Internat. J. Math. & Math. Sci.
VOL. 12 NO. 2 (1989) 209-233

PRODUCTS OF STOCHASTIC MATRICES AND APPLICATIONS

HARRY COHN

Department of Statistics
The University of Melbourne
Parkville, Victoria 3052
Australia

(Received November 20, 1987)

ABSTRACT. This paper deals with aspects of the limit behaviour of products of nonidentical finite or countable stochastic matrices (P_n). Applications are given to nonhomogeneous Markov models as positive chains, some classes of finite chains considered by Doeblin and weakly ergodic chains.

KEY WORDS AND PHRASES. Stochastic matrix, nonhomogeneity, Markov chains, weak ergodicity, tail σ-field, atomic set.
1980 AMS SUBJECTS CLASSIFICATION CODES. 15A51, 60J10.

1. INTRODUCTION.

Let $P_0, P_1, \ldots$, be a sequence of finite or countable stochastic matrices, $P_{i,j}^{(n)}$ the (i,j) entry of P_n, $P_{m,n} = P_m \ldots P_n^{(m,n)}$ the (i,j) entry of $P_{m,n}$. In the 'homogeneous' case, when $P = P_0 = \ldots$, the classical Markov chains theory provides a detailed analysis of P^n: for ergodic chains P^n converges as $n \to \infty$, whereas otherwise P^{nd+r} converges as $n \to \infty$ for some $d > 1$ and $r = 1,\ldots,d-1$. It turns out that in the 'nonhomogeneous' case, when (P_n) are nonidentical, $\liminf_{n\to\infty} P_{i,j}^{(m,n)} > 0$ and $P_{\ell,j}^{(m,n)} > 0$ imply that $\{P_{i,j}^{(m,n)}/P_{\ell,j}^{(m,n)}\}$ converges as $n \to \infty$ in a case that may be thought of as aperiodic whereas otherwise $\{P_{i,j}^{(m,n)}/P_{\ell,j}^{(m,n)} : n \geq m\}$ assume a finite number of limit points. Both $\lim_{n\to\infty} P_{i,j}^{(m,n)}/P_{\ell,j}^{(m,n)}$ and the limit points of $\{P_{i,j}^{(m,n)}/P_{\ell,j}^{(m,n)} : n = m+1,\ldots\}$ will be identified in terms of $\alpha_i^{(m)}(k)/\alpha_\ell^{(m)}(k)$ where $\alpha_u^{(m)}(k) = \lim_{n\to\infty} \sum_{i\in E_n^{(k)}} P_{u,j}^{(m,n)}$ for some sequence of sets $\{E_n^{(k)}\}$. Our results may be understood without reference to Markov chains, but the proofs will consider a Markov chain $\{X_n : n \geq m\}$ with finite or countable state spaces assuming a strictly positive initial probability vector $\pi^{(m)}$ and the one-step transition probability matrices (P_n) $n \geq m$ as the starting point. It will turn out that the structure of the tail σ-field of $\{X_n : n \geq m\}$ is crucial for the asymptotic behaviour of $\{P_{m,n}\}$ and that $\alpha_u^{(m)}(k) = P^{(m)}(T_k | X_m = u)$ where T_k is an atomic set of the tail σ-field of $\{X_n : n \geq m\}$. We first consider the countable case where a number of results are obtained under the assumption that $\liminf_{n\to\infty} P_{i,j}^{(m,n)} > 0$. A particular case is that of convergent $\{P_{i,j}^{(m,n)}\}$ where $\lim_{n\to\infty} P_{i,j}^{(m,n)}$ will be identified. Then we look at the case of finite S where more powerful results are obtained without any assumption on (P_n). Further, we specialize our results to some classes of finite and countable nonhomogeneous chains and explore some connections with the notion of weak ergodicity.

We do not include in this paper specific applications of products of stochastic matrices which seem to be numerous, ranging from demography as shown by Seneta [21], to recent developments in the theory of Markovian random fields assuming phase transitions (see Kemeny et al [12] and Winkler [22]).

Our paper is a streamlined survey of the literature of nonhomogeneous Markov models from the viewpoint of tail σ-fields.

2. TAIL σ-FIELDS.

Let $(\Omega, \mathcal{F}, P)$ be a probability space and Λ a set in $\mathcal{F}$. We shall say that Λ is a *P-atomic* set of $\mathcal{F}$ if $P(\Lambda) > 0$ and Λ does not contain any subset Λ' with $\Lambda' \in \mathcal{F}$ and $0 < P(\Lambda') < P(\Lambda)$. A nonatomic set Λ in $\mathcal{F}$ is said to be a *P-completely nonatomic* set of $\mathcal{F}$ if $P(\Lambda) > 0$ and Λ does not contain any P-atomic subsets of $\mathcal{F}$. It is easy to see that, in general, Ω may be represented as $\Omega = \cup_{n=0}^{\infty}\Lambda_n$ where Λ_0 is P-completely nonatomic and $\Lambda_1, \Lambda_2, \ldots$ are P-atomic sets of $\mathcal{F}$. This representation is unique modulo null probability sets of $\mathcal{F}$. Of course, some of $\{\Lambda_i\}$ may be absent. If Λ_0 is present, we shall say that $\mathcal{F}$ is *nonatomic* whereas if Λ_0 is absent $\mathcal{F}$ is called *atomic*. If Λ_0 is absent and there is only a finite number of atomic sets $\{\Lambda_i\}$ we shall say that $\mathcal{F}$ is *finite*. Finally, $\mathcal{F}$ is said to be *trivial* if $\Lambda_1 = \Omega$.

Take now $\Omega = S \times S \times \ldots$ where S is finite or countable, $X_n(\omega) = \omega_n$ for $\omega = (\omega_1, \ldots, \omega_n, \ldots)$ and write $\mathcal{F}_n$ for the σ-field generated by $\{X_k : k \geq n\}$. A strictly positive distribution $\pi^{(m)} = (\pi_i^{(m)}; i \in S)$ and a sequence of $S \times S$ stochastic matrices $(P_n)_{n \geq m}$ uniquely determine a probability measure $P^{(m)}$ on $\mathcal{F}_m$ such that $\{X_n : n \geq m\}$ is a nonhomogeneous Markov chain on $(\Omega, \mathcal{F}_m, P^{(m)})$ with $P^{(m)}(X_m = i) = \pi_i^{(m)}$ and $P^{(m)}(X_{n+1} = j | X_n = i) = p_{i,j}^{(n)}$ for $i, j \in S$ and $n \geq m$. This model will allow us to use probabilistic arguments on all $P_{i,j}^{(m,n)}$, since the equality $P_{i,j}^{(m,n)} = P^{(m)}(X_n = j | X_m = i)$ makes sense in view of $P^{(m)}(X_m = i) > 0$. The usual model $\{X_n : n \geq 0\}$ may not always lead to $P^{(o)}(X_m = i) > 0$ even if we take $\pi^{(o)}$ to be strictly positive.

We shall write $\{A_n \text{ i.o.}\}$ for $\cap_{n=1}^{\infty}\cup_{m=n}^{\infty} A_m$ and $\{A_n \text{ ult.}\}$ for $\cup_{n=1}^{\infty}\cap_{m=n}^{\infty} A_m$. Here 'i.o.' stands for 'infinitely often' and 'ult.' stands for 'ultimately'. Further, $\lim_{n\to\infty} A_n = A$ a.s. will mean that $\lim_{n\to\infty} 1_{A_n} = 1_A$ a.s. where 1 stands for the indicator function of a set. We shall say that $A = B$ a.s. if $1_A = 1_B$ a.s. The σ-field generated by $X_m, \ldots, X_n$ will be denoted by $\mathcal{F}_m^{(n)}$.

A key tool for out study will be provided by the tail σ-field of $\{X_n : n \geq m\}$ defined as $\mathcal{T}^{(m)} = \cap_{n=m}^{\infty}\mathcal{F}_n$.

PROPOSITION 2.1. Suppose that Λ is a set in $\mathcal{T}^{(m)}$. Then there exists a sequence $\{L_n\}$ of subsets of S such that $\lim_{n\to\infty}\{X_n \in L_n\} = \Lambda$ a.s. with respect to $P^{(m)}$.

PROOF. Since Λ belongs to $\mathcal{F}_m$, the martingale convergence theorem implies that $\lim_{n\to\infty} P^{(m)}(\Lambda | \mathcal{F}_m^{(n)}) = 1_\Lambda$ a.s. with respect to $P^{(m)}$. By the Markov property of $\{X_n : n \geq m\}$ we get $P^{(m)}(\Lambda | \mathcal{F}_m^{(n)}) = P^{(m)}(\Lambda | X_n)$. Thus taking $L_n = \{i : P^{(m)}(\Lambda | X_n = i) > \lambda\}$ with $0 < \lambda < 1$ yields $\lim_{n\to\infty}\{X_n \in L_n\} = \Lambda$ $P^{(m)}$ a.s., completing the proof.

The argument used above goes back, essentially, to Blackwell [1].

3. POSITIVE STATES.

Consider the Markov chain $\{X_n : n \geq m\}$ with a strictly positive initial probability vector $\pi^{(m)}$ and write $\pi_j^{(n)} = P(X_n = j)$ for $n \geq m$. We shall say that j is *positive* if $\liminf_{n\to\infty} \pi_j^{(n)} > 0$.

PROPOSITION 3.1. A state j is positive if and only if for any subsequence (n_k) with $\lim_{n\to\infty} n_k = \infty$ there exists a state i (possibly depending on (n_k)) such that $\limsup_{k\to\infty} P_{i,j}^{(m,n_k)} > 0$.

PROOF. Since $\pi_j^{(n)} = \sum_{i\in S} \pi_i^{(m)} P_{i,j}^{(m,n)}$ and $\pi_i^{(m)} > 0$ for $i \in S$, it suffices to notice that $\lim_{k\to\infty} \pi_j^{(n_k)} = 0$ if and only if $\lim_{k\to\infty} P_{i,j}^{(m,n_k)} = 0$ for all $i \in S$.

Proposition 3.1 shows that the definition of positivity for j depends only on $\{P_{i,j}^{(m,n)} ; n \geq m,\ i \in S\}$.

THEOREM 3.1. If j is positive then

$$\{X_n = j \text{ i.o.}\} = \bigcup_{k=1}^{d} T_k \quad P^{(m)} \text{ a.s.}$$

where T_k, $k=1,\dots,d$ are $P^{(m)}$-atomic sets of $\mathcal{J}^{(m)}$ with $d < \infty$.

PROOF. Notice first that $\{X_n = j \text{ i.o.}\} \in \mathcal{J}^{(m)}$. Assume by way of contradiction that $\{X_n = j \text{ i.o.}\}$ does not equal a finite union of atomic sets and therefore we may find some infinite sequence of disjoint sets $T_1, T_2, \dots \in \mathcal{J}^{(m)}$ with $P^{(m)}(T_i) > 0$ for all i such that

$$\{X_n = j \text{ i.o.}\} = \bigcup_{k=1}^{\infty} T_k \quad P^{(m)} \text{a.s.} \tag{3.1}$$

Take $\delta = \liminf_{n\to\infty} \pi_j^{(n)}$. If (3.1) were true, there would exist a set T_k with $P^{(m)}(T_k) < \delta$. Let $\{E_n^{(k)} : n \geq m\}$ be some subsets of S such that $\lim_{n\to\infty} \{X_n \in E_n^{(k)}\} = T_k$ $P^{(m)}$ a.s. as ensured by Proposition 3.1. In view of (3.1) j must belong to an infinity of sets $\{E_n^{(k)}\}$. This entails

$$\liminf_{n\to\infty} \pi_j^{(n)} \geq \lim_{n\to\infty} P^{(m)}(E_n^{(k)}) = P^{(m)}(T_k) < \delta$$

which is absurd. Thus $\{X_n = j \text{ i.o.}\}$ consists of a finite number of $P^{(m)}$-atomic sets of $\mathcal{J}^{(m)}$.

THEOREM 3.2. Suppose that j is positive. Then there exist d disjoint sequences of integers $\{\Gamma_k\}$ where $\Gamma_k = \{n_t^{(k)} : t=1,2,\dots\}$ and d sequences of sets $\{E_n^{(k)}\}$ such that $\bigcup_{k=1}^{d} \Gamma_k = \{m+1,\dots\}$, and for $i, \ell \in S$

$$\lim_{\substack{n'\to\infty \\ n'\in\Gamma_k}} P_{i,j}^{(m,n')} / P_{\ell,j}^{(m,n')} = \alpha_i^{(m)}(k) / \alpha_\ell^{(m)}(k) \tag{3.2}$$

provided that $P_{\ell,j}^{(m,n')} > 0$ where $\alpha_u^{(m)}(k) = \lim_{n\to\infty} \sum_{j\in E_n^{(k)}} P_{u,j}^{(m,n)}$ for $u \in S$.

