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REGULARITY OF "F" METHOD OF SUMMABILITY OF SEQUENCES

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ABSTRACT. This paper is to develop theorems concerning the REGULARITY of the method " F " which is more general than Cesaro's, Able's and Riemann's methods in the theory of summability.

KEY WORDS AND PHRASES. Regularity of a Method of Summability of Sequences.
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1. INTRODUCTION.

There are many well-known methods in the theory of summability which has many uses throughout analysis and applied mathematics, for example, Cesaro's, Able's, Riemann's, etc.. Mathematicians have contributed much to the study of these methods which all can be found in books that provide an introduction to Summability Theory. The "F" method is one of methods of Summability, and more general than those mentioned above. But there is less information about Regularity available covering the research in this method. This note concerns the regularity of the "F" method. Five theorems will be given.

2. MAIN RESULTS.

DEFINITION 2.1. Suppose that $\{F_n(x)\}_{n=1}^{\infty}$ is a sequence of functions defined in an interval $0 < x \leqq b$ and that for each n

$$\lim_{x\to 0} F_n(x) = 1,$$

and suppose that

$$F(x) = \sum a_n F_n(x)$$

is convergent in some interval $0 < x \leqq c < b$ and

$$\lim_{x\to 0} F(x) = S.$$

Then we say that $\sum a_n$ is summable (F) to S.

It is not hard to see that if $\sum a_n$ is Cesaro or Abel or Riemann summable, then $\sum a_n$ is summable (F) for suitable functions $F_n(x)$, respectively.

There is a well-known theorem about the regularity of the "F" method. [1,2]:

THEOREM. (REGULARITY) In order that the "F" method should be regular, it is necessary and sufficient that

$$\sum |F_n(x) - F_{n+1}(x)| < H, \tag{2.1}$$

where H is independent of x, in some interval $0 < x \leqq c < b$.

It is clear that method "F" is regular if $\{F_n(x)\}$ is monotone and uniformly bounded in some interval $0 < x \leqq c < b$. Next first theorem will prove that the "F" method should be regular for some sequence of functions $\{F_n(x)\}$ without monotonicity.

THEOREM 2.1. The condition (2.1) is satisfied if there are two positive sequences $\{m_n\}_{n=0}^{\infty}$ and $\{M_n\}_{n=0}^{\infty}$ such that

$$0 < m_n < m_{n+1}, \quad \text{for all } n$$

and

$$0 < M_n, \qquad \sum_{n=0}^{\infty} M_n < \infty$$

and for each n

$$|F_n(x) - 1| < M_n x^{m_n}, \quad 0 < x \leqq c < b .$$

PROOF. Since

$$\begin{aligned} &|F_n(x) - F_{n+1}(x)| \\ \leqq\ &|F_n(x) - 1| + |F_{n+1}(x) - 1| \\ \leqq\ &M_n x^{m_n} + M_{n+1} x^{m_{n+1}}, \end{aligned}$$

we have

$$\begin{aligned} &\sum |F_n(x) - F_{n+1}(x)| \\ \leqq\ &\sum (M_n x^{m_n} + M_{n+1} x^{m_{n+1}}) \\ =\ &M_0 x^{m_0} + 2 \sum_{n=1}^{\infty} M_n x^{m_n} . \end{aligned}$$

If $0 < x < r < 1$, then $x^{m_n} \geqq x^{m_{n+1}} \geqq \ldots > 0$, and for any N

$$\left|\sum_{n=0}^{N} M_n\right| \leqq A . \quad \text{(A is a constant)}$$

For each such x, by Abel's inequality

$$\left|\sum_{n=1}^{N} M_n x^{m_n}\right| \leqq A x^{m_1} \leqq A r^{m_1}$$

for any N. Let $N \to \infty$, we have

$$\left| \sum_{n=1}^{\infty} M_n x^{m_n} \right| \leq A r^{m_1} .$$

Thus

$$\sum |F_n(x) - F_{n+1}(x)| \leq M_0 r^{m_0} + 2 A r^{m_1} .$$

Let $H = M_0 r^{m_0} + 2Ar^{m_1} + 1$. H is independent of x and

$$\sum |F_n(x) - F_{n+1}(x)| < H$$

in some interval $0 < x \leq c < b$.

THEOREM 2.2. Suppose "F" is regular. Then $\sum a_n F_n(x)$ convergent implies that $\sum a_n F_n^2(x)$ convergent.

PROOF. It follows from the regularity of "F" and $\lim_{x\to 0} F_n(x) = 1$ for each n, that

$$\sum |F_n(x) - F_{n+1}(x)| < H ,$$

and

$$|F_0(x)| < H_1, \quad |F_n(x)| < H_1 + H . \quad n = 1,\ldots \tag{2.2}$$

where H, H_1 are independent of x, in some interval $0 < x \leq c < b$. $\sum a_n F_n(x) = F(x)$, for any $\varepsilon > 0$ and each x, we can choose $N_0(\varepsilon,x)$, such that

$$\left| \sum_{n=0}^{N-1} a_n F_n(x) - F(x) \right| < \varepsilon,$$

$$N > N_0(\varepsilon,x) + 1 .$$

Let

$$S_n = \sum_{i=0}^{n} a_i F_i(x) ,$$

also

$$\left| \sum_N^P a_n F_n^2(x) \right| = \left| \sum_N^P a_n F_n(x) F_n(x) \right|$$

$$= \left| \sum_N^P (S_n(x) - S_{n-1}(x)) F_n(x) \right|$$

$$= \left| \sum_N^P [(S_n(x) - F(x)) + (F(x) - S_{n-1}(x))] F_n(x) \right|$$

$$< |F(x) - S_{N-1}(x)| \, |F_N(x)| + \sum_N^{P-1} [| S_n(x) - F(x)| \, | F_n(x) - F_{n+1}(x) |]$$

$$+ |(S_P(x) - F(x)| \, |F_P(x)| .$$

For $P > N > N_0(\varepsilon,x) + 1$, it follows

$$\left|\sum_N^P a_n F_n^2(x)\right| < \varepsilon(H_1 + H) + \varepsilon H + \varepsilon(H_1 + H) = \varepsilon(2H_1 + 3H),$$

Therefore, $\sum a_n F_n^2(x)$ is a convergent in the interval $0 < x \leqq c < b$.

