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ON INTEGERS n WITH $J_t(n) < J_t(m)$ For $m > n$

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ABSTRACT. Let F_t be the set of all positive integers n such that $J_t(n) < J_t(m)$ for all $m > n$, $J_t(n)$ being the Jordan totient function of order t. In this paper, it has been proved that (1) every postive integer d divides infinitely many members of F_t (2) if n and n' are consecutive members of F_t, $\frac{n}{n'} \to 1$ as $n \to \infty$ in F_t (3) every prime p divides n for all sufficiently large $n \in F_t$ and (4) Log $F_t(x) \ll \log^{\frac{1}{2}} x$ where $F_t(x)$ is the number of $n \in F_t$ that $n \leqq x$.

KEYS WORDS AND PHRASES. Euler totient function, Jordan totient function of order t.
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1. INTRODUCTION.

In [1] Masser and Shiu consider the set F of positive integers n such that $\phi(n) < \phi(m)$ for all $m > n$, (n) being the Euler totient function. Calling members of F sparsely totient numbers, they prove, among other results, that (1) every integer divides some member of F (2) every prime divides all sufficiently large members of F (3) the ratio of consecutive members of F approaches 1 and (4) log $F(x) \ll \log^{\frac{1}{2}}x$ where F(x) is the counting function of F; that is, the number of members of F which do not exceed x.

In this paper, using similar methods, we extend the above results to the set F_t of all positive integers n such that $J_t(n) < J_t(m)$ for all $m > n$, $J_t(n)$ being the well known Jordan totient function of order t [4]. We recall that $J_t(n)$ is defined as the number of incongruent t-vectors $(a_1,\ldots,a_t)$ mod n such that $((a_1,\ldots,a_t)n) = 1$, it being understood that t-vectors $(a_1,\ldots,a_t)$ and $(b_1,\ldots,b_t)$ are congruent mod n if $a_i \equiv b_i (\mathrm{mod}\ n)$ for $1 \leqq i \leqq t$ and that $J_t(n)$ is given by the formula

$$J_t(n) = n^t \prod_{p|n} (1 - p^{-t}) \tag{1.1}$$

Moreover this function $J_t(n)$ coincides with Cohen's [3] generalization $\phi_t(n)$ of the Euler totient function, defines as the number of positive integers $a \leq n^t$ such that

$(a,n^t)_t = 1$ where $(x,y)_t$ denotes the largest t th power common divisor of x and y. Clearly $J_1(n)$ (and hence $\phi_1(n)$) is the same as $\phi(n)$.

Denoting by $\omega(n)$ the number of distinct prime factors of n, we order these prime factors as $P_1 > P_2 > \dots > P_{\omega(n)}$; thus $P_r = P_r(n)$ is the r th largest prime factor of n. Likewise we order the primes not dividing n as $Q_1 < Q_2 < \dots$. Further we write p_r for the r th prime in the ascending sequence of all primes. For positive integral n, we write n_* to denote the quotient of n by its largest square free divisor. For a positive integer u, we write α_u for the unique positive root of the equation

$$t\, x^{t+u} + (t+u)\, x^t - u = 0. \tag{1.2}$$

It may be directly verified that

$$1 - \frac{2u}{t+u} \le \alpha_u < 1 \tag{1.3}$$

Finally we write $F_t(x)$ for the number of members of F_t that do not exceed x.

2. MAIN RESULTS.

We prove the following results:

THEOREM 2.1. Let $k \ge 2$, $d \ge 1$, $\ell \ge 0$ be integers such that

$$d < p_{k+1} - 1 \tag{2.1}$$

and

$$d^t(p^t_{k+\ell} - 1) < (d+1)^t(p^t_k - 1). \tag{2.2}$$

Then $d\, p_1 \dots p_{k-1} p_{k+}$ is a member of F_t.

COROLLARY 2.1. Every positive integer d divides infinitely many members of F_t.

COROLLARY 2.2. If n and n' are consecutive members of F_t, then $n^{-1}n' \to 1$ as $n \to \infty$ in F_t.

THEOREM 2.2. Every prime p divides n for all sufficiently large $n \in F_t$.