PROOF. Let $\{E_n^{(k)}\}$ be some sequences of sets with $\lim_{n\to\infty}\{X_n \in E_n^{(k)}\} = T_k$ $P^{(m)}$ a.s. for k=1,2,...,d whose existence is ensured by Proposition 2.1.

We first show that $j \in \cup_{k=1}^d E_n^{(k)}$ for n sufficiently large.

Indeed, if we assume the contrary, i.e. that $j \notin \cup_{k=1}^d E_{n_t}^{(k)}$ for a sequence $\{n_k\}$ with $\lim_{t\to\infty} n_t = \infty$, the positivity of j leads to $P^{(m)}(X_{n_t}=j \text{ i.o.}) > 0$ and contradicts $\{X_n \in \cup_{k=1}^d E_n^{(k)} \text{ i.o.}\} = \{X_n \in \cup_{k=1}^d E_n^{(k)} \text{ ult.}\} = \cup_{k=1}^d T_k$ $P^{(m)}$ a.s. Thus, we may assume, if necessary by modifying a finite number of sets $\{E_n^{(k)}\}$, that $j \in \cup_{k=1}^d E_n^{(k)}$ for n = m+1,...

Define now the random variable $P_{i,X_n}^{(m,n)}/P_{\ell,X_n}^{(m,n)}$ to equal $P_{i,u}^{(m,n)}/P_{\ell,u}^{(m,n)}$ for $\omega \in \{X_n=u\}$ if $P_{\ell,u}^{(m,n)} > 0$ and 0 otherwise. Since

$$P_{i,u}^{(m,n)}/P_{\ell,u}^{(m,n)} = \frac{P^{(m)}(X_m=i|X_n=u)\pi_\ell^{(m)}}{P^{(n)}(X_m=\ell|X_n=u)\pi_i^{(m)}} \tag{3.3}$$

the Markov property of the reversed chain yields

$$\lim_{n\to\infty} P_{i,X_n}^{(m,n)}/P_{\ell,X_n}^{(m,n)} = \frac{P^{(m)}(X_m=i|\mathcal{F}_m^{(n)})}{P^{(m)}(X_m=\ell|\mathcal{F}_m^{(n)})}\,\frac{\pi_\ell^{(m)}}{\pi_i^{(m)}} \tag{3.4}$$

Using the martingale convergence theorem and Theorem 2.1 in (3.4) leads to

$$\lim_{n\to\infty} P_{i,X_n}^{(m,n)}/P_{\ell,X_n}^{(m,n)} = \frac{P^{(m)}(X_m=i|T_k)}{P^{(m)}(X_m=\ell|T_k)}\,\frac{\pi_\ell^{(m)}}{\pi_i^{(m)}} = \frac{P^{(m)}(T_k|X_m=i)}{P^{(m)}(T_k|X_m=\ell)} \tag{3.5}$$

for almost all $\omega \in T_k$, k=1,...,2, provided that $P^{(m)}(X_m=\ell|T_k) > 0$. Notice that $P^{(m)}(T_k|X_m=u) = \lim_{n\to\infty} P^{(m)}(X_n \in E_n^{(k)}|X_m=u) = \lim_{n\to\infty} \sum_{j\in E_n^{(k)}} P_{u,j}^{(m,n)} = \alpha_u^{(m)}(k)$.

Let $\{n_t^{(k)}: t=1,2,...\}$ be the set of values of n with $j \in E_n^{(k)}$. We show now that (3.5) holds if n runs through $\{n_t^{(k)}: t=1,...\}$ and X_n is replaced by j. Indeed, suppose the contrary. Then there exists $\{n_t\} \subseteq \{n_t^{(k)}\}$ with $\lim_{t\to\infty} n_t = \infty$ such that

$$\lim_{t\to\infty} P_{i,j}^{(m,n)}/P_{\ell,j}^{(m,n_t)} \neq \frac{P^{(m)}(T_k|X_m=i)}{P^{(m)}(T_k|X_m=\ell)} \tag{3.6}$$

for some $k \in \{1,...,d\}$. But $\{X_{n_t}=j \text{ i.o.}\} \subseteq T_k$ a.s.

$P^{(m)}(\{X_{n_t}=j \text{ i.o.}\}) \geq \liminf_{n\to\infty} \pi_j^{(n)} > 0$ makes (3.6) contradict (3.5) and completes the proof.

REMARK 3.1. Theorem 3.2 was stated as a result about products of stochastic matrices without reference to Markov chains theory. We have seen in the proof that $\alpha_u^{(m)}(k)$ may be expressed in terms of the tail σ-field

$\mathcal{J}^{(m)}$ as $P^{(m)}(T_k|X_m=u)$. We shall further consider some characteristic properties of the sets $\{E_n^{(k)}\}$ which, for some special cases, may lead to the actual identification of $\{\alpha_u^{(m)}(k)\}$.

THEOREM 3.3. Suppose that j is positive and $\{X_n=j \text{ i.o.}\} = \cup_{k=1}^d T_k$ $P^{(m)}$ a.s. where $\{T_k\}$ are $P^{(m)}$-atomic sets of $\mathcal{J}^{(m)}$, and let $\{E_n^{(k)}\}$ be some sets such that $\lim_{n\to\infty}\{X_n \in E_n^{(k)}\} = T_k$ $P^{(m)}$ a.s. Let $\Gamma_k = \{n_k^{(t)}\}$ be the set of values n such that $j \in E_n^{(k)}$, $k=1,\ldots,d$. Then

$$\sum_{n\in\Gamma_k} \sum_{i\notin E_{n+1}^{(k)}} p_{i,j}^{(n)} < \infty \tag{3.7}$$

PROOF. Define $A_n = \{X_n=j, X_{n+1} \notin E_{n+1}^{(k)}\}$ for $n \geqslant m$. It is clear that

$$P^{(m)}(A_n, n\in\Gamma_k \text{ i.o.}) = 0 . \tag{3.8}$$

Recall $\mathcal{F}_m^{(n)}$ is the σ-field generated by $X_m,\ldots,X_n$ with $m<n$. According to the Borel-Cantelli-Levy lemma

$$P^{(m)}(A_n, n\in\Gamma_k \text{ i.o.}) = 0 \text{ if and only if} \qquad P^{(m)}\Big(\sum_{n\in\Gamma_k} P^{(m)}(A_n|\mathcal{F}_m^{(n)}) = \infty\Big) = 0 . \tag{3.9}$$

Notice now that the Markov property of $\{X_n : n\geqslant m\}$ yields

$$P^{(m)}(A_n|\mathcal{F}_m^{(n)}) = P^{(m)}(X_{n+1} \notin E_{n+1}^{(k)}|X_n=j)1_{\{X_n=j\}} . \tag{3.10}$$

Thus, it will suffice to show that $P^{(m)}(\sum_{n\in\Gamma_k} P^{(m)}(A_n|\mathcal{F}_m^{(n)}) = \infty) = 0$ implies (3.7). Assume, by way of contradiction, that

$$\sum_{n\in\Gamma_k} \sum_{i\notin E_{n+1}^{(k)}} p_{j,i}^{(n)} = \sum_{n\in\Gamma_k} P^{(m)}(X_{n+1} \notin E_{n+1}^{(k)}|X_n=j) = \infty \tag{3.11}$$

and write

$$Y_n = \frac{\sum_{n'\in\Gamma_k \cap \{1,\ldots,n\}} P^{(m)}(X_{n'+1} \notin E_{n'+1}^{(k)}|X_{n'}=j)1_{\{X_{n'}=j\}}}{\sum_{n'\in\Gamma_k \cap \{1,\ldots,n\}} P^{(m)}(X_{n'+1} \notin E_{n'+1}^{(k)}|X_{n'}=j)} \tag{3.12}$$

Since $0 \leqslant Y_n \leqslant 1$, for $\{Y_n\}$ to converge in probability to 0 as $n\to\infty$, it is necessary that $E^{(m)}(Y_n) \to 0$ as $n\to\infty$, where $E^{(m)}(Y_n)$ denotes the expectation of Y_n under $P^{(m)}$. However, $\liminf_{n\to\infty} E^{(m)}(Y_n) \geqslant \liminf_{n\to\infty}\pi_j^{(n)} > 0$ and therefore $P^{(m)}(\limsup_{n\to\infty} Y_n > 0) > 0$. Since the denominator of Y_n in (3.12) tends to ∞ as $n\to\infty$, it follows that the numerator of Y_n will tend to ∞ as $n\to\infty$ on a set of positive $P^{(m)}$ probability. Taking account of (3.9) and (3.10) yields $P^{(m)}(\sum_{n\in\Gamma_k} P^{(m)}(A_n|\mathcal{F}_m^{(n)}) = \infty) > 0$, a contradiction that completes the proof.

REMARK 3.2. For homogeneous Markov chains with period d, the sets $\{E_n^{(k)}\}$ become $\{C_{u+n(\text{mod } d)}\}$, i.e. the cyclically moving subclasses of the recurrent class to which j belongs. In this case
$P^{(m)}(\{X_n \in C_{u+n(\text{mod } d)}\} | X_{n-1} \in C_{u+n-1(\text{mod } d)}\}) = 1$,
$\lim_{n\to\infty}\{X_n \in C_{u+n(\text{mod } d)}\} = T_u$, $u=1,\ldots,d$, and (3.7) holds trivially in view of $p_{j,i}^{(n)} = 0$ for $i \notin C_{u+n+1(\text{mod } d)}$.

The results and proofs of this Section rely on Cohn [5], [6] and [8].

4. A CLASS OF COUNTABLE CHAINS.

We shall next consider a class of stochastic matrices (P_n) satisfying the following condition

(A) S admits the decomposition $S = T \cup C$ where $j \in T$ if $\lim_{n\to\infty} P_{i,j}^{(m,n)} = 0$ for all m and i, and $j \in C$ if there exists m such that j is positive for $\{X_n : n \geq m\}$.

A partition $T, C_1, C_2, \ldots$ of S will be said to be a *basis* for (P_n) if for any $j \in C_k$ and m large enough, $\{X_n = j \text{ i.o.}\} = T_k$ a.s. is a $P^{(m)}$-atomic set of $\mathcal{J}^{(m)}$

THEOREM 4.1. Suppose that (P_n) satisfies condition (A). Then the existence of $\lim_{n\to\infty} P_{i,j}^{(m,n)}/P_{\ell,j}^{(m,n)}$ for $i,\ell \in S$, $m \geq 0$ and $j \in C$ with $P_{\ell,j}^{(m,n)} > 0$ is a necessary and sufficient condition for the existence of a basis T, $C_1, C_2, \ldots$.

PROOF. We know from Proposition 3.1 that $\Lambda = \{X_n = j \text{ i.o.}\}$ is a finite union of $P^{(m)}$-atomic sets of $\mathcal{J}^{(m)}$, i.e. $\Lambda = \cup_{k=1}^d T_k$. We shall first prove that $d=1$. Suppose otherwise, i.e. $d \geq 2$. Since $\{X_n = j \text{ i.o.}\} \supseteq T_1 \cup T_2$, Theorem 3.1 implies that there must exist two sequences $\{n_k\}$ and $\{n_k'\}$ such that

$$\begin{aligned}
\lim_{k\to\infty} P_{i,j}^{(m,n_k)}/P_{\ell,j}^{(m,n_k)} &= P(T_1|X_m=i)/P(T_1|X_m=\ell) \\
\lim_{k\to\infty} P_{i,j}^{(m,n_k)}/P_{\ell,j}^{(m,n_k)} &= P(T_2|X_m=i)/P(T_2|X_m=\ell)
\end{aligned} \tag{4.1}$$

However, $\lim_{n\to\infty} P_{i,j}^{(m,n)}/P_{\ell,j}^{(m,n)}$ was supposed to exist. Thus (4.1) entails

$$P^{(m)}(T_1|X_m=i)/P^{(m)}(T_1|X_m=\ell) = P^{(m)}(T_2|X_m=i)/P^{(m)}(T_2|X_m=\ell) \tag{4.2}$$

Further, for any set A in $\mathcal{J}^{(m)}$ with $P^{(m)}(A) > 0$, the martingale convergence theorem in conjunction with the time reversibility of the Markov property

$P^{(m)}(A|X_n)$ may be made as close as desired to 1_A for n

large enough. It is easy to see that if j is positive for $\{X_n : n \geq m\}$ it will stay positive for any $\{X_n : n \geq m'\}$ with $m' > m$, and
$P^{(m)}(A|X_n=j) = P^{(m')}(A|X_n=j)$ if $P^{(m)}(X_n=j)P^{(m')}(X_n=j) > 0$. Thus, one may choose i, ℓ and m such that for $\varepsilon < \frac{1}{2}$ $P^{(m)}(T_1|X_m=i) > 1-\varepsilon$ and

$P^{(m)}(T_2|X_m=\ell) > 1-\varepsilon$; since T_1 and T_2 are disjoint we also get $P^{(m)}(T_2|X_m=i) < \varepsilon$ and $P^{(m)}(T_1|X_m=\ell) < \varepsilon$. It is easy to see that these four inequalities are in contradiction with (4.2). We have reached a contradiction that proves that $\Lambda = \{X_n=j \text{ i.o.}\}$ is a $P^{(m)}$-atomic set of $\mathcal{T}^{(m)}$.

