

FIXED POINT THEOREMS FOR COMPATIBLE MULTI-VALUED AND SINGLE-VALUED MAPPINGS

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ABSTRACT The notion of compatibility for point-to-point mappings recently defined by Jungck is generalized to include multi-valued mappings. This idea is used to establish a fixed point theorem for a generalized contractive multi-valued mapping and a single-valued mapping.

KEY WORDS AND PHRASES: Compatible Multi-Valued Mappings, Fixed Point.

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1. INTRODUCTION Jungck [2] proved the following theorem for f -contractive point-to-point mappings.

THEOREM 1. A continuous self-mapping f of a complete metric space (X, d) has a fixed point iff there exists a mapping $g : X \rightarrow X$ which commutes with f and such that $g(X) \subseteq f(X)$, $d(gx, gy) \leq hd(fx, fy)$ for all $x, y \in X$, where $0 \leq h \leq 1$. Furthermore, f and g have a unique common fixed point.

The present authors generalized this theorem in two different directions. Sessa [8], generalizing the notion of commutativity for point-to-point mappings, established the idea of weak commutativity for two self-mappings f and g of a metric space (X, d) , i.e. $d(fgx, gfx) \leq d(fx, gx)$ for all x in X . Under this concept, he extended theorem 1. On the other hand, Kaneko [5] proved that theorem 1 can be extended to the setting of multi-valued mappings, generalizing a famous result of Nadler [7]. Recently Jungck [3] made an extension of weak commutativity in the following way.

DEFINITION 1. Two self-mappings f and g of metric space (X,d) are compatible iff $d(fgx_n, gfx_n) \rightarrow 0$ whenever x_n is a sequence in X such that $fx_n \rightarrow t, gx_n \rightarrow t$ for some point t in X .

It can be seen that two weakly commuting mappings are compatible but the converse is false. Examples supporting this fact can be found in [3].

The purpose of this paper is to extend the definition of compatibility to include multi-valued mappings.

2. RESULTS Let (X,d) be a metric space and $CB(X)$ the family of all closed bounded subsets of X . Let H be the Hausdorff metric on $CB(X)$ induced by d , i.e.

$$H(A, B) = \max \{ \sup d(x, B) : x \in A, \sup d(x, A) : x \in B \}$$

for all A, B in $CB(X)$, where $d(x, A) = \inf \{ d(x, y) : y \in A \}$. It is well known [10] that $(CB(X), H)$ is a metric space. It is indeed a complete metric space in the event that (X, d) is also a complete metric space.

DEFINITION 2. The mappings $f : X \rightarrow X$ and $T : X \rightarrow CB(X)$ are compatible iff $fTx \in CB(X)$ for all $x \in X$ and $H(Tfx_n, fTx_n) \rightarrow 0$ whenever x_n is a sequence in X such that $Tx_n \rightarrow M \in CB(X)$ and $fx_n \rightarrow t \in M$.

Definition 2 extends definition 1 above. We are now in a position to prove our results.

Theorem 2. Let (X,d) be a complete metric space, $f : X \rightarrow X$ and $T : X \rightarrow CB(X)$ be compatible continuous mappings such that $T(X) \subseteq f(X)$ and

$$(1) \quad H(Tx, Ty) \leq h \max \left\{ d(fx, fy), d(fx, Tx), d(fy, Ty), \frac{1}{2} [d(fx, Ty) + d(fy, Tx)] \right\}$$

for all x, y in X , where $0 \leq h < 1$. Then there exists a point $t \in X$ such that $ft \in Tt$.

Proof We draw inspiration from [6]. Let $x_0 \in X$ be arbitrary and choose $x_1 \in X$ such that $fx_1 \in Tx_0$. This is possible since $Tx_0 \subseteq f(X)$. If $h = 0$, then $d(fx_1, Tx_1) \leq H(Tx_0, Tx_1) = 0$, i.e. $fx_1 \in Tx_1$ since Tx_1 is closed. Now assume that $h > 0$ and for $k = \frac{1}{\sqrt{h}} > 1$, by the definition of H , there exists a point $y_1 \in Tx_1$ such that $d(y_1, fx_1) \leq kH(Tx_1, Tx_0)$ (this is not generally true if $k \leq 1$ as it is seen in [7, remark at p. 480]). Since $Tx_1 \subseteq f(X)$, let $x_2 \in X$ be such that $y_1 = fx_2$. In general, if x_n has been selected, choose $x_{n+1} \in X$ so that $y_n = fx_{n+1} \in Tx_n$ and $d(y_n, fx_n) \leq kH(Tx_n, Tx_{n-1})$ for each $n \geq 1$. Using (1), we have that