THEOREM 2.3. Let u be a fixed positive integer. As $n \to \infty$ in F_t we have

(a) $\liminf_n P_u(n) \log^{-1} n = 1$

(b) $\limsup_n P_1(n) \log^{-1} n \ge 2$

(c) $$\alpha_u \le \liminf_n Q_u(n) \log^{-1} n \le \limsup_n Q_u(n) \log^{-1} n = 1 \tag{2.3}$$

(d) $$\limsup_n P_{t+u}(n) \log^{-1} n \le \alpha_u^{-1} \tag{2.4}$$

and

(e) $$\limsup_n P_1(n) \log^{-t-1} n \le t \tag{2.5}$$

THEOREM 2.4. $\log F_t(x) \ll \log^{\frac{1}{2}} x$.

3. FOR THE PROOFS OF THE THEOREMS WE NEED THE FOLLOWING LEMMAS.

LEMMA 3.1. Let r be a positive integer and $x_1,\dots,x_r$, $y_1,\dots,y_r$, x and y be real numbers satisfying (i) $1 \le x_i \le y_i$ and $y \ge x_i$ for $i = 1,\dots,r$ and (ii) $x_1 \dots x_r\, x < y_1 \dots y_r\, y$. Then $(x_1-1) \dots\dots (x_r-1)(x-1) < (y_1-1)\dots(y_r-1)(y-1)$.

This is lemma 3.1 of [6].

For positive integers a and b we write $f(a,b)$ to denote the smallest multiple of b that exceeds a. Clearly

$$a < f(a,b) \leqq a + b. \tag{3.1}$$

LEMMA 3.2. If $n \in F_t$ and

$$p_1 \dots p_k \leqq n < p_1 \dots p_k \, p_{k+1} \tag{3.2}$$

then $k-2t < \omega(n) \leqq k$.

PROOF. The second inequality in (3.2) implies that $\omega(n) \leqq k$. Suppose if possible, that $0 < \omega(n) \leqq k - 2t$. It follows from (1.1) that

$$J_t(n)m^{-t} \geqq (1 - p_1^{-t}) \dots (1 - p_{k-2t}^{-t}) . \tag{3.3}$$

Choosing $m = f(n, p_1 \dots p_{k-t})$, we have, by (3.1) and (3.2),

$$1 < \frac{m}{n} \leqq 1 + \frac{p_1 \cdots p_{k-t}}{n} \leqq 1 + \frac{1}{p_{k-t+1} \cdots p_k} < 1 + \frac{1}{p_{k-t}} \tag{3.4}$$

On the other hand, since $p_1, \dots, p_{k-t}$ divide m, we have, in virtue of (3.3),

$$\begin{aligned} J_t(m)m^{-t} &\leqq (1-p_1^{-t}) \dots (1-p_{k-t}^{-t}) \\ &\leqq J_t(n)n^{-t} \, (1-p_{k-2t+1}^{-t}) \dots (1-p_{k-t}^{-t}) \\ &\leqq J_t(n)n^{-t} \, (1-p_{k-t}^{-t})^t \end{aligned}$$

so that, (3.4) now yields

$$\frac{J_t(m)}{J_t(n)} < (1 + p_{k-t}^{-t}) \, (1 - p_{k-t}^{-t})^t < 1$$

contrary to the hypothesis that $n \in F_t$. Hence the lemma follows.

REMARK 1. Since for each n there is a unique k such that (3.2) holds, lemma 3.2 implies that $\omega(n) \to \infty$ as $n \to \infty$ in F_t.

LEMMA 3.3. If u is a positive integer, $n \in F_t$ and $\omega(n) \geqq t + u$ then $\alpha_u \, P_{t+u} < Q_u$.

PROOF. We write $a = P_1 \dots P_{t+u}$, $b = Q_1 \dots Q_u$ and $m = na^{-1}f(a,b)$ so that

$$1 \quad \frac{m}{n} \quad 1 + \frac{b}{a} \quad 1 + Q_u^u \, P_{t+u}^{-t-u} . \tag{3.5}$$