Conversely, if we assume that $\Lambda = \{X_n=j \text{ i.o.}\}$ is a $P^{(m)}$-atomic set of $\mathcal{T}^{(m)}$, then Theorem 3.2 yields that $\lim_{n\to\infty} P_{i,j}^{(m,n)}/P_{\ell,j}^{(m,n)} = P^{(m)}(\Lambda|X_m=i)/P^{(m)}(\Lambda|X_m=\ell)$, completing the proof.

THEOREM 4.2. Suppose that (P_n) assumes a basis $T, C_1, C_2, \ldots$. Then (i) for any $j \in C_k$, $i, \ell \in S$ with $P_{\ell,j}^{(m,n)} > 0$ and n large enough

$$\lim_{n\to\infty} P_{i,j}^{(m,n)}/P_{\ell,j}^{(m,n)} = \alpha_i^{(m)}(k)/\alpha_\ell^{(m)}(k)$$

where $\alpha_u^{(m)}(k) = \lim_{n\to\infty}\sum_{j\in E_n^{(k)}} P_{u,j}^{(m,n)}$ with $\limsup_{n\to\infty}\sum_{j\in C_k} P_{u,j}^{(m,n)} \leqslant \alpha_u^{(m)}(k)$.

(ii) if for any C_k there exists λ with $0<\lambda<1$ such that $\alpha_i^{(m)}(k) < 1-\lambda$ for $j\in C_{k'}$, with $k' \neq k$ and m large enough, then $i \in C_k$ implies

$$\sum_{n=1}^{\infty} \sum_{i\notin C_k \cup T} P_{j,i}^{(n)} < \infty.$$

PROOF. Part (i) follows from Theorem 4.1 and Theorem 3.2 if we notice that $\limsup_{n\to\infty}\sum_{j\in C_k} P_{u,j}^{(m,n)} \leqslant \lim_{n\to\infty}\sum_{j\in E^{(k)}} P_{u,j}^{(m,n)}$. Indeed, although it is not true in general, that $P^{(m)}(X_n \in C_k \text{ i.o.}) = P^{(m)}(T_k)$, $\{X_n \in F \text{ i.o.}\} = T_k$ $P^{(m)}$ a.s. obtains for any finite set $F \subset C_k$. This being true for any k and the states of C_k being positive necessarily imply that $F \subset E_n^{(k)}$ for n large enough and therefore $\lim_{n\to\infty} {}_{j\in C_k} P_{u,j}^{(m,n)} \leqslant \alpha_u^{(m)}(k)$, which proves (i).

For part (ii) we may invoke Theorem 3.3 provided that we show that $C_k \cup T \supseteq E_n^{(k)}$. The latter follows from taking $E_n^{(k)} = \{j : P^{(m)}(T_k|X_n=j) < \lambda\}$, noticing that, as shown before, $P^{(m)}(T_k|X_n=i) = \lim_{n\to\infty} P^{(m)}(X_n \in E_n{}^{(k)}\}|X_n=i)$ and arguing as in the proof of Proposition 1.1.

REMARK 4.1. It is easy to see that in case that $\lim_{n\to\infty}\sum_{j\in T} P_{u,j}^{(m,n)} = 0$ for all $u \in S$ we get that $\alpha_u^{(m)}(k) = \lim_{n\to\infty}\sum_{j\in C_k} P_{u,j}^{(m,n)}$. This happens when $\lim_{n\to\infty} P^{(m)}(X_n \in T) = 0$. In particular, the finite chains always satisfy this condition.

The results in this Section were derived in Cohn [8].

5. CONVERGENT CHAINS.

A chain $\{X_n\}$ will be said to be *convergent* if $\lim_{n\to\infty} P_{i,j}^{(m,n)}$ exists for all m, i and j (see Maksimov [15], Iosifescu [11], and Mukherjea [17]). Of course, this definition depends only on (P_n) and does not need to involve a chain $\{X_n\}$. As before our results may be read off without reference to a Markov chain structure. The class of matrices (P_n) for which $\{P_{i,j}^{(m,n)}\}$ converge is a subclass of that considered in Section 4 and, naturally, will lead to stronger properties. For convergence chains we shall identify the limits of $\{P_{i,j}^{(m,n)}\}$ rather than those of their ratios as was done in the previous sections.

THEOREM 5.1. Suppose that $\{X_n\}$ is a convergent chain. Then

(i) there exists a basis $\{T, C_1, C_2, \ldots\}$

(ii) for any m, $i \in S$ and $j \in C_k$

$$\lim_{n\to\infty} P_{i,j}^{(m,n)} = \pi_j \alpha_i^{(m)}(k)/\alpha_k$$

where $\alpha_i^{(m)}(k) = \lim_{n\to\infty} \sum_{j\in E_n(k)} P_{i,j}^{(m,n)}$, $\alpha_k = \lim_{n\to\infty} \sum_{j\in E_n(k)} \pi_j^{(n)}$ and $\pi_j = \lim_{n\to\infty} \pi_j^{(n)}$ with $\pi_j^{(n)} = \sum_{i\in S} \pi_i^{(m)} P_{i,j}^{(m,n)}$, $n \geqslant m$.

REMARK 5.1. Since $\{P_{i,j}^{(m,n)}\}$ do not depend on the choice of $\pi^{(m)}$ provided that $\pi_i^{(m)} > 0$, we notice that π_j/α_k is also indepedent of $\pi^{(m)}$.

PROOF. Theorem 4.1 ensures the existence of a basis $\{T, C_1, C_2, \ldots\}$. Further, the martingale convergence theorem yields

$$P^{(m)}(X_m=i|X_n) = P^{(m)}(X_m=i|\mathcal{F}_m^{(n)}) \to P^{(m)}(X_m=i|\mathcal{T}^{(m)}) \text{ as } n \to \infty.$$

But

$$P^{(m)}(X_m=i|X_n=j) = P_{i,j}^{(m,n)} \pi_i^{(m)}/\pi_j^{(n)}$$

and taking into account that $\{X_n=j \text{ i.o.}\} = T_k$ a.s. is a $P^{(m)}$-atomic set of $\mathcal{T}^{(m)}$ we get

$$\lim_{n\to\infty} \pi_i^{(m)} P_{i,X_n}^{(m,n)}/\pi_{X_n}^{(n)} = P^{(m)}(X_m=i|T_k) \tag{5.1}$$

for almost all $\omega \in T_k$. Further we can argue as in the proof of Theorem 4.2(i) to conclude (ii) on using (5.1).

COROLLARY 5.1. Suppose that $\{X_n\}$ is a countable convergent chain assuming a basis $\{T, C_1, C_2, \ldots\}$ and denote $q_{i,j}^{(m)} = \lim_{n\to\infty} P_{i,j}^{(m,n)}$ and $q_{i,j} = \lim_{m\to\infty} q_{i,j}^{(m)}$. Then

$$q_{i,j} = \begin{cases} \pi_j/\alpha_k & \text{for } i \in C_k,\ j \in C_k \\ \\ 0 & \end{cases}$$

PROOF. According to Theorem 5.1, $q_{i,j}^{(m)} = \pi_j \alpha_i^{(m)}(k)/\alpha_k$ and since $i \in C_k$ $\text{im}_{m\to\infty} \pi_i^{(m)} > 0$ and $\{X_n=i \text{ i.o.}\} = T_k$ $P^{(m)}$ a.s. we easily conclude that $P^{(m)}(T_k|X_m=i) \to 1$ as $m \to \infty$, and the case $i \in C_k$ and $j \in C_k$ follows. Assume now that $j \notin C_k$. Then either $j \in T$ which yields $q_{ij} = 0$ for $i \in C_k$, or $j \in C_{k'}$, with $k' \neq k$, in which case $P^{(m)}(T_{k'}|X_n=i) \to 0$ as $n \to \infty$ is a consequence of

$P^{(m)}(T_k|X_m=i) \to 1$ as $m\to\infty$ and T_k and $T_{k'}$ being disjoint. Using now Theorem 5.1 (ii) yields $q_{ij} = 0$ and completes the proof.

REMARK 5.2. Theorem 4.2(ii) holds, of course, for convergent chains as well. The additional condition imposed on convergent chains does not seem to make this result any stronger.

The convergent chain concept goes back to Maksimov [15], who considered the case of bistochastic matrices (P_n). Extensions to finite and countable convergent chains were given by Mukherjea [16-18], etc. The methods used in these papers are matricial and the sets of the basis are characterized by means of some limit points of matrices $\{Q_k\}$ with $Q_k = \lim_{n\to\infty} P_{k,n}$. However, such matricial methods yielded much weaker results and do not seem to permit the identification of $\{Q_k\}$.

The results of this section were derived in Cohn [8].

6. FINITE MARKOV CHAINS: THE GENERAL CASE.

We shall now consider the case when s, the number of elements of S, is finite. For such chains we shall derive stronger results under less restrictive or no assumptions on $\{P_n\}$. As before we need start off by considering the structure of $\mathcal{T}^{(m)}$.

THEOREM 6.1. The tail σ-field $\mathcal{T}^{(m)}$ of $\{X_n : n \geq m\}$ is finite and the number of $P^{(m)}$-atomic sets of $\mathcal{T}^{(m)}$ does not exceed s.

PROOF. Let $T_1,\ldots,T_d$ be some disjoint sets in $\mathcal{T}^{(m)}$ with $P^{(m)}(T_k) > 0$ for $k=1,\ldots,d$. As shown in the proof of Proposition 1.1, letting $E_n^{(k)} = \{j : P^{(m)}(T_k|X_n=j) > 0.5\}$ yields $\lim_{n\to\infty}\{X_n \in E_n^{(k)}\} = T_k$ $P^{(m)}$ a.s. for $k=1,\ldots,d$. It is easy to check that $\{E_n^{(k)}, k=1,\ldots,d\}$ are disjoint for any n. Since $P^{(m)}(T_k) > 0$, $\{E_n^{(k)}\}$ must be non-empty for n large. However, there are s states in S which requires $d \leq s$ and completes the proof.

LEMMA 6.1. Let $\{X_n : n \geq m\}$ and $\{X'_n : n \geq m'\}$ be two finite Markov chains with strictly positive initial probability vectors $\pi^{(m)}$ and $\pi'^{(m)}$ and sequences of transition probability matrices $(P_n)_{n\geq m}$ and $(P_n)_{n\geq m'}$ respectively. Write $\pi_i^{(n)} = P^{(m)}(X_n=i)$, $\pi_i'^{(n)} = P^{(m')}(X'_n=i)$, $E_n^+ = \{i : \pi_i^{(n)} > 0\}$ and $E_n'^+ = \{i : \pi_i'^{(n)} > 0\}$. Then there exists a number $N > 0$ such that $E_n^+ = E_n'^+$ for $m, m' \geq N$ and $n \geq \max(m,m')$.

PROOF. Suppose that $m' > m$ and let $j \in E_n^+$. Then $\pi_j^{(n)} \geq \pi_i^{(m)} P_{i,j}^{(m,n)} > 0$ for some $i \in S$ and since $\pi_i^{(m)} > 0$, it follows that $P_{i,j}^{(m,n)} > 0$. By the Chapman-Kolmogorov formula there must exist $\ell \in S$ such that $P_{i,\ell}^{(m,m')} P_{\ell,j}^{(m'.n)} > 0$. Thus $\pi_j'^{(n)} \geq \pi'^{(m')} P_{\ell,j}^{(m',n)} > 0$. We have shown that $E_n^+ \subseteq E_n'^+$. Notice that the finiteness of S makes it impossible that $E_n^+ \subset E_n'^+$ with strict inclusion for all m,n and $m' > m$. Thus there must exist m and N such that for $m' > m$ $E_N^+ = E_N'^+$. Choose now $n > N$ and $j \in E_n'^+$. Then there is a state $i \in E_N'^+$ such that $\pi'^{(n)}_j \geqslant \pi'^{(N)}_i P^{(N,n)}_{i,j} > 0$. Thus $P^{(N,n)}_{i,j} > 0$ for $i \in E'^{+}_N$. But $E'^{+}_N = E^{+}_N$ leads to $\pi^{(n)}_j \geqslant \pi^{(N,n)}_i > 0$ and $E^{+}_n = E'^{+}_n$ obtains for $n > N$, completing the proof.