$$\begin{aligned} d(fx_n, fx_{n+1}) &\leq kH(Tx_{n-1}, Tx_n) \\ &\leq \sqrt{h} \max \left\{ d(fx_{n-1}, fx_n), \frac{1}{2} [d(fx_{n-1}, fx_n) + d(fx_n, fx_{n+1})] \right\} \\ &\leq \sqrt{h} \max \{ d(fx_{n-1}, fx_n), d(fx_n, fx_{n+1}) \} \end{aligned}$$

i.e. $d(fx_n, fx_{n+1}) \leq \sqrt{h}d(fx_n, fx_{n-1})$ for each n . Since $\sqrt{h} < 1$, this shows that $\{fx_n\}$ is a Cauchy sequence, hence it converges to some point $t \in X$ using the completeness of X . Furthermore, the above inequalities also show that $H(Tx_{n-1}, Tx_n) \leq hd(fx_{n-1}, fx_n)$. Since $\{fx_n\}$ is Cauchy, this must imply that $\{Tx_n\}$ is a Cauchy sequence in the complete metric space $(CB(X), H)$. Now let $Tx_n \rightarrow M \in CB(X)$. Thus

$$\begin{aligned} d(t, M) &\leq d(t, fx_n) + d(fx_n, M) \\ &\leq d(t, fx_n) + H(Tx_{n-1}, M) \rightarrow 0 \text{ as } n \rightarrow \infty. \end{aligned}$$

Since M is closed, $t \in M$ and the compatibility of f and T implies that $H(Tfx_n, fTx_n) \rightarrow 0$ as $n \rightarrow \infty$. This along with the continuities of f and T imply that

$$\begin{aligned} d(ft, Tt) &\leq d(ft, ffx_{n+1}) + d(ffx_{n+1}, Tt) \\ &\leq d(ft, ffx_{n+1}) + H(fTx_n, Tt) \\ &\leq d(ft, ffx_{n+1}) + H(fTx_n, Tfx_n) + H(Tfx_n, Tt) \rightarrow 0 \end{aligned}$$

as $n \rightarrow \infty$, i.e. $ft \in Tt$ since Tt is closed.

Q.E.D.

Remark 1. A nonempty subset S of X is proximal if, for each $x \in X$, there exists a point $y \in S$ such that $d(x, y) = d(x, S)$. Let $PB(X)$ be the family of all bounded proximal subsets of X . If $T : X \rightarrow PB(X)$, then the iterative process $\{y_n\}$ in the above proof can be simplified in the following way: if x_n has been selected, let $x_{n+1} \in X$ be such that $y_n = fx_{n+1} \in Tx_n$ and $d(fx_n, y_n) = d(fx_n, Tx_n)$. Note that an iteration scheme of Smithson [9], where Tx is compact (hence proximal) for all $x \in X$, is included here. Since a proximal set is closed, we have $PB(X) \subseteq CB(X)$. The results of [4] and [5] follow as corollaries.

COROLLARY 1. Let (X, d) be a complete metric space, $f : X \rightarrow X$ and $T : X \rightarrow PB(X)$ be continuous mappings such that $fTx \in PB(X)$ and $H(Tfx, fTx) \leq d(fx, Tx)$ for all $x \in X$. If (1) is satisfied for all x, y in X , $0 \leq h < 1$, and $T(X) \subseteq f(X)$, then there exists $t \in X$ such that $ft \in Tt$.

COROLLARY 2. Let (X, d) be a complete metric space, $T : X \rightarrow CB(X)$ and f be a continuous self-mapping of X such that $H(Tx, Ty) \leq hd(fx, fy)$ and for all x, y in X , where $0 \leq h < 1$ and $Tfx = fTx$. If $T(X) \subseteq f(X)$, then there exists $t \in X$ such that $ft \in Tt$.

Note that the continuity of f implies the continuity of T in corollary 2. The following example shows theorem 2 is indeed a proper generalization over corollaries 1 and 2.

EXAMPLE Let $X = [1, \infty)$ be endowed with the Euclidean metric d . Let $fx = 2x^4 - 1$ and $Tx = [1, x^2]$ for each $x \geq 1$. T and f are clearly continuous and $T(X) = f(X) = X$. Since $fx_n \rightarrow 1$ and $Tx_n \rightarrow \{1\}$ iff $x_n \rightarrow 1$, $H(fTx_n, Tfx_n) = 2(x_n^4 - 1) \rightarrow 0$ iff $x_n \rightarrow 1$, $fTx =$

$[1, (2x^4 - 1)^2]$ for all $x \geq 1$, f and T are compatible. Since $H(Tx, Ty) = |x^2 - y^2| \leq 2|x^2 - y^2|$
 $|x^2 + y^2|/2 = d(fx, fy)/2$, all the conditions of theorem 2 hold with $h = \frac{1}{2}$, but the above
 corollaries are not applicable since f and T do not weakly commute (for $x = 2$), hence do not
 commute either.

In the sequel, we use the following lemma which is a slight generalization of proposition
 2.2 (part 1) of [3].