Since $Q_1, \dots, Q_u$ are prime factors of m but not of n where as the prime factors $P_1, \dots, P_{t+u}$ of n may or may not divide m, we have

$$J_t(m)m^{-1} \quad (1-Q_1^{-t}) \dots (1-Q_u^{-t})(1-P_1^{-t})^{-1} \dots (1-P_{t+n}^{-t})^{-1} \quad J_t(n)n^{-t}$$

so that (3.5) and the hypothesis that n F_t together imply

$$\begin{aligned} 1 < J_t(m)(J_t(n))^{-1} &< (1+Q_u^u P_{t+u}^{-t-u}) \prod_{i=1}^{u} (1-Q_i^{-t}) \prod_{i=1}^{t+u} (1-P_1^{-t})^{-1} \\ &< (1+Q_u^u P_{t+u}^{-t-u})^t \, (1 - Q_u^{-t})^u \, (1 - P_{t+u}^{-t})^{-t-u} \end{aligned}$$

Taking (t + u) th roots and employing the well known inequality $x^r \leq 1 + r(x-1)$ for $x > 0$, $0 < r < 1$, we obtain

$$1-P_{t+u}^{-t} < (1-Q_u^{-t})^{u/t+u} \ (1+Q_u^u \ P_{t+u}^{-t-u})^{t/t+u}$$

$$< \{1-uQ_u^{-t}(t+u)^{-1}\} \ \{1+tQ_u^u \ P_{t+u}^{-t-u} \ (t+u)^{-1}\}$$

$$< 1-uQ_u^{-t}(t+u)^{-1} + tQ_u^u \ P_{t+u}^{-t-u} \ (t+u)^{-1}.$$

Cancelling 1 in the above and multiplying by $(t+u)Q_u^t$ we arrive at

$$t(Q_u \ P_{t+u}^{-1})^{t+u} + (t+u) \ (Q_u \ P_{t+u}^{-1})^t - u > 0$$

which, in virtue of (1.2), implies that $Q_u \ P_{t+u}^{-1} > \alpha_u$.

REMARK 2. When $t = u = 1$ this lemma 4 of [1] and when $t = 1$ this yields a slight improvement of lemma 7 of [1].

LEMMA 3.4. For $n \in F_t$ we have $P_1(n) < t(Q_1(n))^{t+1}$.

PROOF. We write P for $P_1(n)$ and Q for $Q_1(n)$. Suppose $P \geq t \ Q^{t+\pm}$ choosing $m = n \ f(P,Q)P^{-1}$ we see that

$$1 < \frac{m}{n} \leq 1 + \frac{Q}{P} \leq 1 + \frac{1}{tQ^t} \quad . \tag{3.6}$$

Arguing as in lemma 3.3, we have, since $P \geq Q^{t+1}$ by our assumption,

$$J_t(m)m^{-t} \leq (1-Q^{-t})(1-P^{-t})^{-1} \ J_t(n)n^{-t}$$

$$\leq (1-Q^{-t})(1-Q^{-t(t+1)})^{-1} \ J_t(n)n^{-t}$$

$$= (1+Q^{-t} + \dots + Q^{-t^2})^{-1} \ J_t(n)n^{-t}$$

so that by (3.6)

$$J_t(m)(J_t(n))^{-1} \leq (1+Q^{-t} + \dots + Q^{-t^2})^{-1} \ (1+t^{-1}Q^{-t})^t \leq 1,$$

contrary to the hypothesis. This establishes the lemma.

LEMMA 3.5. For $n \in F_t$, $n_* < t(Q_1(n))^{t+1}$.

PROOF. Writing $m = n \ f(n_*,Q)n_*^{-1}$, we note that

$$1 < \frac{m}{n} \leq 1 + \frac{Q}{n_*}$$

so that, since $n \in F_t$,

$$\exp(0) = 1 < J_t(m)(J_t(n))^{-1} \leq (1-Q^{-t})(1+Qn_*^{-1})^t$$

$$< \exp(-Q^{-t})\exp(tQn_*^{-1}).$$

The lemma follows on comparing the exponents.

LEMMA 3.6. Let $A \geq 0$, $M \geq 3$ and $\pi(A;M)$ be the number of primes p with $A < p \leq A+M$ ($\pi(A;M) = \pi(A+M) - \pi(A)$).

Then

$$\pi(A;M) \leq 2M(\log M)^{-1} \{1 + 0 (\log\log M \log^{-1} M)\}$$

and the 0-constant is independent of A.

This is Theorem 4.5, Chapter 19 of [2].

REMARK 3. Lemma 3.6 implies that $\pi(A;M) \leq 3M \log^{-1} M$ for all $A \geq 0$ and sufficiently large M.