THEOREM 6.2. Suppose that S is finite. Then there exist some sequences of subsets of S, $\{E^{(k)}_n\}$, k=1,...,S, such that for m=0,1,...

$$\lim_{n\to\infty} P^{(m,n)}_{i,j}/P^{(m,n)}_{\ell,j} = \alpha^{(k)}_i(m)/\alpha^{(k)}_\ell(m) \tag{6.1}$$

for $j \in E^{(k)}_n$, k=1,...,d, where $\alpha^{(k)}_u = \lim_{n\to\infty} \sum_{j\in E^{(k)}_n} P^{(m,n)}_{u,j}$ provided that $\alpha^{(k)}_\ell > 0$, and

$$\lim_{n\to\infty} P^{(m,n)}_{i,j} = 0 \tag{6.2}$$

for $j \in E_n = S - \cup^{d}_{k=1} E^{(k)}_n$.

REMARK. Throughout the paper we will drop the explicit dependence of j upon n ; and so j is not kept fixed, but in general varying with $E^{(k)}_n$. The same convention will be valid for all the sets dependent on n to be further considered.

PROOF. Choose m such that the sets $\{E^{+}_n\}$ attached to $\{X_n : n \geqslant m\}$ are maximal. Such a choice is always possible in view of Lemma 6.1.

According to (3.5)

$$\lim_{n\to\infty} P^{(m,n)}_{i,X_n}/P^{(m,n)}_{\ell,X_n} = P^{(m)}(T_k|X_m=i)/P^{(m)}(T_k|X_m=\ell) \tag{6.3}$$

for almost all $\omega \in T_k$, k=1,...,

Write now

$$E^{(r)}_{n,k} = \{j:P(T_k|X_n=j) > 0.5\} \cap \bigcap_{i,\ell\in S, m=0,1,\dots,r-1} \{j:|P^{(m,n)}_{i,j}/P^{(m,n)}_{\ell,j} - $$
$$- P^{(m)}(T_k|X_m=i)/P^{(m)}(T_k|X_m=\ )| < \frac{1}{r}\}.$$

But (6.3) implies

$$\lim_{n\to\infty}\{X_n \in \bigcap_{i,\ell\in S, m=0,1,\dots,r-1} \{j:|P^{(m,n)}_{i,j}/P^{(m,n)}_{\ell,j} - $$
$$- P^{(m)}(T_k|X_m=i)/P^{(m)}(T_k\ X_m=\ell)| < \frac{1}{r}\} \supseteq T_k \text{ a.s.}$$

for any r and k . Take $r=2^\nu$, $\nu=1,2,\dots$ For each ν one can find a number $m(\nu)$ such that

$$P^{(m)}(T_k \,\Delta \bigcup_{n\geqslant m(\nu)} \{X_n \in E^{(r)}_{n,k}\}) \leqslant 2^{-\nu},\ k=1,\dots,d$$

$$P^{(m)}(T_k \,\Delta \bigcap_{n\geqslant m(\nu)} \{X_n \in E^{(r)}_{n,k}\}) \leqslant 2^{-\nu},\ k=1,\dots,d$$

Define now $E^{(k)}_n = E^{(r)}_{n,k}$ for $m(\nu) \leqslant n < m(\nu+1)$ and consider the elementary set properties

$$\left.\begin{array}{l} A \,\Delta\, (B \cap C) \\ \\ A \,\Delta\, (B \cup C) \end{array}\right\} \subseteq (A \,\Delta\, B) \cup (A \,\Delta\, C) \tag{6.4}$$

Using (6.4) yields

$$P^{(m)}(T_k \,\Delta \bigcup_{n\geq m(\nu)} \{X_n \in E_n^{(k)}\}) = P^{(m)}(T_k \,\Delta \bigcup_{u=\nu}^{\infty} (\bigcup_{m(u)\leq n<m(u+1)} \{X_n \in E_{n,k}^{(2^u)}\})$$

$$\leq \sum_{u=\nu}^{\infty} P^{(m)}(T_k \,\Delta \bigcup_{n(u)\leq n<m(u+1)} \{X_n \in E_{n,k}^{(2^u)}\})$$

$$\leq \sum_{u=\nu}^{\infty} P^{(m)}(T_k \,\Delta \bigcup_{n\geq m(u)} \{X_n \in E_{n,k}^{(2^u)}\}) + \sum_{u=}^{\infty} P^{(m)}(T_k \,\Delta \bigcap_{n\geq m(u)} \{X_n \in E_{n,k}^{(2^u)}\})$$

$$\leq 2^{-(\nu-2)}.$$

Similarly, one can show

$$P^{(m)}(T_k \,\Delta \bigcap_{n\geq m(\nu)} \{X_n \in E_n^{(k)}\}) \leq 2^{-(\nu-1)}$$

which boils down to

$$\lim_{n\to\infty} P_{i,j}^{(t,n)}/P_{\ell,j}^{(t,n)} = P^{(m)}(T_k|X_t=i)/P^{(m)}(T_k|X_t=\ell) \qquad (6.5)$$

for $j \in E_n^{(k)}$ and $i,j \in E_t^+$. If we notice now that

$$\alpha_u^{(k)}(m) = P^{(m)}(T_k|X_m=u) = \lim_{n\to\infty} P^{(m)}(\{X_n \in E_n^{(k)}\}|X_m=u)$$

we get that (6.1) holds for states i, j in E_t^+ which depend on the m that was chosen at the beginning of the proof. We shall nest show that (6.1) holds for every t with the same sequences $\{E_n^{(k)}\}$ constructed above for a particular m . Indeed, assume $m' \neq m$. Then with our choice of m $E_n^+ \supseteq E_n'^+$ for $n > \max(m,m')$, and

$$P_{i,j}^{(m',n)}/P_{\ell,j}^{(m',n)} = \frac{\sum_{u\in E_{n'}^+} P_{i,u}^{(m'n')}P_{u,j}^{(n',n)}/P_{u',j}^{(n',n)}}{\sum_{u\in E_{n'}^+} P_{\ell,u}^{(m',n')}P_{u,j}^{(n',n)}/P_{u',j}^{(n',n)}} \qquad (6.6)$$

where u' is a state in $E_{n'}^+$ with $P_{u',j}^{(m',n)} > 0$. Since $\pi_u^{(n')}\pi_{u'}^{(n')} > 0$ the ratio $P_{u,j}^{(n',n)}/P_{u',j}^{(n',n)}$ makes sense on the probability space attached to the chain $\{X_n : n \geq m\}$ and (6.5) implies

$$\lim_{n\to\infty} P_{u,j}^{(n',m)}/P_{u',j}^{(n',n)} = P^{(m)}(T_k|X_{n'}=u)/P^{(m)}(T_k|X_{n'}=u')$$

where $j \in E_n^{(k)}$. Using this in (6.6) yields

$$\lim_{n\to\infty} P_{i,j}^{(m',n)}/P_{\ell,j}^{(m',n)} = \frac{\sum_{u\in E_{n'}^+} P_{i,u}^{(m',n')}P^{(m)}(T_k|X_n=u)}{\sum_{u\in E_{n'}^+} P_{\ell,u}^{(m',n')}P^{(m)}(T_k|X_{n'}=u)}$$

$$= \frac{P^{(m)}(T_k|X_{m'}=i)}{P^{(m)}(T_k|X_{m'}=\ell)}$$

which completes the proof of (6.1).

To prove (6.2) notice that $\{X_n \in \bigcup_{k=1}^d E_n^{(k)} \text{ i.o.}\} = \Omega$ $P^{(m)}$ a.s. which entails $P^{(m)}(X_n \in E_n \text{ i.o.}) = 0$. This, in turn implies $\lim_{n\to\infty} P^{(m)}(X_n \in E_n) = 0$ and proves (6.2).

The tail σ-field of a finite Markov chain was proven to be finite in Cohn [2]. Further, Senchenko [19] and Cohn [3] have independently shown by different methods that the number of atomic sets does not exceed the number of states. The proof of Theorem 6.1 given here was taken from Cohn [4]. Iosifescu [11] has studied the tail σ-field structure of continuous time Markov processes. Kingman [13] has given a geometrical representation of the transition matrices of a nonhomogeneous Markov chain from which the tail σ-field structure may be derived.

As far as the asymptotic behaviour of transition probabilities is concerned, it seems that the first result in the case of nonasymptotical independent chains was given by Blackwell [1] who derived the existence of limits for the reverse transition probabilities. However, Blackwell's paper does not refer to the tail σ-field notion. The results of this section on the asymptotics of $\{P_{m,n}\}$ are derived in slightly different forms in Cohn [5] and [7].

7. SOME CLASSES OF CHAINS CONSIDERED BY DOEBLIN.

In an important, but little known paper published in 1937, Doeblin [9] introduced a number of nonhomogeneous Markov chain models and gave without proofs several results concerning their asymptotic behaviour. One model was defined by the following

CONDITION (D_1). There exists a strictly positive number δ such that for any fixed states (i,j) either $p_{i,j}^{(n)} \geq \delta$ for all n or $p_{i,j}^{(n)} = 0$ for all n.

Doeblin asserted that in this case it is possible to decompose S into disjoint 'final classes' $G_0, G_1, \ldots, G_v$, and each final class G_α, $1 \leq \alpha \leq v$ may be further decomposed into 'cyclical subclasses' $\{C_i(\alpha);\ i=1,\ldots,d(\alpha)\}$. These have the following asymptotic properties (according to Doeblin) as $n \to \infty$

(i) $P_{i,j}^{(m,n)} \to 0$ for every $i \in S$ and $j \in G_0$;

(ii) $P_{i,j}^{(m,n)} = 0$ for every $i \in G_\alpha$ and $j \notin G_\alpha$;

(iii) if $i, j \in G_\alpha$ with $i \in C_\ell(\alpha)$ and $j \in C_{\ell'}(\alpha)$ then $P_{i,j}^{(m,n)} = 0$ provided that $n - m \neq (\ell' - \ell) \bmod d(\alpha)$;

(iv) if $i, j \in G_\alpha$ with $i \in C_\ell(\alpha)$, $j \in C_{\ell'}(\alpha)$ then $P_{i,j}^{(m,n)} = P_j^{(n)} + \varepsilon_{i,j}^{(m,n)}$ provided that $n - m = (\ell' - \ell) \bmod d(\alpha)$.

Here $\varepsilon_{i,j}^{(m,n)} \to 0$ exponentially as $n \to \infty$ for any m, i and j and the limit distribution $\{P^{(n)}\}$ satisfies $\sum_{j \in C_{\ell'}(\alpha)} P_j^{(n)} = 1$.

(v) for $i \in \bigcup_{\alpha=0}^{v} G_\alpha$, $j \in C_\ell(\alpha)$ and some $1 \leq \ell \leq d(\alpha)$, $\alpha = 1, \ldots, v$

$P_{i,j}^{(m,n)} = P^{(m)}[i, \bar{C}_{n,\ell}(\alpha)] P_j^{(n)} + \varepsilon_{i,j}^{(m,n)}$ where $P_j^{(n)}$, $\varepsilon_{i,j}^{(m,n)}$ are as in (iv) and $P^{(m)}[i, \bar{C}_{n,\ell}(\alpha)]$ is the limit as $r \to \infty$ of the probability, given $X_m = i$, that $X_{n+rd(\alpha)} \in C_\ell(\alpha)$.

Doeblin subsequently relaxed assumption (A) allowing positive $P_{i,j}^{(n)}$ to tend to 0 as $n \to \infty$. More precisely, he considered

CONDITION (D_2). There exists a strictly positive number δ and some $N \geq 1$ such that for any fixed pair of states (i,j) either $p_{i,j}^{(n)} \geq \delta$ for $n \geq N$ or $\lim_{n\to\infty} p_{i,j}^{(n)} = 0$.

