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# ON AN INTEGRAL TRANSFORM

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ABSTRACT. A formula of inversion is established for an integral transform whose kernel is the Bessel function $J_u(kr)$ where r varies over the finite interval (0,a) and the order u is taken to be the eigenvalue parameter. When this parameter is large the Bessel function behaves for varying r like the power function $r^u$ and by relating the Bessel functions to their corresponding power functions the proof of the inversion formula can be reduced to one depending on the Mellin inversion theorem.

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## 1. INTRODUCTION

This paper considers the problem of inverting the integral transform defined by the equation

$$F(u) = \int_0^a J_u(kr)f(r)\frac{dr}{r} \tag{1.1}$$

where k, a are positive constants and u is a complex parameter. This transform, which does not appear to have been used previously, is useful to remove the group of terms $(r^2f_{rr} + rf_r + k^2r^2f)$ that occurs when the space form of the wave equation is expressed in polar coordinates and r varies over the finite interval $0 < r < a$. The transform is related to another transform defined on the same finite interval by the equation

$$F_1(u) = \int_0^a [J_u(kr)Y_u(ka) - J_u(ka)Y_u(kr)]f(r)\frac{dr}{r} \tag{1.2}$$

This transform was introduced in [1] where a formula of inversion for it was stated. However because of the behaviour of the Bessel function terms in the vicinity of the origin the transform (1.2) is more complicated than that defined by (1.1) and exists for a smaller class of functions f(r). For the transform (1.2) to exist the function f(r) must tend to zero as $r \to 0$, an unnecessarily restrictive condition. Also, if the transform (1.2) does exist for a certain class of functions it may be expected to do so in some strip $|Re(u)| < \gamma$ of the complex u-plane, in which case, for the same class of functions, the transform (1.1) will exist in the half plane $Re(u) > -\gamma$ containing the strip. To see this we may suppose that f(r) is continuous for $0 < r \leq a$ and that $f(r) = O(r^b)$ as $r \to 0$, then since $J_u(kr) \sim (kr/2)^u/\Gamma(u+1)$, it follows from (1.1) that F(u) will exist in the half plane $Re(u) > -b$. However, since

$$Y_u(x) = [J_u(x)\cos u\pi - J_{-u}(x)]\operatorname{cosec} u\pi \tag{1.3}$$

then

$$F_1(u) = -\operatorname{cosec} u\pi \int_0^a [J_u(kr)J_{-u}(ka) - J_u(ka)J_{-u}(kr)]f(r)\frac{dr}{r}$$

$$= -\operatorname{cosec} u\pi[F(u)J_{-u}(ka) - F(-u)J_u(ka)] \tag{1.4}$$

so that $F_1(u)$ will exist only if $F(u)$ and $F(-u)$ possess a common domain of definition. If $F(u)$ exists only for $\operatorname{Re}(u) > -b$ and $F(-u)$ for $\operatorname{Re}(u) < b$, where $b < 0$, no such domain exists. However if $b > 0$, $F(u)$ and $F(-u)$ will exist in overlapping half planes and $F_1(u)$ will exist in the common strip, that is for $|\operatorname{Re}(u)| < b$. If $f(r) \sim c_1 r^b$ as $r \to 0$, where $c_1$ is a constant and $b \leq 0$, then $F_1(u)$ fails to exist for any value of u.

In this paper a formula of inversion is established for the integral transform defined by (1.1). Before stating this formula it is necessary to summarize certain results relating to the zeros $u_n$ of the function $J_u(ka)$, regarded as a function of the order u when the quantity ka is supposed prescribed and positive. It is known, [2],[3] that there is an infinite number of such zeros, all real and simple, and that they all lie in the interval $-\infty < u \leq ka$. The large u-zeros are asymptotic to the negative integers and, depending on the value of ka, only a finite number, or none, of the zeros is positive. An extensive table of values of the zeros of $J_u(ka)$, regarded as a function of u, has been compiled in [4] where the first ten such zeros are calculated for a large number of values of ka ranging from $10^{-3}$ to $10^{-6}$.

