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SET-SET TOPOLOGIES AND SEMITOPOLOGICAL GROUPS

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ABSTRACT. Let G be a group with binary operation,·. Let T be a topology for G such that for all $g \in G$ the maps, m_g: $G \to G$ and ${}_g m$: $G \to G$, defined by $m_g(f) = f \cdot g$ and ${}_g m(f) = g \cdot f$, respectively, are continuous. Then (G,T) is called a semitopological group. Some specific set-set topologies for function spaces are discussed and the concept of topologically determined collections of sets is introduced and used to classify some set-set topologies as semitopological groups.

KEY WORDS AND PHRASES. Point-open topology, compact-open topology, g-topology, B-topology, topologically determined collection of sets.
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1. INTRODUCTION.

Husain [1] defined a semitopological group as a group, G, with a binary operation,·, and a topology, T, such that both right and left "multiplication" are continuous. We shall define the concept of topologically determined collections of sets and present some results which help classify some set-set topologies on function spaces as semitopological groups.

2. SOME TOPOLOGIES FOR FUNCTION SPACES.

Before beginning our discussion of semitopological groups, we present the definitions of some topologies for function spaces to which we will be referring.

DEFINITION 1. [2] Let (X,T) and $(Y,\tilde{T})$ be topological spaces. For $p \in X$ and for $U \in \tilde{T}$, define the set $(\{p\}, U) = \{f \in Y^X: f(p) \in U\}$. Then, we define $S = \{(\{p\},U): p \in X \text{ and } U \in \tilde{T}\}$. S is a subbasis for a topology, T_p, on Y^X, called the <u>point-open topology</u>.

DEFINITION 2. Let (X,T) and $(Y,\tilde{T})$ be topological spaces. For $K \subseteq X$ and for $U \subseteq Y$, define the set $(K,U) = \{f \in Y^X: f(K) \subseteq U\}$. Next, we define the sets $S_{co} = \{(K,U): K \text{ is a compact subset of } X \text{ and } U \in \tilde{T}\}$ and $S_g = \{(K,U): U \in \tilde{T} \text{ and } (X \backslash K) \in T, \text{ and either } K \text{ or } (Y \backslash U) \text{ is compact}\}$. Then S_{co} and S_g are subbases for topologies, T_{co} and T_g, respectively, on Y^X. T_{co} is called the <u>compact-open topology</u> [2], while T_g is named the <u>g-topology</u> [3].

These three topologies are related as follows: $T_p \subseteq T_{co}$ and if X is T_2, then $T_{co} \subseteq T_g$. T_2 is needed so that compact sets in X are closed.

DEFINITION 3. Let (X,T) and $(Y,\tilde{T})$ be topological spaces. For $U \subseteq X$ and $V \subseteq Y$, define the set $B(U,V) = \{f \in Y^X: f(U) \cap V \neq \emptyset\}$. Then we set $S = \{B(U,V): U \in T \text{ and } V \in \tilde{T}\}$. Then S is a subbasis for a topology on Y^X called the B topology which will be denoted by T_B.

THEOREM 1. Let (X,T) and $(Y,\tilde{T})$ be topological spaces and let $F \subseteq C(X,Y)$. Then $T_B \subseteq T_p$.

PROOF. Let $B(U,V)$ be a subbasic open set in T_B. Then $B(U,V) = \bigcup_{x \in U} (\{x\},V)$ and $\bigcup_{x \in U} (\{x\},V) \in T_p$. Thus, $T_B \subseteq T_p$.

Note that if X is discrete or T_p is trivial, then $T_p = T_B$. However, the converse of this statement is not true as Example 1 will show.

EXAMPLE 1. Let $X = \{1,2,3,4\}$. Let $T = \{0, X, \{1,2\}, \{3,4\}\}$ be a topology on X. T is obviously not discrete. $(\{1\}, \{3,4\})$ and $(\{3\}, \{3,4\})$ are non-empty subbasic proper open sets in $(C(X,X), T_p)$ and $(\{1\}, \{3,4\}) \neq (\{3\}, \{3,4\})$. So we see that T_p is not trivial. But $T_B = T_p$.

EXAMPLE 2. Let $X = Z_+$ with the cofinite topology, T_{cof}, i.e., $0 \in T_{cof}$ if and only if either $X \backslash 0$ is finite or $X = \phi$. So, if 0, $V \in T_{cof}$ are not empty, 0 and V are infinite and there exists $j \in X$ such that for all $k > j$, $k \in 0 \cap V$. $B(0,V)=H(X)$ if 0 and V are non-empty. $(H(X),T_B)$ is trivial. But $(H(X),T_p)$ is not, since not every homeomorphism belongs to $(\{1\}, \{10,11,12,...\})$. So we see that T_B is not always the same as T_p.