4. PROOFS OF THEOREMS.

PROOF OF THEOREM 2.1. Let $n = d\, p_1 \cdots p_{k-1}\, p_{k+\ell}$ where d,k,ℓ satisfy (2.1) and (2.2). From (1.1) and (2.2) we have

$$J_t(n) \leq d^t\, (p_1{}^t - 1) \ldots (p_{k-1}^t)(p_{k+}^t \; -1)$$

$$< (d+t)^t (p_1{}^t - 1) \ldots (p_{k-1}^t - 1)(p_k{}^t - 1) \,. \qquad (4.1)$$

Let $m > n$. There is a unique s such that $p_1 \ldots p_s \leq m < p_1 \ldots p_{s+1}$ and the last inequality implies that $\omega(m) \leq s$. Hence

$$J_t(m) \geq m^t\, (1 - p_1{}^{-t}) \ldots (1 - p_s{}^{-t})$$

$$\geq (p_1{}^t - 1) \ldots\ldots (p_s{}^{t-1}) \qquad (4.2)$$

Case (1) $s \geq k+1$. We have, by (4.2),

$$J_t(m) \geq (p_1{}^t - 1) \ldots (p_k{}^t - 1)(p_{k+1}^t - 1)$$

$$> (p_1^t - 1) \ldots (p_k^t - 1)(d+1)^t \qquad \text{by (2.1)}$$

$$\geq J_t(n) \qquad \text{by (4.1)}$$

Case (2) $s \leq k$, $\omega(m) \leq k-1$. In this case

$$J_t(m)m^{-t} \geq (1 - p_1{}^{-t}) \ldots (1 - p_{k-1}^{-t})$$

where as

$$J_t(n)n^{-s} \leq (1 - p_1^{-t}) \ldots\ldots (1 - p_{k-1}^{-t})(1 - p_{k+}^{-t}\,)$$

$$< (1 - p_1^{-t}) \ldots\ldots (1 - p_{k-1}^{-t})$$

so that $J_t(m) > J_t(n)$.

Case (3) $s \leq k = \omega(m)$ and $m_* \geq d+1$. In this case $s = k = \omega(m)$ and

$$J_t(m) = m_*^t \prod_{p|m} (p^t - 1)$$

$$\geq m_*^t\; (p_1^t - 1) \ldots\ldots (p_k^t - 1)$$

$$\geq (d+1)^t\; (p_1^t - 1) \ldots\ldots (p_k^t - 1)$$

$$> J_t(n)$$

in virtue of (4.1).

Case (4) $s \leq k = \omega(m)$ and $m_* \leq d$. Let $m = m_* q_1 \dots q_k$, q_i's being the distinct prime factors of m in ascending order. Then $q_i \geq p_i$ and since $m > n$ we have

$$q_1 \dots q_{k-1} q_k > p_1 \dots\dots p_{k-1} (d\, p_{k+\ell}\, m_*^{-1})$$

Taking $r = k-1$, $y_i = q_i^t$, $y = q_k^t$, $x_i = p_i^t$ and $x = d^t p_{k+\ell}^t m_*^{-t}$ in lemma 3.1 we obtain

$$(q_1^t-1) \dots (q_{k-1}^t)(q_k^t-1) > (p_1^t-1) \dots\dots (p_{k-1}^t-1)\ (d^t p_{k+\ell}^t\, m_*^{-t}-1)$$

so that

$$\begin{aligned} J_t(m) &= m_*^t\ (q_1^t-1) \dots (q_k^t-1) > (p_1^t-1) \dots\dots (p_{k-1}^t-1)(d^t p_{k+\ell}^t - m_*^{\,t}) \\ &> d^t\ (p_1^t-1) \dots (p_{k-1}^t-1)(p_{k+\ell}^t-1) \\ &\geq J_t(n) \end{aligned}$$

by (4.1). This completes the proof of Theorem 2.1.

PROOF OF COROLLARY 2.1. Let d be any positive integer. For each $k \geq 2$ such that (2.1) holds, we can take $\ell = 0$ so that (2.2) holds. Thus $d\, p_1 \dots\dots p_k \in F_t$ for each $k \geq 2$ such that $p_{k+1} > d+1$.

As the proof of Corollary 2.2 is essentially the same as that of the Corollary given in section 3 of [1], we omit it.