The following theorem may now be stated:

THEOREM. Suppose that $f(r)$ is continuous for $0 < r \leq a$ and that $f(r) = O(r^b)$ as $r \to 0$, then the integral transform defined by (1.1) possesses the inverse

$$f(r) = \frac{1}{2i}\int_L \frac{u\phi(u,r)F(u)du}{J_u(ka)} \tag{1.5}$$

where $0 < r < a$,

$$\phi(u,r) = J_u(kr)Y_u(ka) - J_u(ka)Y_u(kr) \tag{1.6}$$

and the path of integration L in the complex u-plane is the line $\operatorname{Re}(u) = c$ positioned so that $c > -b$ and so that all of the zeros of $J_u(ka)$ lie to the left of it.

## 2. THE INTEGRAL THEOREM.

The formula (1.5) can be established by following the method of proof adopted in [5] to derive a related formula in which the kernel of the transform involved the Neumann function $Y_u(kr)$. This method stems from the observation that when u is large the function $J_u(x)$ behaves for varying x like the power function $(x/2)^u/\Gamma(u+1)$. By relating the Bessel functions to their corresponding power functions we may evade invoking many special properties of these functions and reduce the proof of (1.5) to one depending on the Mellin inversion theorem.

Let $f = O(r^b)$ as $r \to 0$ where b is real and let $L(R)$ denote the straight line in the u-plane drawn from the point $c-iR$ to the point $c+iR$, where $c > -b$. The first step in the proof of (1.5) is to form the equation:

$$\int_{L(R)} \frac{u\phi(u,r)F(u)du}{J_u(ka)} = \int_{L(R)} \frac{u\phi(u,r)du}{J_u(ka)} \int_0^a J_u(kt)f(t)\frac{dt}{t} \tag{2.1}$$

This equation follows at once by substituting the expression (1.1) for the function $F(u)$ present on the left hand side of equation (2.1). The next step is to extract the dominant contribution from the combination of Bessel functions appearing on the right hand side of (2.1) when the variable u is large. With this aim in view we appeal to the asymptotic formula

$$J_u(x) = \frac{(\frac{1}{2}x)^u}{\Gamma(u+1)}\left[1 - \frac{x^2}{4(u+1)} + O(u^{-2})\right] \tag{2.2}$$

This formula applies whenever u is large compared to x and bounded away from the negative integers. It follows from (2.2) that

$$J_u(kt)/J_u(ka) = (t/a)^u[1 - k^2(t^2-a^2)/4u + O(u^{-2})] \tag{2.3}$$

The behaviour of the function $\phi(u,r)$ for large values of u can be obtained in a similar fashion if the $Y_u$ type functions present in (1.6) are first eliminated in favour of the $J_u$ type functions by means of the identity (1.3) from which it follows in turn that

$$\phi(u,r) = -[J_u(kr)J_{-u}(ka) - J_u(ka)J_{-u}(kr)]\operatorname{cosec} u\pi \tag{2.4}$$

$$= -\frac{1}{u\pi}[(a/r)^u - (r/a)^u] + \frac{k^2(r^2-a^2)}{4\pi u^2}[(r/a)^u + (a/r)^u] + O[u^{-3}(a/r)^{|\mathrm{Re}(u)|}] \tag{2.5}$$

after estimating the Bessel functions occurring in (2.4) by means of (2.2) and using the identity $\Gamma(1+u)\Gamma(1-u) = u\pi \operatorname{cosec} u\pi$. It follows on multiplying (2.3) and (2.5) that

$$\frac{\pi u\phi(u,r)J_u(kt)}{J_u(ka)} = (t/r)^u - (rt/a^2)^u + h(u,r,t) \tag{2.6}$$

where, for large values of u bounded away from the negative integers,

$$h(u,r,t) = \frac{k^2(r^2-t^2)}{4u}(t/r)^u + \frac{k^2(r^2+t^2-2a^2)}{4u}(rt/a^2)^u + O[u^{-2}(t/r)^u] \tag{2.7}$$

The equation (2.6) is the desired asymptotic formula that determines the behaviour of the integrand present in (2.1) when u is large. The dominant terms in this expression are the power functions $(t/r)^u$ and $(rt/a^2)^u$, the function $h(u,r,t)$ remaining being clearly $O[u^{-1}(t/r)^u]$.