3. SEMITOPOLOGICAL GROUPS.

Note, before we continue, that $H(X)$ is a group with binary operation, o, composition, and identity map, $e(x) = x$, for every $x \in X$.

DEFINITION 4. Let G be a group with binary operation, o. A topology T, for G is called RMC (LMC) provided that, for every $g \in G$, the map $m_g: G \to G$, $({}_gm: G \to G)$, defined by $m_g(f) = fog$, $({}_gm(f) = gof)$, is continuous. Here, "RMC" stands for "right multiplication continuous" and "LMC" for "left multiplication continuous."

The topology of uniform convergence, $T_{\tilde{U}}$, on a subgroup of $H(X)$, is always RMC and is LMC under certain conditions. This topology will give us an example of a topology which is not both RMC and LMC.

THEOREM 2. Let (X,U) be a uniform space. Let G be a subgroup of $H(X)$. Let $\tilde{U}$ be the induced uniformity on G. Then $T_{\tilde{U}}$ is RMC.

PROOF. Recall, given a uniform space, $(X,\mathcal{U})$ for each $U \in \mathcal{U}$, define the set, $\tilde{U} = \{(f,g): (f(x),g(x)) \in U \text{ for all } x \in X\}$. Set $B = \{\tilde{U}: U \in \mathcal{U}\}$. Then B is a basis for a uniformity on G, $\tilde{U}$, which in turn induces a topology, $T_{\tilde{U}}$, on G.

Assume $g \in G$ and let 0 be open in $T_{\tilde{U}}$. Let $f \in m_g^{-1}(0)$, then $fog \in 0$. Hence there exists $U \in \mathcal{U}$ such that $fog \in \tilde{U}[fog] \subseteq 0$. $f \in \tilde{U}[f]$. Now if $h \in \tilde{U}[f]$, $(f(x),h(x)) \in U$, for all $x \in X$. Thus, $(fog(x),hog(x)) \in U$, for all $x \in X$. Hence, $h \circ g \in \tilde{U}[fog] \subseteq 0$, and we have that $h \in m_g^{-1}(0)$. $\tilde{U}[f] \subseteq m_g^{-1}(0)$. Thus, $T_{\tilde{U}}$ is RMC.

DEFINITION 5. [1] Let G be a subgroup of $H(X)$. Let T be a topology for G such that T is both LMC and RMC. Then (G,T) is called a semitopological group. We will denote this by STG.

THEOREM 3. Let $(X,\mathcal{U})$ be a uniform space. Let G be a subgroup of H(X). Let $\check{U}$ be the induced uniformity on G. Then, if $g \in G$ implies that g is uniformly continuous w.r.t. $\mathcal{U}$, then $(G, T_{\check{U}})$ is a STG.

PROOF. Assume $g \in G$ and O is open in G. Let $f \in {}_g m^{-1}(O)$. Then $g \circ f \in O$. Hence there exists a $U \in \mathcal{U}$ such that $gof \in \tilde{U}[gof] = \{h \in G : (gof, h) \in \check{U}\} \subseteq O$. By definition of uniform continuity, there exists $V \in \mathcal{U}$ such that if $(p,q) \in V$ then $(g(p), g(q)) \in U$. Set $W = U \cap V \in U$. Then $gof \in \tilde{W}[gof] \subseteq \tilde{U}[gof] \subseteq O$. $f \in \tilde{W}[f]$. Claim: $\tilde{W}[f] \subseteq {}_g m^{-1}(O)$. If $h \in \tilde{W}[f]$, then $(f(x), h(x))\ W \subseteq V$ for all $x \in X$. So, $(gof(x), goh(x)) \in U$ for all $x \in X$. Hence $goh \in \tilde{U}[gof] \in O$, which means our claim is true and hence $T_{\tilde{U}}$ is LMC. By Theorem 2, $T_{\tilde{U}}$ is RMC, so $(G, T_{\tilde{U}})$ is a STG.

EXAMPLE 3. Let $X = \mathbb{R}$, the reals with the usual uniform structure, i.e., for each $\varepsilon > 0$, we have the basis element $U_\epsilon = \{(x,y) : |x - y| < \varepsilon\}$. Then a basis for the induced uniformity on H(X) is the collection of all sets of the form,
$\tilde{U}_\epsilon = \{(f,g) : (f(x), g(x)) \in U_\epsilon \text{ for all } x \in X\}$.