PROOF OF THEOREM 2.2. Let p be a given prime. Choosing r such that $p_r > p\, \alpha_1^{-1}$, we see that $\omega(n) \geq r + t + 1$ for all n in F_t such that $n \geq p_1 \dots\dots p_{r+3t}$ in virtue of lemma 2. For such n we have

$$P_{t+1} \geq p_r > p\, \alpha_1^{-1}$$

so that

$$Q_1 > \alpha_1 P_{t+1} > p$$

in virtue of lemma 3.3 ($u = 1$), yielding that $p|n$.

REMARK 1. Though this theorem follows immediately from Theorem 2.3 a direct proof seems desirable.

PROOF OF THEOREM 2.3. (a) For any $n \in F_t$, there is a unique integer k satisfying

$$p_1 \dots\dots p_k \leq n < p_1 \dots\dots p_{k+1}\ . \tag{4.3}$$

By lemma 3.2, $\omega = \omega(n) \geq k - 2t + 1$ so that, for a fixed integer u, $P_u(n) \geq p_{\omega-u+1} \geq p_{k-2t-u+2} \sim \log n$ as $n \to \infty$ in F_t since, by the prime numbers theorem $(\theta(x) \sim x)$,

$$p_k \sim \log\,(p_1 \dots p_k) \sim \log n$$

and

$$p_{k-2t-u+2} \sim p_k \quad \text{as} \quad k \to \infty \quad (\text{hence as } n \to \infty \text{ in } F_t).$$

Thus $\liminf_n \frac{P_u(n)}{\log n} \geq 1$. On the other hand, considering members n of F_t of the form $p_1 \dots p_k$ (take d = 1, $\ell = 0$ in theorem 2.1), we have

$$P_u(n) \leq P_1(n) = p_k \sim \log n$$

as $n \to \infty$ through such members of F_t. Hence

$$\liminf_n \frac{P_u(n)}{\log n} = 1.$$

(b) For $k \geq 2$, $\ell \geq 0$ theorem 2.1 (with d = 1) says that

$$p_{k+\ell}^t < 2^t\ (p_k^t - 1) + 1 \Rightarrow p_1 \dots p_{k-1}\, p_{k+\ell} \in F_t.$$

choosing ℓ to be the largest subject to the above condition we have

$$p_{k+\ \ +1}^t \geq 2^t\ (p_k^t - 1) + 1 > p_{k+\ell}^t \tag{4.4}$$

so that

$$p_{k+\ell+1} < 2(p_k - 1) \sim 2 \log n$$

where $n - p_1 \dots p_{k-1}p_{k+\ell}$, since n satisfies (4.3) in virtue of (4.4).
Now $P_1(n) = p_{k+\ell} \sim p_{k+\ell+1}$ and this yields

$$\limsup_n \frac{P_1(n)}{\log n} \geq 2\ .$$

(c) Since, by lemma 3.3, $Q_u > \alpha_u\ P_{t+u}$, we have, from (a)

$$\liminf_n \frac{Q_u(n)}{\log n} \geq \alpha_u.$$

On the other hand, choosing k as in (4.3) we see that $Q_1 \leq p_{k+1}$ and hence $Q_u \leq p_{k+u}$ for all n in F_t, where as, for members of F_t of the form $p_1 \dots p_k$ we have $Q_u = p_{k+u}$. Since $p_{k+u} \sim p_k \sim \log n$ we have

$$\limsup_n \frac{Q_u(n)}{\log n} = 1.$$

(d) and (e) now follow by applying lemmas 3 and 4 respectively.

PROOF. OF THEOREM 2.4. We write $G(x) = F_t(x) - F_t(\frac{x}{2})$ = number of members n of F_t satisfying $\frac{x}{2} < n \leq x$ and show that $\log G(x) \ll \log^{\frac{1}{2}} x$ from which the theorem follows easily. Throughout the proof we assume that x is a sufficiently large positive real number. We write

$$u = [\ \log^{\frac{1}{2}} x\ (\log \log x)^{-1}] - t \tag{4.5}$$

and note that $t+u \geq 2^t + 2t$. For $n \in F_t$, $n > \frac{x}{2}$ we have $Q_1(n) \geq \frac{4}{5}\,\alpha_1 \log n$ by (2.3) Putting $Q_1 = p_{\ell+1}$ and noting that $\ell + 1 \geq \frac{3}{4}\ \frac{Q_1(n)}{\log Q_1(n)}$ by the prime number theorem we conclude that

$$\omega(n) \geq \ell \geq \frac{\alpha_1 \log n}{2 \log \log n} \geq \frac{\log^{\frac{1}{2}} x}{\log \log x} \geq t + u \quad 2^t + 2t.$$

