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A GENERALIZED FRATTINI SUBGROUP OF A FINITE GROUP

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ABSTRACT. For a finite group G and an arbitrary prime p, let $S_p(G)$ denote the intersection of all maximal subgroups M of G such that [G:M] is both composite and not divisible by p; if no such M exists we set $S_p(G) = G$. Some properties of G are considered involving $S_p(G)$. In particular, we obtain a characterization of G when each M in the definition of $S_p(G)$ is nilpotent.

KEY WORDS AND PHRASES. Solvable group, Nilpotent group, Supersolvable group, Frattini subgroup.

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1. INTRODUCTION.

It is an interesting problem to investigate the relationships between the structure of a finite group G and the properties of the maximal subgroups of G. This has been studied by several people (e.g. [4], [5]). In [2] and [7-8] we have considered the family of maximal subgroups whose indices are composite and co-prime to a given prime. In this note we obtain further results in this direction. All groups considered are finite. A maximal subgroup M of a group G will be sometimes denoted by $M <\cdot G$. A maximal subgroup M of G of composite index will be called <u>c-maximal</u>.

2. THE SUBGROUP $S_p(G)$.

Let G be a group and p any prime. Consider the family of subgroups of G:

$$J = \{M : M \text{ is c-maximal}, [G : M]_p = 1\}$$

Define $S_p(G) = \cap \{M : M \in J\}$, if J is empty then set $S_p(G) = G$. This subgroup was introduced by us and several results have been obtained in [2] and [7]. We remark that $S_p(G)$ is a characteristic subgroup containing the Frattini subgroup $\Phi(G)$.

Our first result is motivated by Rose [9] where it was proved that if every non-normal, maximal subgroup of a group is nilpotent then the group is solvable. This result was extended by us ([2, Theorem 1.1]). We now obtain a further result in this direction.

THEOREM 2.1. Let p be the largest prime divisor of the order of a group G. Suppose that each subgroup in the family J (see above for the definition) is nilpotent. Then
(i) either, G is p-nilpotent or, there exists a normal p-subgroup P_0 of G such that G/P_0 is p-nilpotent.
(ii) if $l_p(G)$ denotes the p-length of G then $l_p(G) \leq 2$.
(Note: It follows directly from [2, Theorem 1.1] that G is solvable in this case).

PROOF: (i) we distinguish two cases:
Case 1: G has no normal p-subgroup. Let P be a Sylow p-subgroup of G. Then $N_G(P) \neq G$ and choose $M <\cdot G$ such that $N_G(P) \leq M$. If [G : M] is a prime q, say, then it is easy to see that $q > p$, an impossibility, thus [G : M] is composite and clearly $[G : M]_p = 1$. So $M \in J$ implying that M is nilpotent. Therefore $M = N_G(P)$. Let P_0 be a nontrivial characteristic subgroup of P. As G has no normal p-subgroup, $N_G(P_0) = N_G(P) = M$. Consequently M induces only p-automorphism on P_0 and so by Thompson [10] G is p-nilpotent.
Case 2: G has a normal p-subgroup. Let P_0 be a normal p-subgroup of G of the largest possible order. If P_0 is a Sylow p-subgroup of G then trivially G/P_0 is p-nilpotent. So, assume that P_0 is not a Sylow p-subgroup. We use induction on $|G|$. We note that p is the largest prime dividing $|G/P_0|$. It is easy to see that G/P_0 satisfies the hypothesis of the theorem and G/P_0 has no normal p-subgroup. So by induction hypothesis G/P_0 is p-nilpotent. Thus the proof of (i) is complete and (ii) follows now readily.

Our next result illustrates how under certain conditions the supersolvability of a group is controlled by the structure of certain groups of smaller orders.

THEOREM 2.2. For a group G and any prime p, if $|S_p(G)|$ is co-prime to p, then G is supersolvable $\Leftrightarrow$ $G/S_p(G)$ is supersolvable.

PROOF: The case $\Rightarrow$ is trivial and we consider now the $\Leftarrow$ case. If every maximal subgroup of G is of prime index then G is supersolvable by a well known result of Huppert and so $S_p(G)$ is supersolvable. Now let M be a c-maximal subgroup of G. If M does not contain $S_p(G)$, then $G = MS_p(G)$ and so $[G:M]_p = 1$ since by hypothesis $|S_p(G)|$ is a p'-number. Consequently $M \in J$ and so $S_p(G) \leq M$, a contradiction. Thus $S_p(G)$ is contained in every c-maximal subgroup of G and so $S_p(G)$ is contained in L(G), the intersection of all c-maximal subgroups of G. Now by [1] (see [2] for a published proof) L(G) is supersolvable and so the result now follows.

A group G is called a <u>Sylow tower group of supersolvable type</u> if (i) $p_1 > p_2 \ldots > p_k$ are all the prime divisors of $|G|$ and P_i is a Sylow p_i-subgroup of G and (ii) $P_1 P_2 \ldots P_k \lhd G$, $1 \leq i \leq k$.