The next step in the proof is to insert the expression (2.6) into the integral on the right hand side of (2.1). This yields the equation

$$\int_{L(R)} \frac{u\phi(u,r)F(u)du}{J_u(ka)} = \frac{1}{\pi}\int_{L(R)} r^{-u}du \int_0^a f(t)t^{u-1}dt \tag{2.8}$$

$$- \frac{1}{\pi}\int_{L(R)} (a^2/r)^{-u}du \int_0^a f(t)t^{u-1}dt + \frac{1}{\pi}\int_{L(R)} du \int_0^a f(t)h(u,r,t)t^{-1}dt$$

Now by the Mellin inversion theorem, [6],

$$\lim_{R\to\infty} \int_{L(R)} r^{-u}du \int_0^a t^{u-1}f(t)dt = \begin{cases} 2i\pi f(r), & 0 < r < a \\ 0, & r > a \end{cases}$$

It follows on letting $R\to\infty$ in (2.8) that the first term on the right hand side of that equation tends to $2if(r)$ whilst the second term tends to zero, since $a^2/r > a$ therein.

Therefore, on proceeding to the limit, equation (2.8) yields the formula

$$\lim_{R\to\infty}\int_{L(R)} \frac{u\phi(u,r)F(u)du}{J_u(ka)} = 2if(r) + \frac{1}{\pi}\lim_{R\to\infty}\int_0^a f(t)\frac{dt}{t}\int_{L(R)} h(u,r,t)du \tag{2.9}$$

after interchanging the order of integration in the final term on the right hand side of (2.8). To justify the reversal of the order of integration in this term it is sufficient to verify that the repeated integral in question is, for finite values of R, absolutely convergent. With this aim in view we write $u = \alpha + i\beta$ and utilize the following bound on the function $J_u(kt)$:

$$|J_u(kt)| \leq G(\alpha,\beta)(kt)^\alpha \tag{2.10}$$

The derivation of this bound is sketched in the appendix to this paper. The function $G(\alpha,\beta)$ appearing in the bound (2.10) is, for given $\alpha$, a continuous function of $\beta$. On using (2.10) in conjunction with (2.6) we see that

$$|h(u,r,t)| \leq [1 + (r/a)^{2\alpha} + \left|\frac{\pi u\phi(u,r)}{J_u(ka)}\right| G(\alpha,\beta)(kr)^\alpha](t/r)^\alpha = g(\alpha,\beta,r)t^\alpha$$

where $g(\alpha,\beta,r)$ varies continuously for u on the path L(R). It follows that, on the path L(R) where Re(u) = c,

$$\int_0^a |f(t)h(u,r,t)|t^{-1}dt \leq g(c,\beta,r)\int_0^a |f(t)|t^{c-1}\,dt$$

The t-integral here exists since, by hypothesis, $f = O(t^b)$ and $b + c > 0$. Since the path L(R) is of finite extent it follows that the last repeated integral on the right hand side of (2.8) does exist when the integrand is replaced by its modulus. This establishes the truth of (2.9) and it remains to prove that

$$\lim_{R\to\infty}\int_0^a f(t)t^{-1}dt\int_{L(R)} h(u,r,t)du = 0 \tag{2.11}$$

This result will be established with the aid of Cauchy's theorem which is first used to prove that

$$\int_{L(R)} [h(u,r,t) - h(u,t,r)]du = O(R^{-1}) \tag{2.12}$$

as $R \to \infty$ , uniformly for all values of t such that $r < t < a$ where $r > 0$.

To verify (2.12) it is first noted that the Bessel functions are entire functions of the complex variable u so that the function h(u,r,t) defined by (2.6) is an analytic function of u except for simple poles at the zeros of $J_u(ka)$, regarded as a function of u. Furthermore it follows from (2.6) in conjunction with (1.6), (2.4) and (2.5) in turn, that

$$h(u,r,t) - h(u,t,r) = u[J_u(kr)Y_u(kt) - J_u(kt)Y_u(kr)] + (r/t)^u - (t/r)^u \tag{2.13}$$

$$\begin{aligned} h(u,r,t) - h(u,t,r) = &-u\pi[J_u(kr)J_{-u}(kt) - J_u(kt)J_{-u}(kr)]\operatorname{cosec} u\pi \\ &+ (r/t)^u - (t/r)^u \end{aligned} \tag{2.14}$$

$$h(u,r,t) - h(u,t,r) = O[u^{-1}\{(r/t)^u + (t/r)^u\}] \tag{2.15}$$

It is seen from (2.13) that the function $h(u,r,t) - h(u,t,r)$ is an entire function of $u$ and from (2.14) that the same function is an odd function of $u$, since the same is true of the cross product of Bessel functions as well as of the remaining terms $(r/t)^u - (t/r)^u$.

