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A CONSTRUCTION OF A BASE FOR THE M FOLD TENSOR PRODUCT OF A BRANCH SPACE

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ABSTRACT. Let E be a Banach space with Schauder base $(x_n)_n$. Let $\overset{m}{\otimes}_\alpha E$ denote the completion of the m fold tensor product with respect to a reasonable cross norm α. We show that the set $\{x_{i_1} \otimes \dots \otimes x_{i_m} : x_{i_j} \in (x_n)_n\}$ can be enumerated so that for each positive integer k, the first k^m terms are precisely all the elements of the form $x_{i_1} \otimes \dots \otimes x_{i_m}$ with $i_1,\dots,i_m \in \{1,\dots,k\}$ and the set so arranged is a Schauder base for $\overset{m}{\otimes}_\alpha E$.

KEYWORDS AND PHRASES. m fold tensor product, projective, injective and reasonable cross norms, Schauder bases.
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1. INTRODUCTION.

Given a Banach space E, we will denote by $\overset{m}{\otimes} E$ the m fold algebraic tensor product and by $\overset{m}{\otimes}_\alpha E$ its completion with respect to reasonable norm α. If $(x_n)_n$ is a Schauder base for E then $\overset{m}{\otimes}(x_n)$ will denote the set

$$\{x_{i_1} \otimes \dots \otimes x_{i_m} : x_{i_j} \in (x_n),\ 1 \le j \le m\}.$$

For $m > 2$ the projective (injective) tensor norm on $\overset{m}{\otimes} E$ will be denoted by $\gamma_m(\lambda_m)$. If $m = 2$ this norm will be denoted by $\gamma(\lambda)$.

In [1], [4] it is shown that if E_1, E_2 are Banach spaces with Schauder bases $(x_n)_n$, $(y_n)_n$ then for any reasonable norm α, the space $E_1 \otimes_\alpha E_2$ has a Schauder base (z_n) with the following properties.

(1) $(z_n)_n$ is obtained by enumerating the set $(x_n)_n \otimes (y_n)_n$;

(2) the enumeration is such that for any positive integer k, the first k^2 terms are precisely all elements of the form

$$x_{i_1} \otimes y_{i_2}, \qquad i_1, i_2 \in \{1,\dots,k\}.$$

In this paper we show that for any positive integer m, the space $\overset{m}{\otimes}_\alpha E$ has a Schauder base obtained by enumerating $\overset{m}{\otimes}(x_n)_n$ in such a way that for any positive integer k, the first k^m terms are precisely all elements of the form

$$x_{i_1} \otimes \cdots \otimes x_{i_m} \text{ with } i_1,\ldots,i_m \in \{1,\ldots,k\}.$$

Our proof is via an induction argument and in a forthcoming paper we utilize this construction to derive some properties of the symmetric tensor algebra of a Banach space. Note that even for the case m = 3, iteration to $(E \otimes_\gamma E) \otimes_\gamma E$ of the enumerating scheme described in [1] will not yield a base with the above mentioned properties.

For definitions and terminology regarding Schauder bases, tensor products and tesnor norms we refer the reader to [4], [3], and [2].

We denote the following property (*) of the projective tensor norm. For $m > 2$ let u be an element of $\overset{m}{\otimes}E$. It can easily be shown that

$$(*) \qquad \gamma(u) = \gamma_m(u)$$

where $\gamma(u)$ is the projective norm of u when u is considered as an element of $F \otimes E$ with $F = \overset{m-1}{\otimes} E$ being endowed with the norm γ_{m-1}. We shall make use of this fact subsequently.

We now state our theorem.

THEOREM. Let (x_n,f_n), $x_n \in E$, $f_n \in E'$ be biorthogonal system such that $(x_n)_n$ is a Schauder base for E. Let α be a reasonable norm on $\overset{m}{\otimes}E$. Then there exists for $\overset{m}{\otimes}_\alpha E$ a biorthogonal system (z_n,g_n) with the following properties.