Two subclasses of chains satisfying Condition (D_2) were further considered: those satisfying $\sum_{n=1}^{\infty} \max_{(i,j)\in\Lambda} p_{i,j}^{(n)} < \infty$ (Condition (D_2')) and those satisfying $\sum_{n=1}^{\infty} \max_{(i,j)\in\Lambda} p_{i,j}^{(n)} = \infty$ (Condition (D_2'')) where $\Lambda = \{(i,j): \lim_{n\to\infty} p_{i,j}^{(n)} = 0\}$.

To study chains satisfying Condition (D_2) Doeblin proposed introducing an associated chain, derived from the initial one by taking 0 for the positive one-step transition probabilities tending to 0. However, by so doing the transition matrices become nonstochastic, and it seems to us that Doeblin intended to add the transition probabilities replaced by 0 to the ones bounded away from 0 in the same row to preserve the stachasticity of the matrix. But there is considerable leeway in defining a matrix in this way and Doeblin's details are rather sketchy.

In the case (D_1) it will be easily seen that the associated chain may be defined in any way described above, but in general we shall have to use some arguments based on the tail σ-field structure to justify the definition that we are going to adopt for an associated chain.

We proceed now to define an associated matrix. For the sake of definiteness we shall consider a matrix in which the entries of the initial matrix replaced by 0 are all added to the first entry in their row larger than δ. More precisely, we say that $P_n' = (p_{i,j}'^{(n)})$ is an *associated* matrix of P_n if $p_{i,j}'^{(n)} = 0$ for $(i,j) \in \Lambda$; $p_{i,j(i)}'^{(n)} = p_{i,j(i)}'^{(n)} + \sum_{j\in S_i} p_{i,j}^{(n)}$, $j(i)$ being the first entry in the ith row such that $(i,j) \notin \Lambda$ and $S_i = \{j:(i,j) \in \Lambda\}$, and $p_{i,j}'^{(n)} = p_{i,j}^{(n)}$ for the pairs (i,j) such that $(i,j) \notin \Lambda$ and $j > j(i)$.

A Markov chain assuming the initial probability vector $\pi^{(m)}$ and the transition matrices $(P_n')_{n\geq m}$ will be said to be associated to $\{X_n : n \geq m\}$.

Denote by $\{E_n^*: n=0,1,\ldots\}$ a sequence of sets with the property $\liminf_{n\to\infty} \min_{i\in E_n^*} \pi_i^{(n)} > 0$ and write $E_n^{**} = S - E_n^*$. If $\{E_n^{**}\}$ are present, then $\liminf_{n\to\infty} \max_{i\in E_n^{**}} \pi_i^{(n)} = 0$.

LEMMA 7.1. Suppose that there exists a sequence of positive integers $m_1 < n_1 < m_2 < n_2 < \ldots$ such that $\sum_{u=1}^{\infty} P_{i,j}^{(m_u,n_u)} = \infty$ where $i \in E_{m_u}^*$ and $j \in S$. Then $P^{(m)}(X_{n_u} = j \text{ i.o.}) > 0$.

PROOF. Using the argument employed in the proof of Theorem 3.3 we get that $P^{(m)}(A_u \text{ i.o.}) > 0$ where $A_u = \{X_{m_u} = i, X_{m_u} = j\}$, u=1,2,... But $P^{(m)}(X_{n_u} = j \text{ i.o.}) \geq P^{(m)}(A_u \text{ i.o.}) > 0$ as stated.

In what follows we shall consider a condition that contains (D_1). We call this Condition (D_1^*):

(i) either the $\{E_n^{**}\}$ are empty or $\lim_{n\to\infty} \pi_i^{(n)} = 0$ for $i \in E_n^{**}$, and

(ii) the sequence $\{p_{i,j}^{(n)}: i \quad S, n=1,2,\ldots\}$ may contain 0's but its positive values are bounded away from 0, i.e. there exists $\delta > 0$ such that $\inf'_{i,j,n}(p_{i,j}^{(n)}) > 0$ where inf' means that the infinum is taken over the strictly positive matrices of the sequence.

Notice that (D_1^*)(ii) holds under (D_1). As far as (D_1^*)(i) is concerned an arbitrary associated matrix P_n', say P_0', may be considered and taking into account only the position of its positive and null entries one may derive the periodicity and the cyclically moving subclasses of a homogeneous Markov chain assuming such type of transition probability matrix. Choosing D to be a multiple of $\{d(\alpha), \alpha = 1,\ldots,v\}$ we can easily conclude that $P_{m,n+ND}$ have all the positive and null entries in the same position as P_1^{ND} and that $\{P_{i,j}^{(n,n+ND)}: i,j \in C_k(\alpha), k=1,\ldots,d(\alpha)\}$ are all positive for N sufficiently large, δ^{ND} being a lower bound of these entries. Using this we can show that $C_k(\alpha) \cap E_n^* = C_k(\alpha)$, since for $j \in C_k(\alpha)$, $\pi_j^{(n+ND)} \geqslant c\delta^{ND}$ with $c = \min_{i \in S}\pi_i^{(n)}$. Therefore $E_n^* \supseteq \cup_{\alpha=1}^{v} G_\alpha$ for n sufficiently large. Further, it is easy to see that $E_n^{**} = G_0$, and it will be seen that G_0 plays in this case the role of a 'transient' set.

THEOREM 7.1. Suppose that (D_1^*) holds. Then

(i) there exists a sequence of disjoint events of S, say $E_n^{(1)},\ldots,E_n^{(d)}$, n=0,1,... and a positive integer N such that for any m=0,1,...

$$\bigcup_{n=N}^{\infty} \{X_n \in E_n^{(k)}\} = T_k \qquad P^{(m)} \text{ a.s.} \tag{7.1}$$

for k=1,...,d.

(ii) If $E_n = E - \cup_{k=1}^{d} E_n^{(k)}$ are present, then for $i \in S$, m=0,1,... and $j \in E_n$

$$\lim_{n\to\infty} P_{i,j}^{(m,n)} = 0 \, . \tag{7.2}$$

For $i \in S$, m=0,1,... $j \in E_n^{(k)}$, $n \geqslant N$

$$P_{i,j}^{(m,n)} = \pi_j^{(n)} \frac{P^{(m)}(\bigcup_{n=N}^{\infty}\{X_n \in E_n^{(k)}\} | X_m = i)}{P^{(m)}(\bigcup_{n=N}^{\infty}\{X_n \in E_n^{(k)}\})} + o(\pi_j^{(n)}) \, . \tag{7.3}$$

For $i \in E_m^{(k)}$, $j \in E_n^{(k)}$, $m \geqslant N$

$$P_{i,j}^{(m,n)} = \frac{\pi_j^{(n)}}{P^{(m)}(\bigcup_{n=N}^{\infty}\{X_n \in E_n^{(k)}\})} + o(\pi_j^{(n)}). \tag{7.4}$$

For $i \in E_m^{(k)}$, $j \notin E_n^{(k)}$, $m, n \geqslant N$

$$P_{i,j}^{(m,n)} = 0 \, . \tag{7.5}$$

PROOF. We shall first prove that for n sufficiently large $E^{**}_n = E_n$, where $P^{(m)}(X_n \in E_n \text{ i.o.}) = 0$. Suppose that for an infinity of n's $p^{(n)}_{i,j} \geq \delta$ where $i \in E^*_n$ and $j \in E^{**}_{n+1}$. Then $\pi^{(n+1)}_j \geq \delta\pi^{(n)}_i$. But $\liminf_{n\to\infty}\pi^{(n)}_i > 0$; this entails $\liminf_{n\to\infty}\pi^{(n)}_j > 0$, which is impossible. Thus, $p^{(n)}_{i,j} = 0$ whenever $i \in E^*_n$ and $j \in E^{**}_{n+1}$ for n sufficiently large. It follows that $P^{(m,n)}_{i,j} = 0$ for $i \in E^*_m$ and $j \in E^{**}_n$ and by Theorem 6.2 we get that $E^{(k)}_n \cap E^{**}_n$ must be empty for $n \geq N$ since otherwise (6.2) would be invalidated. Therefore $E^{(k)}_n \cap E^*_n = E^{(k)}_n$ for n sufficiently large and according to Lemma 7.1, $p^{(n)}_{i,j} \geq \delta$ may happen only for finitely many n with $i \in E^{(k)}_n$ and $j \in E^{(k')}_{n+1}$ with $k' \neq k$. It follows that $P^{(m)}(\{X_n \in E^{(k)}_n\} | \{X_{n-1} \in E^{(k)}_{n-1}\}) = 1$ for $k=1,\ldots,d$ and $n \geq N$ which in conjunction with $\lim_{n\to\infty}\{X_n \in E^{(n)}_k\} = T_k$ $P^{(m)}$ a.s. yields $T_k = \cup^{\infty}_{n=N}\{X_n \in E^{(k)}_n\}$ and (7.1) is proved.

It is easy to see that Theorem 6.2 may now be applied to conclude (7.3) and (7.4). Finally, (7.2) follows from Lemma 7.1, and (7.4) is a consequence of the proven part (i).

REMARK 7.1. For the chain satisfying Condition (D_1) $\{E^{(k)}_n\}$ are the subclasses $\{C_k(\alpha)\}$ in their cyclical order, i.e.

$$\{E^{(k)}_n\} = \{C_{u+n(\mathrm{mod}\ d(\alpha))}(\alpha)\}$$

where $u=1,\ldots,d(\alpha)$, $\alpha = 1,\ldots,v$ and $k=1,\ldots,\sum^{v}_{\alpha=1}d(\alpha)$. Indeed, $P^{(m)}(\{X_n \in C_{u+n(\mathrm{mod}\ d(\alpha))}(\alpha)\} | X_{n-1} = C_{u+n-1(\mathrm{mod}\ d(\alpha))}(\alpha)\}) = 1$ and $\lim_{n\to\infty}\{X_n \in C_{u+n(\mathrm{mod}\ d(\alpha))}(\alpha)\} = \Lambda_u$ (say) $P^{(m)}$ a.s. Besides, Λ_u belongs to $\mathcal{J}^{(m)}$ and therefore it is either $P^{(m)}$-atomic or a union of $P^{(m)}$-atomic sets of $\mathcal{J}^{(m)}$. Since $\{P^{(n,n+ND)}_{i,j};\ i,j \in C_k(\alpha),\ k=1,\ldots,d(\alpha)\}$ are positive for all n. If Λ_u were not $P^{(m)}$-atomic then there would exist two $P^{(m)}$-atomic subsets of Λ_u, Λ'_u and Λ''_u and by Theorem 7.1 there would also exist two sequences of sets $\{E'_n\}$ and $\{E''_n\}$ such that $\cup_{n\geq N}\{X_n \in E'_n\} = \Lambda'_u$ $P^{(m)}$ a.s. and $\cup_{n\geq N}\{X_n \in E''_n\} = \Lambda''_u$ $P^{(m)}$ a.s., in which case (7.5) would be contradicted. This proves Doeblin's results stated before.

We next consider conditions which contain as particular cases Doeblin's Conditions (D'_1) and (D''_2). First we consider

CONDITION (B). There exists $\delta > 0$ such that if $\Lambda_n = \{(i,j):\ p^{(n)}_{i,j} < \delta\}$ then $\max_{(i,j)\in\Lambda_n} p^{(n)}_{i,j} \to 0$ as $n \to \infty$.

For chains satisfying Condition (B) we consider an associated chain in the same way as in the case (D^*_1), i.e., we define the associated matrices (P'_n) in which the entries of the initial matrix replaced by 0 are all added to the first entry larger than δ in their row. In fact the whole definition of an associated matrix given before may be copies here word for word, the only difference being that Λ is replaced by Λ_n, where $\Lambda_n = \{(i,j):\ p^{(n)}_{i,j} < \delta\}$;

S_i and $j(i)$ also depend on n and should be denoted by $S_{i,n}$ and $j(i,n)$ respectively.

Let us next consider

CONDITION (C). There exist sequences of disjoint sets $\{F_n^{(\alpha)}: n=0,1,\ldots\}$ $\alpha=1,\ldots,d'$ with $d' \geq 2$ and a number N such that $p_{i,j}^{(n)} = 0$ for $i \in F_n^{(\alpha)}$ and $j \notin F_{n+1}^{(\alpha)}$ for $n \geq N$, $\alpha = 1,\ldots,d'$.

Theorem 7.1 shows that (D_1^*) is a particular case of (C); the difference between these two conditions lies in that $\lim_{n\to\infty}\{X_n \in F_\alpha^{(n)}\} = \bigcup_{n=N}^{\infty}\{X_n \in F_\alpha^{(n)}\}$ $P^{(m)}$ a.s. need not be an $P^{(m)}$-atomic set of $\mathcal{J}^{(m)}$, i.e. it may be a union of $P^{(m)}$-atomic sets of $\mathcal{J}^{(m)}$.