If we now apply Cauchy's theorem to the integral of $h(u,r,t) - h(u,t,r)$ around the boundary of the rectangle whose corners are the points $\pm iR$, $c \pm iR$ we find, since the contributions of the top and bottom sides are $O[R^{-1}(r/t)^c] + O[R^{-1}(t/r)^c]$, that

$$\int_{L(R)} [h(u,r,t) - h(u,t,r)]du = \int_{-iR}^{iR} [h(u,r,t) - h(u,t,r)]du + O[R^{-1}(r/t)^c] + O[R^{-1}(t/r)^c]$$
$$= O[R^{-1}(r/t)^c] + O[R^{-1}(t/r)^c]$$

by virtue of the odd property of the function $h(u,r,t) - h(u,t,r)$. The result (2.12) now follows, for $t$ such that $r < t < a$, where $r > 0$.

It is now possible to return to the proof of (2.11). The $t$-integration present in this equation is decomposed into the parts $(0,r)$ and $(r,a)$ and the result (2.12) used to convert the contribution from the second such part. This leads to the equation

$$\int_0^a f(t)t^{-1}dt \int_{L(R)} h(u,r,t)du = \int_0^r f(t)t^{-1}dt \int_{L(R)} h(u,r,t)du + \int_r^a f(t)t^{-1}dt \int_{L(R)} h(u,t,r)du + O(R^{-1}) \qquad (2.16)$$

We now show that both of the integrals taken along the path $L(R)$ on the right hand side of (2.16) are $O(R^{-\frac{1}{2}})$ so that all three terms on the right hand side of this equation tend to zero as $R \to \infty$. We consider the first such integral, that in which $0 < t < r$. By Cauchy's theorem the path $L(R)$ may be deformed onto the part $C(R)$ of the circle $u = c + Re^{i\theta}$ lying to the right of, and joining the end points of, $L(R)$, since $h(u,r,t)$ is analytic in the region traversed. Since $h = O[u^{-1}(t/r)^u]$ it follows, if $0 < t < r$, that

$$\left|\int_{L(R)} h(u,r,t)du\right| = \left|\int_{C(R)} h(u,r,t)du\right| \leq (t/r)^c \int_{-\pi/2}^{\pi/2} (t/r)^{R\cos\theta} d\theta$$
$$\leq \frac{\pi(t/r)^c}{R\log(r/t)} \qquad (2.17)$$

Since this expression breaks down at the value $t = r$, we decompose the $t$-interval $(0,r)$ into the parts $(0, r - R^{-\frac{1}{2}})$ and $(r - R^{-\frac{1}{2}}, r)$ and apply the bound (2.17) to the first such part only. Since $\log(r/t)$ exceeds $-\log(1 - r^{-1}R^{-\frac{1}{2}}) > r^{-1}R^{-\frac{1}{2}}$ throughout this part it follows that

$$\left|\int_{L(R)} h(u,r,t)du\right| \leq \frac{r\pi(t/r)^c}{R^{\frac{1}{2}}} \quad \text{for} \quad 0<t<r-R^{\frac{1}{2}}. \qquad (2.18)$$

In the remaining interval $(r - R^{-\frac{1}{2}}, r)$, we still have $h = O(u^{-1})$ and deforming $L(R)$

ontc C(R) as before we have

$$\Big|\int_{L(R)} h(u,r,t)du\Big| = \Big|\int_{C(R)} h(u,r,t)du\Big| \le \int_{-\pi/2}^{\pi/2} d\theta = \pi \tag{2.19}$$

It follows on adding the contributions from the stated parts that

$$\Big|\int_0^r f(t)t^{-1}dt \int_{L(R)} h(u,r,t)du\Big| \le \pi r^{1-c}\, R^{-\frac{1}{2}} \int_0^{r-R^{-\frac{1}{2}}} |f(t)|t^{c-1}dt + \pi \int_{r-R^{-\frac{1}{2}}}^{r} |f(t)|t^{-1}dt \tag{2.20}$$