(1) As sets $\{z_n: n \in N\} = \overset{m}{\otimes}(x_n)_n$
$\{g_n: n \in N\} = \overset{m}{\otimes}(f_n)_n$.

(2) The enumeration of (z_n,g_n) is such that for each positive integer k the first k^m terms are precisely all the tensors of the form
$x_{i_1} \otimes \cdots \otimes x_{i_m}$,
$f_{i_1} \otimes \cdots \otimes f_{i_m}$ with $i_1,\ldots,i_m \in \{1,\ldots,k\}$.

(3) The sequence $(z_n)_n$ is a Schauder base for $\overset{m}{\otimes}_\alpha E$.

PROOF. It is known that [2] $\lambda_m \leq \alpha \leq \gamma_m$ for any reasonable norm α. Also $\overset{m}{\otimes}E' \subseteq (\overset{m}{\otimes}_{\lambda_m} E)'$. Thus any enumeration of $\overset{m}{\otimes}(x_n)_n$ together with the corresponding enumeration of $\otimes(f_n)_n$ yields a biorthogonal system for $\overset{m}{\otimes}_\alpha E$. It is easily shown that the linear span of $\overset{m}{\otimes}(x_n)_n$ is dense in $\overset{m}{\otimes}_{\gamma_m} E$ and we shall first establish the assertions of the theorem for $\alpha = \gamma_m$.

Now for m = 2 the base constructed in [1] has the stated properties.

Suppose that a biorthogonal system (z_n,g_n) with the stated properties can be constructed for $\overset{m}{\otimes}_{\gamma_m} E$.

Consider the sequence $(w_n)_n$ in $\overset{m+1}{\otimes} E$ defined by the scheme in the accompanying figure. For each positive integer k, the terms

$$w_{k^{m+1}+1},\ w_{k^{m+1}+2},\ldots,w_{(k+1)^{m+1}}$$

are described by the tensor beneath each term.

Figure 1

k, a positive integer; ℓ an integer $0 \leq \ell < k$

$w_{k^{m+1}+1}$,	$w_{k^{m+1}+2}$,	$w_{(k+1)^m+k^m(k-1)}$
$x_1 \otimes z_{k^m+1}$	$x_1 \otimes z_{k^m+2}$	$x_1 \otimes z_{(k+1)^m}$
$w_{(k+1)^m+k^m(k-1)+1}$,	$w_{(k+1)^m+k^m(k-1)+2}$,	$w_{2(k+1)^m+k^m(k-2)}$
$x_2 \otimes z_{k^m+1}$	$x_2 \otimes z_{k^m+2}$	$x_2 \otimes z_{(k+1)^m}$
⋮	⋮	⋮
$w_{\ell(k+1)^m+k^m(k-\ell)+1}$,	....	$w_{(\ell+1)(k+1)^m+k^m(k-(\ell+1))}$
$x_{\ell+1} \otimes z_{k^m+1}$	....	$x_{\ell+1} \otimes z_{(k+1)^m}$
$w_{(k-1)(k+1)^m+k^m+1}$,	....	$w_{k(k+1)^m}$
$x_k \otimes z_{k^m+1}$	....	$x_k \otimes z_{(k+1)^m}$
$w_{k(k+1)^m+1}$,	$w_{k(k+1)^m+2}$,	$w_{(k+1)^{m+1}}$
$x_{k+1} \otimes z_1$	$x_{k+1} \otimes z_2$	$x_{k+1} \otimes z_{(k+1)^m}$

In the preceding tableau each of the first k rows contains $(k+1)^m - k^m$ entries. The last row has $(k+1)^m$ entries, and so altogether we have exhibited $k((k+1)^m - k^m) + (k+1)^m = (k+1)^{m+1} - k^{m+1}$ entries.

The sequence (h_n) is obtained by enumerating the set $\{g_p \otimes f_q | g_p \in (g_n),\ f_p \in (f_n)\}$ according to the same scheme.

