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COMPLETENESS OF REGULAR INDUCTIVE LIMITS

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ABSTRACT. Regular LB-space is fast complete but may not be quasi-complete. Regular inductive limit of a sequence of fast complete, resp. weakly quasi-complete, resp. reflexive Banach, spaces is fast complete, resp. weakly quasi-complete, resp. reflexive complete, space.

KEY WORDS AND PHRASES. Regular locally convex inductive limit, complete, quasi-complete, fast complete space.
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1. INTRODUCTION.

In [1, §31.6] Kothe has a sequence of Banach spaces $E_1 \subset E_2 \subset \ldots$ whose inductive limit is not quasi-complete. In [2] there is an example of reflexive Frechet spaces E_n whose inductive limit is not even fast complete. Since an LF-space is fast complete iff it is regular, see [3], there is a natural question asked by Jorge Mujica in [4]: Is every regular LB-space complete?

Throughout the paper $E_1 \subset E_2 \subset \ldots$ is a sequence of locally convex spaces with continuous inclusions $E_n \to E_{n+1}$, $n \in N$. Their locally convex inductive limit is denoted by E. The space E is called regular if every set bounded in E is bounded in some E_n.

2. MAIN RESULTS.

Let F be a locally convex space and $A \subset F$ absolutely convex. We denote by F_A the seminormed space $\cup\{nA; n\varepsilon N\}$ whose topology is generated by the Minkowski functional of A. If F_A is Banach space, A is called Banach disk. The space F is called fast complete if every set bounded in F is contained in a bounded Banach disk. Every sequentially complete space is fast complete and there are fast complete spaces which are sequentially incomplete, see [5].

EXAMPLE. For each $n \varepsilon N$ and $x : NxN \to C$, put

$$\| x \|_n = \max \{\sup\{j^{-i}|x_{ij}|;\ i \leq n,\ j \varepsilon N\},\ \sup \{|x_{ij}|;\ i > n,\ j \varepsilon N\}\},$$

$E_n = \{x;\ \| x \|_n < +\infty \ \&\ \lim_{j \to \infty} x_{ij} = 0 \text{ for } i > n\}$, $B_n = \{x \varepsilon E_n;\ \| x \|_n \leq 1\}$, and

$E = \text{indlim } E_n$. We prove that each E_n is a Banach space, $E_1 \subset E_2 \subset \dots$, inclusions $E_n \to E_{n+1}$, $n \in N$, are continuous, E is regular and not quasi-complete.

CLAIM 1. Each space E_n is Banach.

PROOF. Let $\{x(k)\}$ be a Cauchy sequence in E_n. For each $i, j \in N$ the sequence $\{x(k)_{ij}\}$ is Cauchy in C and has a limit x_{ij}. Let x be the matrix with the entries x_{ij}. Given $\varepsilon > 0$, there is k such that $p, r \geq k$ implies $\|x(p) - x(r)\|_n \leq \varepsilon$. Hence,

$\|x(p) - x\|_n \leq \limsup_{r \to \infty} \|x(p) - x(r)\|_n \leq \varepsilon$ and

$\|x\|_n \leq \|x - x(p)\|_n + \|x(p)\|_n < +\infty$.

Take $i > n$ and choose j_i so that $|x(k)_{ij}| < \varepsilon$ for $j > j_i$.

Then $|x_{ij}| \leq |x_{ij} - x(k)_{ij}| + |x(k)_{ij}| \leq \|x - x(k)\|_n + |x(k)_{ij}| < 2\varepsilon$ and

$\lim_{j \to \infty} x_{ij} = 0$.

CLAIM 2. $E_1 \subset E_2 \subset \dots$ and each inclusion $E_n \to E_{n+1}$ is continuous. Proof follows from the inequalities $\|x\|_1 \geq \|x\|_2 \geq \dots$, $x \in \cup \{E_n; n \in N\}$.

CLAIM 3. E is regular.