We are now in the position to formulate the following two conditions.

CONDITION (D_2^*). (i) (B) is satisfied with $\sum_{n=1}^{\infty}\max_{(i,j)\in\Lambda_n} p_{i,j}^{(n)} < \infty$; (ii) the associated matrices satisfy (D_1^*).

CONDITION (D_2^{**}). (i) (B) is satisfied with $\sum_{n=1}^{\infty}\max_{(i,j)\in\Lambda_n} p_{i,j}^{(n)} = \infty$; (ii) the associated matrices satisfy (C).

THEOREM 7.2. If (D_2^*) holds, then for $i \in S$, m=0,1,... and $j \in E_n'$

$$\lim_{n\to\infty} P_{i,j}^{(m,n)} = 0 \tag{7.6}$$

and for $i \in S$ and $j \in E_n'^{(k)}$, k=1,...,d'

$$P_{i,j}^{(m,n)} = \pi_j^{(n)} \frac{P^{(m)}(T_k|X_m=i)}{P^{(m)}(T_k)} + o(\pi_j^{(n)}). \tag{7.7}$$

PROOF. It is easy to see that if $A = \{(X_n,\ldots,X_{n+N}) \in B\}$ for $B \subset \underbrace{S\times\ldots\times S}_{N+1 \text{ times}}$, and if $i \in \{j: \pi_i^{(m)}\pi_j'^{(m)} > 0\}$, then

$$|P^{(m)}(A|X_n=i) - P'^{(m)}(A|X_n'=i)| \leq \varepsilon_n \tag{7.8}$$

where $\varepsilon_n = \sum_{k=n}^{\infty}\sum_{(i,j)\in\Lambda_k} p_{i,j}^{(k)}$, for $n > m$ and any $N > 1$. The standard monotone class argument extends (7.8) to any $A \in \mathcal{F}_n$.

Suppose that we choose an associated chain $\{X_n':n \geq u\}$ such that $\pi_i^{(u)} > 0$ for $i \in E_u^*$, and $\liminf_{n\to\infty}\min_{i\in E_n^*}\pi_i^{(n)} > \varepsilon_u$. Then if we write (7.8) for $m = u$ and $i \in \{j: \pi_j^{(u)} > 0\}$, and take $A = \{X_n=j\}$ we get

$$|P_{i,j}^{(u,n)} - P_{i,j}'^{(u,n)}| \leq \varepsilon_u \tag{7.9}$$

But $\pi_j^{(n)} = \sum_{i\in S}\pi_i^{(u)} P_{i,j}^{(u,n)}$, $\pi_j'^{(n)} = \sum_{i\in S}\pi_i^{(u)} P_{i,j}'^{(u,n)}$ and therefore

$$|\pi_j^{(n)} - \pi_j'^{(n)}| \leq \sum_{i\in S}\pi_i^{(u)}|P_{i,j}^{(u,n)} - P_{i,j}'^{(u,n)}| \leq \varepsilon_u \tag{7.10}$$

On the other hand (7.9) implies

$\liminf_{n\to\infty}\min_{j\in E_n^*}\pi_j'^{(n)} \geq \liminf_{n\to\infty}\min_{j\in E_n^*}\pi_j^{(n)} - \varepsilon_u > 0$ which entails $E_n'^* \supseteq E_n^*$ for n sufficiently large. Take now $A = \{X_{n'} \in E_{n'}'^{**}\}$ in (7.8). Since $\lim_{n'\to\infty}P'^{(m)}(X_{n'}' \in E_{n'}'^{**}|X_n'=i) = 0$ we get $\limsup_{n'\to\infty}P^{(m)}(X_{n'} \in E_n'^{**}|X_m=i) \leq \varepsilon_n$. Further it is easy to see that $\{j:\pi_j'^{(n)} > 0\} \subseteq \{j:\pi_j^{(n)} > 0\}$ and since $\pi_j'^{(m)} > 0$ for $j \in E_m^*$ and m sufficiently large, we get $\limsup_{n\to\infty}P^{(m)}(\{X_n \in E_n'^{**}\} \cap \{X_m \in E_m^*\}) \leq \varepsilon_m$. But the sequence $\{E_n^*\}$ has the property $\limsup_{n\to\infty}P^{(m)}(X_n \in E_n^*) = 1$. This leads to $\lim_{n\to\infty}P^{(m)}(X_n \in E_n'^{**}) = 0$ and therefore $E_n^* = E_n^{*\prime}$ and $E_n'^{**} = E_n^{**}$ for n sufficiently large.

Consider now a $P^{(m)}$-atomic set of $\mathcal{J}^{(m)}$, say T_k. Then by Theorem 7.1 $\lim_{n\to\infty}\{X_n \in E_n^{(k)}\} = T_k$ $P^{(m)}$ a.s. and (7.8) implies

$$|P^{(m)}(\bigcup_{r=n}^{\infty}\{X_r \in E_r^{(k)}\}|X_m=i) - P'^{(m)}(\bigcup_{r=n}^{\infty}\{X_r' \in E_r^{(k)}\}|X_m'=i)| \leq \varepsilon_m \tag{7.11}$$

and

$$|P^{(m)}(\bigcap_{r=n}^{\infty}\{X_r \in E_r^{(k)}\}|X_m=i) - P'^{(m)}(\bigcap_{r=n}^{\infty}\{X_r' \in E_r^{(k)}\}|X_m'=i)| \leq \varepsilon_m . \tag{7.12}$$

Taking the limit over n in (7.11) and (7.12) and using the triangle inequality yields

$$|P'^{(m)}(\limsup_{n'\to\infty}\{X_n' \in E_n^{(k)}\}|X_m'=i) - P'^{(m)}(\liminf_{n'\to\infty}\{X_n' \in E_n^{(k)}\}|X_m'=i)| \leq 2\varepsilon_n . \tag{7.13}$$

Multiplying (7.13) by $\pi_i'^{(m)}$, assuming over i and taking the limit over m we get that $\lim_{n\to\infty}\{X_n' \in E_n^{(k)}\}$ a.s. with respect to $P'^{(m)}$ exists, has positive probability and is either $P'^{(m)}$-atomic or a union of $P'^{(m)}$-atomic sets of $\mathcal{J}'^{(m)}$. We may now interchange $P^{(m)}$ and $P'^{(m)}$ and get that

$$|P^{(m)}(\limsup_{n\to\infty}\{X_n' \in E_{n'}'^{(k)}\}|X_m=i) - P^{(m)}(\liminf_{n\to\infty}\{X_n' \in E_{n'}'^{(k)}\}|X_m=i)| \leq 2\varepsilon_m . \tag{7.14}$$

Now because $\mathcal{J}^{(m)}$ and $\mathcal{J}'^{(m)}$ are finite, we conclude that their atomic sets are in a one-to-one correspondence. Therefore $d = d'$ and

$$P^{(m)}(\lim_{n\to\infty}\{X_n \in E_n^{(k)}\} \,\Delta\, \lim_{n\to\infty}\{X_n \in E_n'^{(k')}\}) = 0$$

$$P'^{(m)}(\lim_{n\to\infty}\{X_n' \in E_n^{(k)}\} \,\Delta\, \lim_{n\to\infty}\{X_n' \in E_n'^{(k)}\}) = 0$$

for $k,k' \in \{1,\ldots,d\}$. Since we have seen in the proof of Theorem 7.1 that $E_n'^{(k)} \cap E_n'^* = E_n'^*$ for n large enough it follows that $E_n'^{(k')} \subseteq E_n^{(k)}$ for k sufficiently large. Now it is easy to see that $P^{(m)}(\{X_n \in E_n' \text{ i.o.}\}) = 0$ and complete the proof by an already familiar reasoning.

REMARK 7.2. The sets $E_n'^{(k)}$ corresponding to the associated chain $\{X_n':n \geq m\}$ are in general smaller than the sets $\{E_n^{(k)}\}$ for which Theorem 7.1 guarantees the same convergence property (7.7). It is therefore possible that there exists states $i_u \in E_n'$ with $i_u \in E_n^{(k)}$ for some k and n. However such states have the property $P^{(m)}(X_n = i_u \text{ i.o.}) = 0$. The usefulness of using $\{E'^{(k)}_n\}$ instead of $\{E^{(k)}_n\}$ lies in the fact that the former are more easily obtainable. For example in case (D'_2) we have seen that such sets may be identified by means of an arbitrary one-step transition probability matrix.

We turn now to the case (D^{**}_2) which is considerably more complicated than the ones considered so far. We shall first need the following

LEMMA 7.2. Suppose that the sequence of sets $\{A_n\}$ is such that for $m_1 < n_1 < m_2 < n_2 < \ldots$ and a sequence $\{i_k\}$ with $i_k \in E^*_{m_k}$, k=1,2,... and some $N \geq 1$

$$\sum_{k=1}^{\infty} \min_{i \in A_{n_k}} P^{(m_k,n_k)}_{i_k,j} = \infty \tag{7.15}$$

and $p^{(n)}_{i,j} = 0$ for $i \in A_n$ and $j \in A_{n+1}$ with $n \geq N$. Then

(i) $P^{(m)}(X_{n_k} = j_k \text{ i.o.}) > 0$ for $j_k \in A_{n_k}$, k=1,2,...

(ii) $$\lim_{n\to\infty} P^{(m,n)}_{i,j}/\pi^{(n)}_j = \frac{P^{(m)}(\bigcup_{n=N}^{\infty}\{X_n \in A_n\}\,|\,X_m = i)}{P^{(m)}(\bigcup_{n=N}^{\infty}\{X_n \in A_n\})}$$

for $i \in S$, m=0,1,..., $j \in A_n$, n=1,2,...

PROOF. (i) Let us consider the subchain $X_{m_1}, X_{n_1}, X_{m_2}, X_{n_2}, \ldots$ We shall show that $\lim_{k\to\infty}\{X_{n_k} \in A_{n_k}\}$ a.s. is a $P^{(m)}$-atomic set of $\mathcal{J}'^{(m)}$, where $\mathcal{J}'^{(m)}$ is the tail σ-field of $\{X_{m_1}, X_{n_1}, \ldots\}$. Since $p^{(n)}_{i,j} = 0$ for $i \in A_n$ and $j \in A_{n+1}$ with $n \geq N$, we get $\bigcup_{n=N}^{\infty}\{X_n \in A_n\} = \lim_{k\to\infty}\{X_{n_k} \in A_{n_k}\}$ $P^{(m)}$ a.s. If $\lim_{k\to\infty}\{X_{n_k} \in A_{n_k}\}$ is not a $P^{(m)}$-atomic set of $\mathcal{J}'^{(m)}$ then it must be a union of atomic sets of $\mathcal{J}'^{(m)}$ In the latter case, by Proposition 2.1, there must exist sequences $\{A^{(1)}_n, \ldots, A^{(v)}_{n'}\}$ with $n' \in \{m_1, n_1, m_2, n_2, \ldots\}$ such that $\lim_{n'\to\infty}\{X_{n'} \in A^{(r)}_{n'}\} = T^{(r)}$ (say) $P^{(m)}$ a.s. for r=1,...,v. Further, since $i_k \in E^*_{m_k}$ there must be a number M such that $i_k \in \bigcup_{r=1}^{v} A^{(r)}_{m_k}$ for $k \geq M$. But $\{A^{(1)}_{n'}, \ldots, A^{(v)}_{n'}\}$ are disjoint for all n and therefore there exist the sets $\Lambda_1, \ldots, \Lambda_v$ with $\Lambda_r = \{m_k : i_k \in A^{(r)}_{n_m}\}$, r=1,...,v and $\bigcup_{r=1}^{v}\Lambda_r \supseteq \{m_k : k \geq M\}$. Take $j_k \notin A^{(r)}_{n_k}$ for $m_k \in \Lambda_r$. Since

$$\sum_{k=M}^{\infty} P^{(m_k,n_k)}_{i_k,j_k} \leq \sum_{r=1}^{v}\sum_{m_k \in \Lambda_k} P^{(m_k,n_k)}_{i_k,j_k} \quad \text{if we apply Lemma 7.1 we get that}$$

$\sum_{m_k \in \Lambda_k} P^{(m_k,n_k)}_{i_k,j_k} < \infty$ for r=1,...,v and therefore $\sum_{k=M}^{\infty} P^{(m_k,n_k)}_{i_k,j_k} < \infty$ which contradicts (7.15). Hence $\lim_{k\to\infty}\{X_{n_k} \in A_{n_k}\}$ a.s. is a $P^{(m)}$-atomic set of $\mathcal{J}'^{(m)}$. Further $\lim_{k\to\infty}\{X_{n_k} \in A_{n_k}\} = T'$ a.s. is also a $P^{(m)}$-atomic set of $\mathcal{J}^{(m)}$, since otherwise $\mathcal{T}^{(m)}$ would contain at least two $P^{(m)}$-atomic sets, T_1 and T_2. If $\{E_n^{(1)}\}$ and $\{E_n^{(2)}\}$ are some sequences corresponding to T_1 and T_2 such that $\lim_{n\to\infty}\{X_n \in E_n^{(1)}\} = T_1$ $P^{(m)}$ a.s. and $\lim_{n\to\infty}\{X_n \in E_n^{(1)}\} = T_1$ $P^{(m)}$ a.s. and $\lim_{n\to\infty}\{X_n \in E_n^{(2)}\} = T_2$ $P^{(m)}$ a.s., we also get $\lim_{k\to\infty}\{X_{n_k} \in E_{n_k}^{(1)}\} = T_1$ $P^{(m)}$ a.s. and $\lim_{k\to\infty}\{X_{n_k} \in E_{n_k}^{(2)}\} = T_2$ $P^{(m)}$ a.s. contradicting the atomicity of T' with respect to $\mathcal{T}'^{(m)}$.