In view of the inductive hypothesis the system (w_h,h_n) is a biorthogonal system with the property that the first k^{m+1} terms are all of the form

$$x_{i_1} \otimes \cdots \otimes x_{i_{m+1}}, \; f_{i_1} \otimes \cdots \otimes f_{i_{m+1}}$$

with $i_1,\ldots,i_{m+1} \in \{1,\ldots,k\}$.

Let S_n, T_n, W_n respectively denote the n^{th} partial sum operators of the systems (x_n,f_n), (z_n,g_n) and (w_n,h_n).

Since (x_n), (z_n) are bases the sequences (f_n), (T_n) are bounded in operator norm by some constant M. To show that (w_n) is a base it thus suffices to show that for some positive M', $\| W_n \| \leq M'$ for all n([4], p. 25).

Now, given a positive integer n the defining scheme for (w_j) (see fig. 1) shows that W_n can be expressed as sums of tensor products of the operators S_m, T_m and f_m.

Indeed, let us consider the three possible cases.

Case 1. If $n = k^{m+1}$ for some positive integer k then

$$W_n = \overset{m+1}{\otimes} S_k, \text{ the } m+1 \text{ fold tensor product of } S_k.$$

Case 2. If $n = \ell\,(k+1)^m + k^m(k-\ell) + r$, with $0 \leq \ell < k$ and

$$1 \leq r \leq (k+1)^m - k^m \text{ then}$$

$$W_n = \overset{m+1}{\otimes} S_k + (T_{(k+1)^m} - T_{k^m}) \otimes S_\ell + (T_{k^m+r} - T_{k^m}) \otimes f_{\ell+1} \, .$$

Case 3. If $n = k(k+1)^m + r$ with $1 \leq r < (k+1)^m$ then

$$W_n = \overset{m+1}{\otimes} S_k + (T_{(k+1)^m} - T_{k^m}) \otimes S_k + T_r \otimes f_{k+1}$$

In u is in $\overset{m+1}{\otimes} E$ then property (*) together with the fact that γ_{m+1} is reasonable yields the following inequalities

(1) $\gamma_{m+1}(\overset{m+1}{\otimes} S_k(u)) \leq \|S_k\|^{m+1} \; \gamma_{m+1}(u)$

(2) $\gamma_{m+1}((T_{(k+1)^m} - T_{k^m}) \otimes S_\ell \,(u))$
$\leq \|T_{(k+1)^m} - T_{k^m}\| \, \|S_\ell\| \; \gamma_{m+1}(u)$

(3) $\gamma_{m+1}((T_{k^m+r} - T_{k^m}) \otimes f_{\ell+1}(u))$
$\leq \|T_{k^m+r} - T_{k^m}\| \; \|f_{\ell+1}\| \gamma_{m+1}(u).$

We utilized (*) in deriving (2) and (3). Hence,

$$\gamma_{m+1}(W_n(u)) \leq (M^{m+1} + 2M^2 + 2M^2)\gamma_{m+1}(u)$$

in all three cases. It now follows from the uniform boundedness principle that $(\|W_n\|)$ is bounded.

To complete the proof let us recall that if α is a reasonable norm then $\lambda_m \leq \alpha \leq \gamma_m$. Furthermore, $\overset{m}{\otimes}_\alpha E$ is the completion of $\overset{m}{\otimes} E$ with respect to α. Consequently for all such α, the system (z_n, g_n) is a complete biorthogonal system whose sequence of partial sums T_n is pointwise bounded and hence bounded in operator norm. This means that $(z_n)_n$ is a Schauder base for $\overset{m}{\otimes}_\alpha E$.

CONCLUDING REMARKS.

It would be interesting to find out whether the base for $\overset{m}{\otimes}_{\gamma_m} E$ obtained by iteration of the process in [1] is equivalent to the base described in this paper. We hope to investigate this problem in a future paper.

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