PROOF. Let $D \subset E$ be not bounded in any E_n. For each $n \in N$ choose $x(n) \in D$ such that $\|x(n)\|_n > n$. There are $i(n), j(n) \in N$ for which

$$|x(n)_{i(n),j(n)}| > \begin{cases} nj(n)^{i(n)} & \text{if } i(n) \leq n \\ n & \text{if } i(n) > n \end{cases}$$

Put $m(n) = n + \max\{i(k); k \leq n\}$ and $r(n) = \min\{j(k)^{-i(k)}; k \leq m(n)\}$, $n \in N$.

If $k > n$ then $\|x(n)\|_{m(k)} \geq |_{m(k)} \geq |j(n)^{-i(n)} x(n)_{i(n)j(n)}| > nr(k)$,

if $k \leq n$ then $\|x(n)\|_{m(k)} \geq \|x(n)\|_{m(n)} > nr(k)$.

Let $V = \cup \{r(k)B_{m(k)}; k \in N\}$ and $U = \text{co}V$. Assume $x(n) \in nU$. Then

$$x(n) = \sum_{k=1}^{s} \alpha_k y(k),$$

where $\alpha_k \geq 0$, $\Sigma \alpha_k = 1$, and $y(k) \in nr(k) B_{m(k)}$. To prove that $|y(k)_{i(n)j(n)}| \leq n$ for $k \in N$, we have to distinguish three cases:

(a) $k > n$: Then $|y(k)_{i(n),j(n)}| = |y(k)_{i(n),j(n)} j(n)^{i(n)-i(n)}| \leq$

$\leq \|y(k)\|_{m(k)} j(n)^{i(n)} \leq nr(k)j(n)^{i(n)} \leq n$.

(b) $k \leq n$ & $i(n) \leq m(k)$: Then $|y(k)_{i(n),j(n)}| \leq \| y(k) \|_{m(k)}\, j(n)^{i(n)} \leq$
$\leq nr(k)j(n)^{i(n)} \leq n$.

(c) $k \leq n$&$i(n) > m(k)$: Then $|y(k)_{i(n),j(n)}| \leq \|y(k)\|_{m(k)} \leq nr(k) \leq n$.

On the other hand $|x(n)_{i(n),j(n)}| > n$ and $x(n)$ cannot be a convex combination of $y(k)$, $k \leq s$, i.e. $x(n) \notin nU$. Since U is a 0-neighborhood in E, D is not bounded in E.

CLAIM 4. E is not quasi-complete.

PROOF. Let $\Delta = \{\delta \subset N\times N;\ \{j \in N;\ (i,j) \in \delta\}$ is finite, $i \in N\}$ be ordered by set inclusion. Denote by $x(\delta)$ the set characteristic function of $\delta \in \Delta$. Then $\{x(\delta);\ \delta \in \Delta\} \subset B_1$ and the filter associated with $\delta \to x(\delta)$ is bounded in E_1, hence also bounded in E.
Let P_n: $C^{N\times N} \to C^{N\times N}$ be the projection of an $N\times N$ matrix on its n-th row. Take a a closed absolutely convex 0-neighborhood V in E. For each $n \in N$ choose $m(n) \in N$ and $r(n) > 0$ so that $r(n)B_n \subset V$, $m(n) \geq 2r(n)^{-1/n}$, and put $\sigma = \{(i,j) \in N\times N;\ j \leq m(i)\}$. If $\gamma, \delta \in \Delta$, $\gamma, \delta \geq \sigma$, then $x(\gamma)_{ij} - x(\delta)_{ij} = 0$ for $j \leq m(i)$ and $\|P_n(x(\gamma) - x(\delta))\|_n = \sup\{j^{-n}|\ x(\gamma)_{nj} - x(\delta)_{nj}\ |; j > m(n)\} < m(n)^{-n} \leq 2^{-n}r(n)$. Hence $2^nP_n(x(\gamma) - x(\delta)) \in r(n)\ B_n \subset V$. Since V is absolutely convex, the sequence