THEOREM 2.3. Let q be the largest prime divisor of a group G and assume that $S_q(G) = G$. (In other words, the family J in the definition of $S_q(G)$ is empty). Then G is a Sylow tower group of supersolvable type.

PROOF. We use induction on $|G|$. If Q is a Sylow q-subgroup of $S_q(G)$ then by [7, Proposition 5] $Q \lhd G$. Consider the following two families of subgroups:

$$J = \langle M : M \text{ is c-maximal in } G, [G : M]_q = 1\rangle$$
$$J_1 = \langle M/Q: M/Q \text{ is c-maximal in } G/Q, [G/Q : M/Q]_q = 1\rangle$$

Since $S_q(G) = G$, J is empty. This implies that J_1 is also empty. For, if J_1 is nonempty and M/Q belongs to J_1 then clearly $M \varepsilon J$, contradicting the fact that J is empty. Hence $S_q(G/Q) = G/Q$. This implies that if M/Q is an arbitrary maximal subgroup of G/Q then clearly $[G/Q : M/Q]_q = 1$ and $[G/Q : M/Q]$ must be a prime. Thus every maximal subgroup of G/Q is of prime index. So, by a well known result of Huppert G/Q is supersolvable. Hence G is a Sylow tower group of supersolvable type.

If G is supersolvable then every maximal subgroup of G is of prime index by a well known result of Huppert, and so $S_p(G) = G$. Thus, if G is supersolvable then for $H \leq G$, we have that $S_p(G) \cap H = S_p(H)$. A simple example will show that the converse is not always true. (Take $G = A_4$ and $p = 2$. Here $S_2(H) = H$ for every subgroup H but A_4 is not supersolvable). However, we have the following partial converse:

PROPOSITION 2.4. Let p be the largest prime dividing the order of a group G. Suppose that $S_p(G) \cap H = S_p(H)$ for every subgroup H of G. Then G is a Sylow tower group of supersolvable type.

PROOF: Let Q be any Sylow q-subgroup of G where q is any prime dividing $|G|$. By hypothesis, $S_p(G) \cap Q = S_p(Q)$. Further since any maximal subgroup of Q is of prime index in Q, $S_p(Q) = Q$ irrespective of the fact that p may or may not be equal to q. Thus $S_p(G)$ contains every Sylow q-subgroup of G for every prime q dividing $|G|$. Therefore $S_p(G) = G$. The result now follows by applying Theorem 2.3.

We omit the proof of the following standard result:

LEMMA 2.5. Let G be a supersolvable group in which for every maximal subgroup M, $[G:M] = p$ where p is a fixed prime. Then G is a p-group.

We now prove:

PROPOSITION 2.6. Let p be the largest prime dividing the order of a group G. (i) Assume that $[G:M]_p = 1$ implies that $[G:M]$ is a prime for any $M <\cdot G$. Then G is a Sylow-tower group of supersolvable type. Further if P is a Sylow p-subgroup of G then $P \lhd G$ and G/P is supersolvable.

(ii) Let q be any prime such that q is not equal to p. Assume that $[G:M]_p = 1$ implies that $[G:M]=q$ for any $M <\cdot G$ and furthermore $[G:M_1]_q = 1$ implies that $[G:M_1] = p$ for any $M_1 <\cdot G$. Then G is supersolvable.

PROOF: We omit the proof of (i) which is a direct consequence of Theorem 2.3. Now consider (ii). If P is a Sylow p-subgroup of G then by (i), $P \lhd G$ and G/P is supersolvable. Now q divides $|G/P|$. Suppose if possible $M/P <\cdot G/P$ such that $[G/P:M/P]_q = 1$. Then $[G:M]_q = 1$ and so by hypothesis $[G:M] = p$ which is impossible since M contains P. Thus no maximal subgroup of G/P has index co-prime to q and since G/P is supersolvable, this gives, by using a well known result of Huppert, that every maximal subgroup of G/P is of prime index, and so has index q. By Lemma 2.5 it now follows that G/P is a q-group and so $|G|$ is of the form $p^{\alpha} q^{\beta}$. By using a well-known result of Burnside, G is solvable. We now show that G is supersolvable. Suppose if possible that there exists a maximal subgroup M such that [G:M] is divisible by both p and q. Then G = P M and it follows that $[G:M] = |P|/|P \cap M|$ is a power of p, a contradiction. Therefore, for any $M <\cdot G$, we have that the index of M in G is either co-prime to p, or co-prime to q. Consequently, by the hypothesis it follows that every maximal subgroup of G is of prime index and hence G is supersolvable by using a well-known result of Huppert.

REMARK: Under the hypothesis of Theorem 2.6 (ii) it might be tempting to conjecture that G is nilpotent. However, S_3 satisfies the hypothesis but is not nilpotent.

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