122 + 0.231t + .0355p_x + .0005p_x^2 - .0231p_y^2)^{1/2} \end{aligned} \tag{3.13}$$

$$y = -p_y \left[.232 + .308p_x + 2(1.122 + .0231t + .0355p_x + .0005p_x^2 - .0231p_y^2) \right] \tag{3.14}$$

An integration leads to the phase

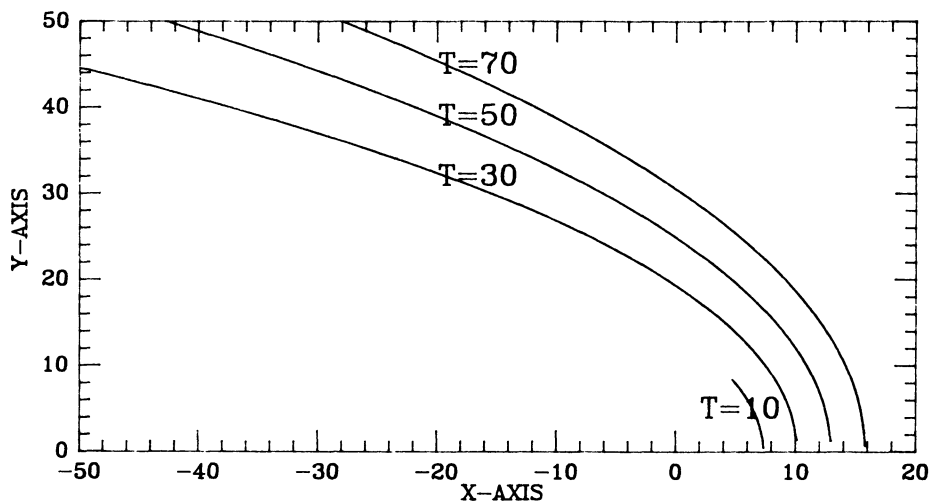
$$\begin{aligned} \Phi(\bar{r}, \bar{p}, t) = & \bar{r} \cdot \bar{p} - 5.116t - 7.556p_x - 0.121p_x^2 + .116p_y^2 - .154tp_x \\ & + .049p_x^3 + .154p_x p_y^2 + 28.86 \left[1.122 + .0355p_x + .0231t + .0005p_x^2 - .0231p_y^2 \right]^{3/2} \end{aligned} \tag{3.15}$$

Then at $r=1$, i.e., the space-time point $(x,y,t) = (6.87, 3.62, 10.)$, the classical map becomes singular, as does the Hessian determinant of the phase (Equation (2.10)) at the corresponding point in wavevector space $\bar{p} = (.106, 1.812)$. This illustrates the level-equivalence of the classical map and the transformation specified by the Lagrange Manifold. To find the caustic at $t=10$, we find those sets of wavevectors (\bar{p}) satisfying Equation (2.10); substituting these wavevectors

into the Lagrange Manifold determines the caustic. The time evolution of the caustic proceeds by substituting successive values of time into the phase and repeating the procedure, (Figure 1).

We note that even over long time duration, the topological type of this caustic does not change, cf. Mather[6]. The determination of the field at the caustic proceeds largely as in [5]. For completeness, we note that at $(x,y,t) = (6.87, 3.62, 10.)$, the first two terms in the asymptotic series are

$$\psi(6.87, 3.62, 10.) \approx \exp(i\tau(\pi/4+9.8))\pi^{1/2} \left[-1.53\Gamma\left(\frac{1}{3}\right)\tau^{-5/6} \cos \frac{\pi}{6} - 1.74\Gamma\left(\frac{2}{3}\right)\tau^{-7/8} \sin \frac{1}{3} \right]$$



TIME EVOLUTION OF A CAUSTIC

FIGURE 1.

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