LEMMA. Let $T : X \rightarrow CB(X)$ and $f : X \rightarrow X$ be compatible. If $fw \in Tw$ for some $w \in X$,
 then $fTw = Tfw$.

PROOF. Let $x_n = w$ for each n . Then $fx_n = fw \rightarrow fw$ and $Tx_n \rightarrow M \equiv Tw$. Hence
 if $fw \in Tw$, then $H(fTw, Tfw) = H(fTx_n, Tfx_n) \rightarrow 0$ by the compatibility. Hence we must
 have $fTw = Tfw$. Q.E.D.

In order to obtain a fixed point result, we need additional assumptions as those given in
 [4] and [5].

THEOREM 3. Let f and T have the same meanings as in theorem 2. Assume also that
 for each $x \in X$ either (i) $fx \neq f^2x$ implies $fx \notin Tx$ or (ii) $fx \in Tx$ implies that $f^n x \rightarrow z$ for some
 $z \in X$. Then f and T have a common fixed point in X .

PROOF. By Theorem 2, $ft \in Tt$ for some $t \in X$ and $fTt = Tft$ by Lemma. Assuming (i),
 we deduce that $ft = f^2t \in fTt = Tft$. Assuming (ii), it is clear that $fz = z$ by the continuity of
 f . We claim that $f^n t \in Tf^{n-1}t$ for each n . To see this, we have that $f^2t = ffteft = Tft$. By
 lemma ($w = ft$), $f^3t = ff^2t \in fTft = Tf^2t$. Repeating this argument, we obtain $f^n t \in Tf^{n-1}t$
 and the continuity of T implies that

$$d(z, Tz) \leq d(z, f^n t) + d(f^n t, Tz) \leq d(z, f^n t) + H(Tf^{n-1}, Tz) \rightarrow 0,$$

i.e. $z \in Tz$ since Tz is closed. Hence z is a common fixed point of f and T . Q.E.D.

REMARK 2. Simple examples prove that the conditions " $T(X) \subseteq f(X)$ " and the
 compatibility of f and T are necessary in theorem 2. Unfortunately, it is not yet known if
 the continuity of both mappings f and T is necessary in theorem 2. However in the case that
 f and T are single-valued mappings, it suffices only the continuity of at least one of them.
 Moreover, the inequality (1) can be weakened as it is proved in the following results, which
 extends theorem 2.1 of [1].

THEOREM 4. Let (X, d) be a complete metric space and $f, T : X \rightarrow X$ be two
 compatible mappings (def.1) such that $T(X) \subseteq f(X)$ and

$$(2) \quad d(Tx, Ty) \leq H \max\{d(fx, fy), d(fx, Tx), d(fy, Ty), d(fx, Ty), d(fy, Tx)\}$$

for all x, y in X , where $0 \leq h < 1$. If one of f or T is continuous, then there exists a unique
 common fixed point of f and T .

PROOF. As in [1], it is seen that the sequence $\{Tx_n\}$ where $Tx_n = fx_{n+1}$ for each n , is a Cauchy sequence. Hence it converges to some point $z \in X$. If T is continuous, then $T^2x_n \rightarrow Tz$ and $Tfx_n \rightarrow Tz$. Since $d(fTx_n, Tfx_n) \rightarrow 0$ by compatibility, $fTx_n \rightarrow Tz$, hence using (2),

$$d(T^2x_n, Tx_n) \leq h \max\{d(fTx_n, fx_n), d(fTx_n, T^2x_n), d(fx_n, Tx_n), \\ d(fTx_n, Tx_n), d(fx_n, T^2x_n)\}$$

which implies, as $n \rightarrow \infty$, that $d(z, Tz) \leq hd(z, Tz)$, i.e. $Tz=z$. Since $T(X) \subseteq f(X)$, there exists a point z' such that $z = Tz = fz'$ and using (2) again,

$$d(T^2x_n, Tz') \leq h \max\{d(fTx_n, z), d(fTx_n, T^2x_n), d(z, Tz'), d(fTx_n, Tz'), d(z, T^2x_n)\}.$$

As $n \rightarrow \infty$, we deduce that $d(z, Tz') \leq hd(z, Tz')$, i.e. $z = Tz' = fz'$ and by Lemma, $fz = fTz' = Tfz' = Tz = z$.

A similar proof can be made if we assume the continuity of f instead of T . The uniqueness follows easily from (2). Q.E.D.

REMARK 3. Note that our proof is different from that of [1], in which it is proved that $fz = Tz$ is the unique fixed point (only if f is continuous). In this paper, it is just the case that $fz = Tz = z$.

REMARK 4. If neither T nor f is continuous, e.g. $X = [0, 1]$, $T0 = \frac{1}{4}$, $f0 = \frac{1}{2}$, $Tx = \frac{x}{8}$, $fx = \frac{x}{4}$ for $x \neq 0$, theorem 3 fails. Similar remarks can be made on the results of [8].

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