$$y_k = \sum_{n=1}^{k} 2^{-n}\ 2^{+n}P_n(x(\gamma) - x(\delta)),\ k \in N$$

is contained in V. It is also contained in B_1 and converges coordinate-wise to $x(\gamma) - x(\delta)$ in E_1. Hence $x(\gamma) - x(\delta)$ is in the weak closure of V. Since V is closed and convex, it is also weakly closed and $x(\gamma) - x(\delta) \in V$. So $\{x(\delta);\ \delta \in \Delta\}$ is a base of a bounded Cauchy filter in E. If it had a limit $x \in E$, then $x_{ij} = 1$ for all $i, j \in N$. This would imply $x \notin E_n$ for any $n \in N$ and $x \notin E$, q.e.d.

LEMMA. Regular inductive limit of a sequence of semireflexive, resp. reflexive, spaces is semireflexive, resp. reflexive.

PROOF. Let each E_n be semireflexive. Since $E = \text{indlim}\ E_n$ is regular, its strong dual E'_b equals to projlim $(E_n)'_b$ and $(E'_b)' \subset \cup\ \{((E_n)'_b)';\ n \in N\} = \cup\ \{E_n;\ n \in N\} = E$.

Let each E_n be reflexive. By [7;IV, 5.6] it suffices to show that E is semireflexive and barreled. Take a barrel B in E. For each $n \in N$, $B \cap E_n$ is a barrel in E_n. Since E_n is reflexive, the barrel $B \cap E_n$ is a neighborhood in E_n, which implies that B is a neighborhood in E and E is barreled.

CONSEQUENCE. Inductive limit of a sequence of reflexive Banach spaces is reflexive.

PROOF. By [6; Th. 4] the inductive limit of reflexive Banach spaces is regular.

THEOREM. Let $E = \text{indlim}\, E_n$ be regular. Then:

(a) Each E_n fast complete $\Rightarrow$ E fast complete.

(b) Each E_n weakly quasi-complete $\Rightarrow$ E weakly quasi-complete.

(c) Each E_n semireflexive $\Rightarrow$ E quasi-complete.

(d) Each E_n reflexive Banach $\Rightarrow$ E complete.

PROOF.

(a) Let $B \subset E$ be bounded, then it is bounded in some E_n and contained in a bounded Banach disk in E_n. Since any Banach disk bounded in E_n is also bounded in E, the proof is complete.

(b) Follows from Lemma since any locally convex space is weakly quasi-complete iff it is semireflexive, [7;IV, 5.5].

(c) Follows from (b) since every weakly quasi-complete space is quasi-complete.

(d) Let $\mathcal{F}$ be a Cauchy filter in E. Then $\mathcal{F}$, as a filter of continuous linear functionals on E_b', converges uniformly on bounded sets in E_b' to a linear, not necessarily continuous, functional $h: E_b' \to C$. Since E is reflexive, it suffices to show that h is continuous.

h is continuous iff $h^{-1}(0)$ is closed in E_b'. The space E is regular, [6; Th. 4], hence $E_b' = \text{projlim}\, E_n'$ is Frechet. Take a sequence $\{x_n;\ n = 1,2...\} \subset h^{-1}(0)$ which converges to x_o in E_b'. We have to show that $h(x_o) = 0$. Choose $\varepsilon > 0$. The set $B = \{x_n;\ n = 0,1,2,...\}$ is bounded in E_b', hence there is $F \in \mathcal{F}$ such that

$\sup\{|f(x_n) - h(x_n)|;\ f \in F,\ x_n \in B\} < \varepsilon$. Fix an $f \in F$ and choose $n \in N$ so that $|f(x_n) - f(x_o)| < \varepsilon$. Then $|h(x_o)| = |h(x_o) - h(x_n)| \le |h(x_o) - f(x_o)| + |f(x_o) - f(x_n)| + |f(x_n) - h(x_n)| < 3\varepsilon$, which implies $h(x_o) = 0$.

CONJECTURE. Regular LB-space may not be sequentially complete.

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