Suppose now that $j_k \in E^*_{n_k}$ for an infinity of k's. Then $\{X_{n_k} = j_k \text{ i.o.}\}$ has positive $P^{(m)}$ probability. But $j_k \in A_{n_k}$ and as seen before $\lim_{k\to\infty}\{X_{n_k} \in A_{n_k}\} = T_u$ $P^{(m)}$ a.s., which leads to $\{X_{n_k} = j_k \text{ i.o.}\} = T_u$ $P^{(m)}$ a.s. If $j_k \in E^{**}_{n_k}$ for k large we get (i) by applying Lemma 7.1. Indeed, $i_k \in E^*_{m_k} \cap A_{m_k}$ for k sufficiently large and because $\lim_{k\to\infty}\{X_{m_k} \in A_{m_k}\} = \lim_{k\to\infty}\{X_{m_k} \in E^{(u)}_{m_k}\} = T_u$ $P^{(m)}$ a.s. we get that $i_k \in E^*_{m_k} \cap E^{(u)}_{m_k}$ for k sufficiently large and (i) follows.

To prove (ii) we shall first prove it for a subsequence $\{n'_k : k=1,2,\dots\}$ i.e. we shall show that $A_{n'_k} \cap E^{(u)}_{n'_k} = A_{n'_k}$ for k sufficiently large. Notice that $A_{n_k} \cap E_{n_k}$ is not empty for an infinity of k's. Indeed, by (7.15) and Lemma 7.1 it is impossible for this intersection to be nonempty for all k sufficiently large, since then (i) would imply the positivity of $P^{(m)}(X_{n_k} \in A_{n_k} \cap E_{n_k} \text{ i.o.})$ contradicting $P^{(m)}(X_{n_k} \in E_{n_k} \text{ i.o.}) = 0$. Therefore there must exist a subsequence of $\{n_k\}$, say $\{n'_k\}$, such that $A_{n'_k} \cap E^{(u)}_{n_k} = A_{n'_k}$ for $k=1,2,\dots$ We shall further show that using the existence of such a subsequence $\{n'_k\}$ we deduce (ii).

We first prove (ii) for $m \geq N$ and $i \in A_m$. Write $\gamma_{m,i} = P^{(m)}(T_u | X_m = i)/P^{(m)}(T_u)$ and take $j \in A_n$ to get

$$\left| \frac{P^{(m,n)}_{i,j}}{\pi^{(n)}_j} - \gamma_{m,i} \right| = \left| \frac{\sum_{\ell=1}^{s} P^{(m,n'_k)}_{i,\ell} P^{(n'_k,n)}_{\ell,j}}{\sum_{\ell=1}^{s} \pi^{(n'_k)}_\ell P^{(n'_k,n)}_{\ell,j}} - \gamma_{m,i} \right| \tag{7.16}$$

$$= \left| \frac{\sum_{\ell\in A_{n'_k}} (P^{(m,n'_k)}_{i,\ell} - \gamma_{m,i}\pi^{(n'_k)}_\ell) P^{(n'_k,n)}_{\ell,j}}{\sum_{\ell\in A_{n'_k}} \pi^{(n'_k)}_\ell P^{(n'_k,n)}_{\ell,j}} \right|$$

$$= \left| \sum_{r\epsilon A_{n_k'}}' \frac{(P_{i,r}^{(m,n_k')} - \gamma_{m,i}\pi_r^{(n_k')})P_{r,j}^{(n_k',n)}}{\sum_{\ell\epsilon A_{n_k'}} \pi_\ell^{(n_k')} P_{\ell,j}^{(n_k',n)}} \right|$$

$$\leqslant \sum_{r\epsilon A_{n_k'}}' \left| \frac{P_{i,r}^{(m,n_k')} - \gamma_{m,i}\pi_r^{(n_k')}}{\pi_r^{(n_k')}} \right|$$

where the prime in the last two sums indicates that the sum is restricted to the values r such that $P_{r,j}^{(n_k',n)} > 0$. Now if $n \to \infty$ we may take $k \to \infty$ and Theorem 6.1 implies that the last sum in (7.16) goes to 0 as $k \to \infty$.

We prove now (ii) for arbitrary m and i. Consider the conditional probabilities

$$P_{i,j}^{*(m,r)} = P^{(m)}(X_{m'+1} \notin A_{m'+1},\dots,X_{r-1} \notin A_{r-1}, X_r = j \,|\, X_m = i) \tag{7.17}$$

for $j \in A_\ell$, $\ell > N$, $m' = \max(n-1,m)$ and $r > m' + 1$, and

$$P_{i,j}^{*(m,m+1)} = P_{i,j}^{(m,m+1)} \tag{7.18}$$

Since $\{X_n \in A_n\} \subseteq \{X_{n+1} \in A_{n+1}\}$ for $n \geqslant N$, a slight modification of a standard reasoning from the theory of homogenoeus chains yields

$$P_{i,j}^{(m,n)} = \sum_{\ell=m'+1}^{n-1} \sum_{k\epsilon A_\ell} P_{i,k}^{*(m,\ell)} P_{k,j}^{(\ell,n)} + P_{i,j}^{*(m,n)} \tag{7.19}$$

We recall now that for $\ell \geqslant N$ and $k \in A_\ell$ we have already shown that $\lim_{n\to\infty} \frac{P_{k,j}^{(\ell,n)}}{\pi_j^{(n)}} = \gamma_{\ell,k}$ where $\gamma_{\ell,k} = \frac{1}{P^{(m)}(T_u)}$. It follows that for an arbitrary N' with $N' > m' + 1$

$$\lim_{n\to\infty} \frac{\sum_{\ell=m'}^{N'} \sum_{k\epsilon A_\ell} P_{i,k}^{*(m,\ell)} P_{k,j}^{(\ell,n)}}{\pi_j^{(n)}} = \frac{P^{(m)}(\bigcup_{\ell=m'}^{N'} \{X_\ell \in A_\ell\} \,|\, X_m = i)}{P^{(m)}(T_u)} \tag{7.20}$$

Because N' was arbitrarily chosen (7.19) and (7.20) together imply

$$\liminf_{n\to\infty} \left\{ \frac{P_{i,j}^{(m,n)}}{\pi_j^{(n)}} - \frac{P^{(m)}(T_u|X_m=i)}{P^{(m)}(T_u)} \right\} \geqslant 0 \tag{7.21}$$

for any m and i.

Further

$$\sum_{i\epsilon S} \pi_i^{(m)} \frac{P_{i,j}^{(m,n)}}{\pi_j^{(n)}} = \sum_{i\epsilon S} \pi_i^{(m)} \frac{P^{(m)}(T_u|X_m=i)}{P^{(m)}(T_u)} = 1$$

and (7.21) yields

$$\lim_{n\to\infty} \frac{P_{i,j}^{(m,n)}}{\pi_j^{(n)}} = \frac{P^{(m)}(T_u | X_m = i)}{P^{(m)}(T_u)}$$

completing the proof.

THEOREM 7.3. Suppose that (D_2^*) holds and let $\Omega_1,\ldots,\Omega_d$ be a partition of $\{1,\ldots,d'\}$ and $E_n^{(k)} = \cup_{\alpha\varepsilon\Omega_k} F_n^{(\alpha)}$, k=1,...,d. If for any $k \varepsilon \{1,\ldots,d\}$ there exists a sequence of positive integers $m_1 < n_1 < m_2 < n_2 < \ldots$ and a sequence of states $i_{m_1}^{(k)}, i_{m_2}^{(k)},\ldots$ such that $i_{m_u}^{(k)} \varepsilon E_{m_u}^*$ for u sufficiently large, $\sum_{u=1}^{\infty} \min_{j\varepsilon E_{n_u}^{(k)}} P_{i_{m_u}^{(k)},j}^{(m_u,n_u)} = \infty$ and $\max_{i\varepsilon E_n^{(k)},j\notin E_{n+1}^{(k)}} P_{i,j}^{(n)} = 0$, then

(i) for any $i_n \varepsilon E_n^{(k)}$, $P^{(m)}(X_n = i_n \text{ i.o.}) > 0$, k=1,...,d ;

(ii) for $i \varepsilon S$, m=0,1,... and $j \varepsilon E_n^{(k)}$, n=1,2,...

$$P_{i,j}^{(m,n)} = \pi_j^{(n)} \frac{P^{(m)}(T_k | X_m = i)}{P^{(m)}(T_k)} + o(\pi_j^{(n)})$$

where $T_k = \lim_{n\to\infty}\{X_n \varepsilon E_n^{(k)}\}$ $P^{(m)}$ a.s., k=1,...,d ;

(iii) for any $i \varepsilon S$, m=0,1,... and $j \varepsilon E_n = S - \cup_{k=1}^{d} E_n^{(k)}$, n=1,2,...

$$\lim_{n\to\infty} P_{i,j}^{(m,n)} = 0$$

(iv) if Ω_k contains more than one element and if $F_n^{(\alpha)} \subset E_k^{(n)}$ then the average sojourn time spent in the sequence of sets $\{F_n^{(\alpha)}, F_{n+1}^{(\alpha)},\ldots\}$ given that $X_n = i$ with $i \varepsilon F_n^{(\alpha)}$ goes to ∞ as $n\to\infty$, but $P^{(m)}(\{X_n \varepsilon F_n^{(\alpha)} \text{ ult.}\}) = 0$.

PROOF. It is easy to see that (i) and (ii) follow from Lemma 7.2, whereas (iii) is implied by Theorem 6.2. It remains to prove (iv). Let us denote by $E(Y_\alpha^{(n)} | X_n = i)$ the average sojourn time referred to in the statement of the Theorem. By definition, $Y_\alpha^{(n)} = m$ on the set $\{X_n \varepsilon F_n^{(\alpha)}\} \cap \ldots \cap \{X_{n+m+1} \varepsilon F_{n+m-1}^{(\alpha)}\}$ $\cap \{X_{n+m} \notin F_{n+m}^{(\alpha)}\}$ for m=1,2,... Because Ω_k contains more than one element, and by (i) $P^{(m)}(X_n = i_n \text{ i.o.}) > 0$ for any sequence $\{i_n\}$ with $i_n \varepsilon E_n^{(k)}$, we conclude that $P^{(m)}(Y_\alpha^{(n)} < \infty) = 1$. Further, it is easy to see that since $\sum_{j\varepsilon F_{n+k+1}^{(\alpha)}} P_{i,j}^{\prime(n+k)} = 1$ for $i \varepsilon F_{n+k}^{(\alpha)}$ and any $k \geq 1$, we get $\sum_{j\varepsilon F_{n+m-1}^{(\alpha)}} P_{i,j}^{(n,n+m-1)} = 1$ for $i \varepsilon F_n^{(\alpha)}$ and $m \geq 2$. Taking into account that $|p_{i,j}^{(n)} - p_{i,j}^{\prime(n)}| \to 0$ as $n \to \infty$ for all $i,j \varepsilon S$ we get

$$\lim_{n\to\infty} P^{(n)}(\{X_n \varepsilon F_n^{(\alpha)}\} \cap \ldots \cap \{X_{n+m-1} \varepsilon F_{n+m-1}^{(\alpha)}\} | X_n = i) = 1$$

which implies that $\lim_{n\to\infty} E(Y_\alpha^{(n)} | X_n = i) = \infty$ for $i \varepsilon F_n^{(\alpha)}$ and $\alpha \varepsilon \Omega_k$. Finally, since $P^{(m)}(X_n \varepsilon F_n^{(\alpha)} \text{ i.o.}) = P^{(m)}(X_n \varepsilon (E_n^{(k)} - F_n^{(\alpha)}) \text{ i.o.}) = P^{(m)}(T_k) > 0$ we get that $P^{(m)}(X_n \varepsilon F_n^{(\alpha)}$ ult.) = 0 and the proof is complete.