Each n is specified uniquely when n_* and the prime factors of n are given. We estimate an upperbound for G(x) by estimating upper bounds for the number of choices, consistent with $n \varepsilon F_t \quad (\frac{x}{2}, x]$, for each of (a) n_* (b) the largest t+u prime factors of n, namely $P_1, \ldots, P_{t+u}$ (c) the prime factors of n that are less that $\alpha_u \, P_{t+u}$ and (d) the prime factors of n that lie in $[\alpha_u \, P_{t+u}, \, P_{t+u})$.

(a) By lemma 3.5 and (2.3) the number of choices for n_* does not exceed $t \, (2 \log x)^{t+1}$ and hence does not exceed $(2 \, t \log x)^{t+1}$.

(b) $P_i \leq P_1 \leq 2t \, (\log x)^{t+1}$ for $1 \leq i \leq t+u$ in virtue of (1.8) so that the number of choices for $P_1, \ldots, P_{t+u}$ does not exceed $(2t \log x)^{(t+1)(t+u)}$.

(c) Each choice of $Q_1, \ldots, Q_u$ gives rise to exactly one choice of the prime factors of n that are $< \alpha_u \, P_{t+u}$ and each choice of these prime factors gives rise to at least one choice for $Q_1, \ldots, Q_u$ since $\alpha_u \, P_{t+u} < Q_u$ and all primes in $(1, \alpha_u \, P_{t+u}) \quad \{Q_1, \ldots, Q_u\}$ divide n. If $P_1 = p_r$ then $r < p_r = P_1 \leq 2t \log^{t+1} x$, $Q_1 \leq p_{r+1}, \ldots$ and $Q_u \leq p_{r+u}$. Hence $Q_i \leq p_{r+u} < (r+u)^2 \quad (3t \log^{t+1} x)^2$ so that the number of choices for $Q_1, \ldots, Q_u$ and hence for the prime factors of n in $(1, \alpha_u P_{t+u})$ does not exceed $(3t \log^{t+1} x)^{2u}$.

(d) Let $M = P_{t+u} - \alpha_u \, P_{t+u}$ and note that

$$\begin{aligned} M = (1 - \alpha_u) \, P_{t+u} &\leq 2t(t+u)^{-1} \, P_{t+u} \\ &\leq 3t(t+u)^{-1} \, \alpha_u^{-1} \log x \\ &< 4t \, \alpha_u^{-1} \log^{\frac{1}{2}} x \log \log x \end{aligned}$$

in virtue of (1.3), (2.4) and (4.5) respectively. Hence, by remark 3 following lemma 3.6, the number of primes in $[\alpha_u \, P_{t+u}, P_{t+u})$ does not exceed $24t \, \alpha_u^{-1} \log^{\frac{1}{2}} x$ and consequently the number of choices for the prime factors of n that lie in this interval does not exceed $2^{24t\alpha_u^{-1} \log^{\frac{1}{2}} x}$.

Combining the estimates (a) through (d) we obtain $G(x) \leq (3t \log x)^{3(t+u)(t+1)}$ $\exp \, (24 \, t \, \alpha_u^{-1} \log^{\frac{1}{2}} x \log 2)$ from which the desired order estimate follows in virtue of (4.5).

It would be interesting to investigate whether results of this nature are available for more general totients; e.g. totients with respect to a polynomial [3] and with respect to a set of polynomials [4], introduced by the first author.

REFERENCES

1. MASSER, D.W. and SHIU, P. On sparsely totient numbers, Pacific J. of Math. 121 (1986), 407-426.

2. HUA, L.K. Introduction to Number Theory, Springer-Verlag, Berlin-Heidelberg-New York, 1982.

3. CHIDAMBARASAMY, J. Totients with respect to a polynomial, Indian J. of Pure & Applied Math. 5 (1974), 601-608.

4. ____________, Totients and unitary totients with respect to a set of polynomials, Ibid, 10 (1979), 287-302.