COROLLARY 2.1. Suppose that "F" is regular, then $\sum a_n F_n(x)$ convergent implies that $\sum a_n F_n^m(x)$ convergent, where m is a positive integer.

PROOF. It follows from Theorem 2.2 that m = 2 the assertion is true. Suppose that $\sum a_n F_n^k(x)$ is convergent and

$$\sum a_n F_n^k(x) = G(x) ,$$

and let

$$S_n(x) = \sum_{i=0}^{n} a_i F_i^k(x),$$

then

$$\left|\sum_N^P a_n F_n^{k+1}(x)\right| = \left|\sum_N^P a_n F_n^k(x)\, F_n(x)\right|$$

$$= \left|\sum_N^P (S_n(x) - S_{n-1}(x))\, F_n(x)\right|$$

$$= \left|\sum_N^P [(S_n(x) - G(x)) + (G(x) - S_{n-1}(x))]\, F_n(x)\right| .$$

Repeating the procedure of the proof of Theorem 2.2, the convergence of $\sum a_n F_n^{k+1}(x)$ can be proved. By the Axiom of Mathematical Induction, for all positive integer m the assertion of the corollary is true.

THEOREM 2.3. Suppose that "F" is regular, then "F^m" is regular, where m is a positive integer.

PROOF. Since

$$\left|F_n^m(x) - F_{n+1}^m(x)\right|$$

$$= \left|F_n(x) - F_{n+1}(x)\right| \left|F_n^{m-1}(x) + F_n^{m-2}(x)\, F_{n+1}(x) + \dots + F_{n+1}^{m-1}(x)\right| ,$$

it follows from (2.2) that

$$\left|F_n^m(x) - F_{n+1}^m(x)\right| < m(H_1 + H)^{m-1} \left|F_n(x) - F_{n+1}(x)\right| ,$$

and

$$\sum \left|F_n^m(x) - F_{n+1}^m(x)\right|$$

$$< m\,(H_1 + H)^{m-1} \sum \left|F_n(x) - F_{n+1}(x)\right| .$$

Hence for any positive integer m, if "F" is regular, then "F^m" is regular also.

THEOREM 2.4. Suppose that methods "F" and "G" are regular, then

1) " $F \pm G$ " are regular;

2) " FG " is regular;

3) " F^{-1} " is regular, if $\inf_{\substack{0<x\leq c \\ \text{all } n}} (F_n(x)) \geq e \neq 0$.

PROOF. It follows from (2.1) and (2.2) that

1) $\sum |(F_n \pm G_n) - (F_{n+1} \pm G_{n+1})|$

$\leq \sum |(F_n - F_{n+1}) \pm (G_n - G_{n+1})|$

$< \sum [\,| F_n - F_{n+1} + G_n - G_{n+1} |\,]$

$< H_F + H_G$,

2) $|F_nG_n - F_{n+1}G_{n+1}|$

$= |F_nG_n - F_{n+1}G_n + F_{n+1}G_n - F_{n+1}G_{n+1}|$

$\leq |F_nG_n - F_{n+1}G_n| + |F_{n+1}G_n - F_{n+1}G_{n+1}|$

$\leq |F_n - F_{n+1}|\ |G_n| + |G_n - G_{n+1}|\ |F_{n+1}|$

$< H_G^1\ |F_n - F_{n+1}| + H_F^1\ |G_n - G_{n+1}|$

and

$$\sum |F_nG_n - F_{n+1}G_{n+1}| < H_FH_G^1 + H_F^1H_G ,$$

3) $$|F_n^{-1} - F_{n+1}^{-1}| = \frac{|F_{n+1} - F_n|}{|F_{n+1}|\ |F_n|} \leq \frac{|F_{n+1} - F_n|}{e^2} ,$$

and

$$\sum |F_n^{-1} - F_{n+1}^{-1}| \leq \sum \frac{|F_{n+1} - F_n|}{e^2}$$

$$\leq \frac{1}{e^2} \sum |F_{n+1}(x) - F_n(x)|$$

$$< \frac{1}{e^2}\ H_F ,$$

where H_F, H_G, H_F^1, and H_G^1 are independent of x, in some interval $0 < x \leq c < b$. Therefore, the assertions 1), 2) and 3) are true. The proof is completed.

REFERENCES

1. GOLDBERG, R. Methods of Real Analysis, John Wiley and Sons, Inc., New York, 1976.

2. HARDY, G.H. Divergent Series, Oxford University Press, 1949.

3. POWELL, R.E. and SHAH, S.M. Summability Theory and its Applications, Van Nostrand Reinhold Co., London, 1972.

4. KNOPP, K. Theory and Application of Infinite Series, English translation by R.C. Young, Blackie, London, 1928.