As a corollary to Theorem 7.3 we shall give a result that describes the asymptotic behaviour of a chain that satisfies Condition (D_2^{**}).

COROLLARY 7.1. Suppose that (D_2^{**}) holds and let $\Omega_1,\Omega_2,\ldots,\Omega_d$ be a partition of $\{(\ell,\alpha),\ \ell=1,\ldots,v\,;\alpha=1,\ldots,d(\alpha)\}$. Let $E_k^{(n)} = \cup_{(\ell,\alpha)\varepsilon\Gamma_k} C_\ell(\alpha)$, $k=1,\ldots,d$ and denote $\varepsilon^{(n)}_{(\ell,\alpha);(\ell',\alpha')} = \max_{i\varepsilon C_\ell(\alpha),\ j\varepsilon C_{\ell'}(\alpha)} P^{(n)}_{i,j}$. Assume that $\sum_{n=1}^{\infty}\min\varepsilon^{(n)}_{(\ell,\alpha);(\ell',\alpha')} = \infty$ where the minimum is taken over all $\{(\ell,\alpha);(\ell',\alpha')\ \varepsilon\,\Omega_k\times\Omega_k\ ,\ k=1,..,d\}$, and that $\varepsilon^{(n)}_{(\ell,\alpha):(\ell',\alpha')} = 0$ for $(\ell,\alpha)\ \varepsilon\,\Omega_k$ and $(\ell',\alpha') \notin \Omega_{k'}$. Then the statement of Theorem 7.3 holds.

We shall omit the proof of this result, which may be carried out by arguments already used in this paper.

We notice that in the case of Condition (D_2'') Doeblin's statement is wrong. However, examining Doeblin's formulae makes it clear that he felt that unlike the previous situations, the limit of the conditional probability that the chain will circulate through the cyclical subclasses of a fixed class may not exist here. The analogy to the homogeneous case seems to break down for the chains satisfying (D_2'') since several atomic sets of the tail σ-field of the associated chain may be lumped into one atomic set of the tail σ-field of the original chain.

Theorem 7.3(iv) generalizes a result stated by Doeblin about chains satisfying Condition (D_2''). It is hard to see how Doeblin could have reached his conclusions in this respect, given the knowledge available at the time his paper was written.

The results of this section, in slightly different form, were given in Cohn [7].

8. WEAK ERGODICITY.

One of the main concerns of the theory of finite stochastic matrices has been to characterize sequences of matrices satisfying the so-called 'weak ergodicity' condition, i.e.

$$\lim_{n\to\infty}(P^{(m,n)}_{i,j} - P^{(m,n)}_{\ell,j}) = 0 \tag{8.1}$$

for any i,j,ℓ and m. This condition has been introduced by Kolmogorov [14] and most papers on nonhomogeneous chains are related to it. Doeblin [9] has found necessary and sufficient conditions for (8.1) and Hajnal [10] has derived similar conditions unaware of Doeblin's results. We shall first give a result that relates weak ergodicity to the structure of the tail σ-field.

THEOREM 8.1. The following conditions are equivalent

(i) weak ergodicity;

(ii) any Markov chain $\{X_n : n \geqslant m\}$ with transition probability $(P_n)_{n\geqslant m}$ and arbitrary initial distribution $\pi^{(m)}$ has a $P^{(m)}$-trivial σ-field $\mathcal{J}^{(m)}$.

PROOF. Since S is finite we may assume, if necessary after relabelling the states at successive times $n=0,1,\ldots$, that there is a positive state $j\ \varepsilon\ S$.

Then Theorem 3.2 and Remark 3.1 imply that

$$\lim_{\substack{n'\to\infty \\ n'\in\Gamma_k}} P_{i,j}^{(m,n')}/P_{\ell,j}^{(m,n')} = P^{(m)}(T_k|X_m=i)/P^{(m)}(T_k|X_m=\ell) \tag{8.2}$$

If (8.1) holds then $P^{(m)}(T_k|X_m=i)/P^{(m)}(T_k|X_m=\ell) = 1$ necessarily follows by (8.2). However, if $\mathcal{J}^{(m)}$ is not $P^{(m)}$-trivial this is not possible as by the martingale convergence theorem $P^{(m)}(T_k|X_n=i)$ must have values close to 0 for some i in view of $\lim_{n\to\infty}P^{(m)}(T_k|X_n) = 1_{T_k}$ $P^{(m)}$ a.s. Thus $\mathcal{J}^{(m)}$ is $P^{(m)}$-trivial. Suppose now that $\mathcal{J}^{(m)}$ is $P^{(m)}$-trivial. Then by Theorem 6.2 we know that

$$\lim_{n\to\infty} P_{i,j}^{(m,n)}/P_{\ell,j}^{(m,n)} = 1 \tag{8.3}$$

for $i \notin E_n$. But $\lim_{n\to\infty}P^{(m)}(X_n \in E_n) = 0$ for all m and (8.1) follows.

THEOREM 8.2. Let (P_n) be a sequence of finite stochastic matrices. The following two conditions are equivalent:

(i) weak ergodicity;

(ii) there exists a sequence of sets $\{E_n^{(1)}\}$ such that for $i,\ell \in S$ and $m \in N$

$$\lim_{n\to\infty} P_{i,j}^{(m,n)}/P_{\ell,j}^{(m,n)} = 1$$

for $j \in E_n^{(1)}$, and for any $i \in S$ and $m=0,1,\ldots$

$$\lim_{n\to\infty} \sum_{j\in E_n^{(1)}} P_{i,j}^{(m,n)} = 1$$

This result is a consequence of Theorems 6.2 and 8.1.

A classical type of results in the theory of nonhomogeneous Markov chains establishes weak ergodicity in terms of some coefficients attached to a stochastic matrix. A historical account of such coefficients, that goes back to Doeblin, may be found in Seneta [20]. Kingman [13] has proven a general result of this kind. Usually, the proof is carried out by some inequalities relating the coefficients of the product of two matrices to the coefficients of the matrices themselves. For example, Hajnal [10] considered the following coefficient attached to a matrix P with entries $p_{\alpha,\beta}$, $1 \leq \alpha \leq s$, $1 \leq \beta \leq s$

$$\{P\} = \min_{\alpha,\alpha'} \sum_{\beta=1}^{s} \min(p_{\alpha,\beta}, p_{\alpha',\beta})$$

We show next that such results are immediate consequences of the results given in this paper by proving the following theorem due to Hajnal [10].

THEOREM 8.3. A sequence of stochastic matrices (P_n) is weakly ergodic if there exists an increasing sequence of positive integers $n_1, n_2, \ldots$ such that $\sum_j \{P_{n_j,n_{n+1}}\}$ diverges.

PROOF. According to Theorem 8.1, (P_n) is not weakly ergodic if the tail σ-field $\mathcal{J}^{(m)}$ is not $P^{(m)}$-trivial and thus assumes at least two $P^{(m)}$-atomic sets. According to Lemma 7.1 we must have

$\sum_{j=1}^{\infty} P_{\alpha,\beta}^{(n_j,n_{j+1})} < \infty$ whenever $\alpha \in E_{n_j}^{(k)} \cap E_n^*$ and $\beta \in E_{n_{j+1}}^{(k')}$ with $k \neq k'$ which makes $\sum_j \{P_{n_j,n_{j+1}}\}$ convergent for any sequence $n_1 < n_2 < \ldots$, and finishes the proof.

There is an important result for bounded positive matrices known in the demographic literature as the Coale-Lopez theorem (see Seneta [21]). The result was given in a somewhat more general form in Seneta [21] and its proof seems rather laborious. The specialization of the Coale-Lopez theorem to the case of stochastic matrices reveals a strong asymptotic independence property. We shall state such a property under a less restrictive assumption on the stochastic matrices.

THEOREM 8.4. Let (P_n) be a weakly ergodic sequence of stochastic matrices such that $\liminf_{n\to\infty} \max_{i\in S} P_{i,j}^{(o,n)} > 0$ for any $j \in S$. Then for all $i,\ell \in S$

$$\lim_{n\to\infty} P_{i,j}^{(m,n)}/P_{\ell,j}^{(m,n)} = 1 .$$

The proof follows easily from Theorem 3.2 and Remark 3.1 in view of the fact that $\{E_n\}$ are empty.

The Lopez theorem imposes the condition $P_{i,j}^{(n,n+r_o)} \geq \delta > 0$ for a certain r_o and δ and any n. This clearly implies weak ergodicity, since as seen in the course of the proof of Theorem 8.3, the failure of weak ergodicity prevents $P_{i,j}^{(n,n+r_o)}$ from being bounded away from 0 for all i,j and n.

The results of this section were derived in Cohn [5].

REFERENCES

1. BLACKWELL, D. Finite non-homogeneous Markov chains, Ann. Math. 46, 594-599, 1945.
2. COHN, H. On the tail σ-algebra inhomogeneous Markov chains, Ann. Math. Statist. 41, 2175-2176, 1970.
3. COHN, H. On the tail events of a Markov chain, Z. Wahrscheinlich. 29, 65-72, 1974.
4. COHN, H. A ratio limit theorem for the finite nonhomogeneous Markov chains, Israel J. Math. 19, 329-334, 1974.
5. COHN, H. Finite non-homogeneous Markov chains: Asymptotic behaviour, Adv. Appl. Probab. 8, 502-516, 1976.
6. COHN, H. Countable non-homogeneous Markov chains: Asymptotic behaviour, Adv. Appl. Probab. 9, 542-552, 1977.
7. COHN, H. On a paper by Doeblin on non-homogeneous Markov chains, Adv. Appl. Probab. 13, 388-401, 1981.
8. COHN, H. On a class of non-homogeneous Markov chains, Math. Proc. Camb. Phil. Soc. 92, 527-534, 1982.
9. DOEBLIN, W. Le cas discontinu des probabilités en chaîne, Pub. Fac. Sci. Univ. Massryk (Brno) 236, 1937.
10. HAJNAL, J. Weak ergodicity in non-homogeneous Markov chains, Proc. Camb. Phil. Soc. 54, 233-246, 1958.
11. IOSIFESCU, M. On finite tail σ-algebras, Z. Wahrscheinlich. 24, 159-166, 1982.
12. KEMENY, J.G., SNELL, L.J., and KNAPP, A.W. Denumerable Markov Chains, Springer, New York, 1976.

13. KINGMAN, J.F.C. Geometrical aspects of the theory of non-homogeneous Markov chains, Math. Proc. Camb. Phil. Soc. 77, 171-185, 1975.

14. KOLMOGOROV, A.N. Uber die analytischen Methoden in der Wahrscheinlichkeitsrechnung, Math. Ann. 104, 415-418, 1931.

15. MAKSIMOV, V.M. Convergence of non-homogeneous biostochastic Markov chains, Theor. Probab. Appl. 15, 604-618, 1970.

16. MUKHERJEA, A. Limit theorems: Stochastic matrices, erdogic Markov chains, and measures on semigroups. Probabilistic Analysis and Related Topics, Vol. 2, 143-203, Academic Press.

17. MUKHERJEA, A. A new result on the convergence of non-homogeneous stochastic chains, Trans. Amer. Math. Soc. 262, 505-520, 1980.

18. MUKHERJEA, A. and CHAUDHURI, R. Convergence of non-homogeneous stochastic chains II, Math. Proc. Camb. Phil. Soc. 90, 167-182, 1981.

19. SENCHENKO, D.V. The final σ-algebra of an inhomogeneous Markov chain with finite number of states, Math. Notes 12, 610-613, 1972.

20. SENETA, E. On the historical development of the theory of finite nonhomogeneous Markov chains, Proc. Camb. Phil. Soc. 74, 507-513, 1973.

21. SENETA, E. Non-negative Matrices, George Allen and Unwin Ltd., London, 1973.

22. WINKLER, G. The number of phases in inhomogeneous Markov fields with finite state spaces on N and Z and their behaviour at infinity, Math. Nachr. 104, 101-117, 1981.