Both terms on the right hand side of the preceding inequality are $O(R^{-\frac{1}{2}})$ which verifies that the first of the repeated integrals on the right hand side of (2.16) vanishes as $R \to \infty$. The second repeated integral on the right hand side of (2.16), that for which $r \le t \le a$, is treated in a similar fashion by dividing the interval into the parts $(r, r + R^{-\frac{1}{2}})$ and $(r + R^{-\frac{1}{2}}, a)$. On deforming the path as before and using the fact that $h(u,t,r) = O[u^{-1}\,(r/t)^u]$ we find the following inequality, analogous to (2.17),

$$\Big|\int_{L(R)} h(u,t,r)\, du\Big| \le \frac{\pi(r/t)^c}{R\log(t/r)} \le 2\pi r(r/t)^c\, R^{-\frac{1}{2}}, \tag{2.21}$$

for $r+R^{-\frac{1}{2}} \le t \le a$, since $\log(t/r)$ exceeds $\log(1 + r^{-1}R^{-\frac{1}{2}}$ in the stated interval, whilst in the remaining interval $(r, r + R^{-\frac{1}{2}})$ we still have $h(u,t,r) = O(u^{-1})$ so that

$$\Big|\int_{L(R)} h(u,t,r)\, du\Big| \le \pi \tag{2.22}$$

It follows on adding the contributions from the two parts of the interval $r \le t \le a$ that

$$\Big|\int_r^a f(t)t^{-1}dt \int_{L(R)} h(u,t,r)du\Big| \le 2\pi r^{1+c}\, R^{-\frac{1}{2}} \int_{r+R^{-\frac{1}{2}}}^{a} |f(t)|t^{-1-c}dt + \pi \int_r^{r+R^{-\frac{1}{2}}} |f(t)|t^{-1}dt$$

Both terms on the right hand side of this equation are again $O(R^{-\frac{1}{2}})$ so that the second repeated integral on the right hand side of (2.16) tends to zero as $R \to \infty$. Thus (2.11) is established and in consequence the formula (2.9) reduces to the equation

$$f(r) = \frac{1}{2i}\int_L \frac{u\phi(u,r)F(u)du}{J_u(ka)} \tag{2.23}$$

where L denotes the line $Re(u) = c$ in the complex u-plane.

If $f(r)$ does vanish as $r \to 0$ the formula (1.5) of the theorem may be restated as the equation:

$$f(r) = \frac{1}{2i}\int_I \frac{u\phi(u,r)F(u)\ du}{J_u(ka)} + \sum \frac{uJ_u(kr)Y_u(ka)F(u)}{(\partial/\partial u)J_u(ka)} \tag{2.24}$$

where I denotes the imaginary axis of the complex u-plane and the summation is extended over the positive zeros $u_n$ of the function $J_u(ka)$, if any such zeros exist.

The formula (2.24) is obtained from (2.23) by deforming the path L in the latter equation onto the imaginary axis and taking into account the residues at the poles of the integrand, if any, in the region traversed. To justify this procedure it is sufficient to verify that the integrand appearing in (2.23) tends to zero as $|\mathrm{Im}(u)| \to \infty$ in the strip $0 \le \mathrm{Re}(u) \le c$ and this requires a knowledge of the behaviour of the function F(u) as $|\mathrm{Im}(u)| \to \infty$ in this strip. A suitable expression for F(u) valid when Im(u) is large can be obtained by using (2.2) which applies as $u \to \infty$ uniformly for any bounded interval of values of x. On replacing x by kr and substituting the resulting expression for $J_u(kr)$ into (1.1) we find that

$$F(u) \sim \frac{(k/2)^u}{\Gamma(u+1)} \int_0^a f(r)\, r^{u-1} dr \tag{2.25}$$

Since Re(u) is bounded in the stated strip it follows from (2.5) that $\phi(u,r) = O(u^{-1})$ and on using (2.2) to estimate $J_u(ka)$ and (2.25) to estimate F(u) we find that, for large values of Im(u), the modulus of the integrand present in (2.23) does not exceed that of the quantity

$$C\, a^{-u} \int_0^a f(r)\, r^{\alpha+i\beta-1} dr \tag{2.26}$$

where C is a constant. If $f(r) = O(r^b)$ as $r \to 0$ where $b > 0$ the integral appearing in (2.26) is absolutely convergent whenever $\alpha > 0$ and furthermore tends to zero as $\beta \to \pm\infty$ by the Riemann-Lebesgue lemma, since on taking log r as new variable of integration it becomes one of the Fourier type. Therefore the integrand present in (2.23) tends to zero as $\beta \to \pm\infty$ for all $\alpha > 0$, so that L may be deformed onto I provided allowance is made for the contributions from the poles in the region crossed. Since these poles occur at the zeros of the function $J_u(ka)$, where the function $\phi(u,r)$ defined by (1.6) reduces to the product $J_u(kr)Y_u(ka)$, we find, on calculating the residues at these points, the summation appearing in the formula (2.24) which is therefore proved.

## 3. APPENDIX.

To obtain the bound (2.10) we consider first the integral representation:

$$\Gamma(\tfrac{1}{2})\, \Gamma(u+\tfrac{1}{2}) J_u(x) = 2(x/2)^u \int_0^1 (1-t^2)^{u-\frac{1}{2}} \cos(xt)\, dt$$

This result applies, [7], p. 48, whenever $\mathrm{Re}(u) > -\frac{1}{2}$. On setting $u = \alpha + i\beta$ and taking the modulus of each side we find, for positive values of x, the inequality,

$$|\Gamma(\tfrac{1}{2})\, \Gamma(u+\tfrac{1}{2}) J_u(x)| \le 2(x/2)^\alpha \int_0^1 (1-t^2)^{\alpha-\frac{1}{2}} dt$$

The integral appearing on the right hand side of this inequality equals $\frac{1}{2}\Gamma(\frac{1}{2})\, \Gamma(\alpha+\frac{1}{2})/\Gamma(\alpha+1)$ so that

$$|\Gamma(u+\tfrac{1}{2}) J_u(x)| \le (x/2)^\alpha \Gamma(\alpha+\tfrac{1}{2})/\Gamma(\alpha+1) \tag{3.1}$$

for $\mathrm{Re}(u) > -\frac{1}{2}$. To obtain a suitable bound on $J_u(x)$ valid for other values of u we appeal to the formula, [7], p. 165

$$\Gamma(\tfrac{1}{2})\,\Gamma(\tfrac{1}{2}-u)J_u(x) = 2(x/2)^{-u}\left[\int_0^1 (1-t^2)^{-u-\frac{1}{2}}\cos(xt-u\pi)dt - \sin u\pi\int_0^\infty (1+t^2)^{-u-\frac{1}{2}}e^{-xt}dt\right]$$

Since $|\sin u\pi| \leq \cosh\beta\pi$ and $|\cos(xt-u\pi)| \leq \cosh\beta\pi$ we find that

$$|\,\Gamma(\tfrac{1}{2})\,\Gamma(\tfrac{1}{2}-u)J_u(x)| \leq 2(x/2)^{-\alpha}\cosh\beta\pi\left[\int_0^1 (1-t^2)^{-\alpha-\frac{1}{2}}dt\right.$$

$$\left. + \int_0^\infty (1+t^2)^{-\alpha-\frac{1}{2}}e^{-xt}dt\right] \tag{3.2}$$

Since $\alpha < -\frac{1}{2}$ the first integral on the right hand side of this inequality does not exceed unity. To treat the second integral we divide the domain into the parts $(0,1)$ and $(1,\infty)$ and use the facts that $1+t^2 \leq 2$ in the first part and $1+t^2 \leq 2t^2$ in the second part. We then find that this integral does not exceed the quantity.

$$2^{-\alpha-\frac{1}{2}}\int_0^1 e^{-xt}dt + 2^{-\alpha-\frac{1}{2}}\int_1^\infty t^{-2\alpha-1}e^{-xt}dt$$

$$\leq 2^{-\alpha-\frac{1}{2}} + 2^{-\alpha-\frac{1}{2}}\int_0^\infty t^{-2\alpha-1}e^{-xt}dt$$

$$= 2^{-\alpha-\frac{1}{2}} + 2^{-\alpha-\frac{1}{2}}x^{2\alpha}\Gamma(-2\alpha)$$

The relation (3.2) therefore leads to the bound:

$$|\Gamma(\tfrac{1}{2})\Gamma(\tfrac{1}{2}-u)J_u(x)| \leq [(1 + 2^{\alpha+\frac{1}{2}})a^{-2\alpha} + \Gamma(-2\alpha)]\; 2^{\frac{1}{2}}\, x^{\alpha}\cosh\beta\pi \tag{3.3}$$

after using the fact that $0 < x \leq a$. This bound applies for $\mathrm{Re}(u) < -\frac{1}{2}$. Then (3.1) and (3.3) together show that $|J_u(x)| \leq x^{\alpha}\, G(\alpha,\beta)$ where $G(\alpha,\beta)$ is, for given $\alpha$, a continuous function of $\beta$.

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