Let $g(x) = x^3$ and let $e(x) = x$. Let $\varepsilon > 0$ be given. Then since $goe = g$, $e \in {}_g m^{-1}(\tilde{U}_\epsilon[g])$. Now let δ be a positive number, so $e \in \tilde{U}_\delta[e]$. Then define $h(x) = x + \frac{1}{2}\delta$. Hence, $h \in \tilde{U}_\delta[e]$. But $goh(x) = (x + \frac{1}{2}\delta)^3$, which gives that $|goh(x) - g(x)| = |\frac{3}{2}\delta x^2 + \frac{3}{4}\delta^2 x + \frac{1}{8}\delta^3|$ and this function has no maximum on $\mathbb{R}$, hence $h \notin {}_g m^{-1}(\tilde{U}_\epsilon[g])$. So, ${}_g m$ is not continuous. $T_{\tilde{U}}$ is not LMC.

4. TOPOLOGICALLY DETERMINED COLLECTIONS OF SETS.

DEFINITION 6. Let X be a topological space. Let $O \subseteq P(X)$, the collection of all subsets of X, with the property that for each $f \in H(X)$, if $A \in O$ then $f(A) \in O$. Then O is a <u>topologically determined</u> (TD) <u>collection of sets.</u>

THEOREM 4. Let (X,T) be a topological space and let $\tilde{U}$ and $\tilde{V}$ be collections of subsets of X. Let G be subgroup of H(X). Let $S(\tilde{U},\tilde{V}) = \{(U,V) : U \in \tilde{U} \text{ and } V \in \tilde{V}\}$ where $(U,V) = \{f \in G : f(U) \subseteq V\}$. If $S(\tilde{U},\tilde{V})$ is a subbasis for a topology, $T(\tilde{U},\tilde{V})$, on G, and if $\tilde{U}$ and $\tilde{V}$ are TD collections of sets then $(G, T(\tilde{U},\tilde{V}))$ is a semitopological group.

PROOF. Let (U,V) be a subbasic open set in $T(\tilde{U},\tilde{V})$ and let $f \in G$. Assume $g \in m_f^{-1}(U,V)$ and $h \in {}_f m^{-1}((U,V))$. Then $gof(U) \subseteq V$ and $foh(U) \subseteq V$.

$gof(U) \subseteq V$ implies $g \in (f(U),V) \in T(\tilde{U},\tilde{V})$ since $\tilde{U}$ is TD. Suppose that $\theta \in (f(U),V)$ then $\theta of(U) \subseteq V$. So $\theta \in m_f^{-1}((U,V))$. Therefore, $(f(U),(V)) \subseteq m_f^{-1}((U,V))$, which shows that m_f is continuous.

$foh(U) \subseteq V$ gives us that $h \in (U, f^{-1}(V)) \in T(\tilde{U},\tilde{V})$ since $\tilde{V}$ is TD. Notice that if $\gamma \in (U, f^{-1}(V))$ then $\gamma(U) \subseteq f^{-1}(V)$, so $\gamma \in {}_f m^{-1}((U,V))$. Hence $(U, f^{-1}(V)) \subseteq {}_f m^{-1}(U,V))$, and so ${}_f m$ is continuous.

Note from the proof of Theorem 4, that if $(G, T(\tilde{U},\tilde{V}))$ is as defined in Theorem 4, then if $\tilde{U}$ is TD, we have that $(\tilde{U},\tilde{V})$ is RMC. Similarly, if $\tilde{V}$ is TD then $T(\tilde{U},\tilde{V})$ is LMC.

Some of the TD collections of sets for a topological space, X, are:

i) all the open subsets of X

ii) all the closed subsets of X

iii) all the compact subsets of X

iv) all the singleton subsets of X

v) all the connected subsets of X

vi) all the regular open subsets of X

vii) all the regular closed subsets of X

Considering the above list, we have the following:

COROLLARY 4.1. Let X be a topological space and let G be a subgroup of H(X). Then (G,T) is a STG when $T = T_p$ or T_{co}.

PROOF. Theorem 4 along with i, iii, and iv above give us the desired conclusion.

THEOREM 5. If U_1, U_2, V_1, and V_2 are TD collections of subsets of X, then (G,T) is a STG where T is the smallest topology containing both $T(U_1,V_1)$ and $T(U_2,V_2)$. (We denote this by $T = T(U_1,V_1) \vee T(U_2,V_2)$.)

PROOF. T has as a subbasis, S, the union of the subbases $S(U_1,V_1)$ and $S(U_2,V_2)$ of $T(U_1,V_1)$ and $T(U_2,V_2)$ respectively. So if $(U,V) \in S$ then either $(U,V) \in S(U_1,V_1)$ or $(U,V) \in S(U_2,V_2)$, from which our conclusion follows.

COROLLARY 5.1 Let (X,T) be a T_2 topological space. Let G be a subgroup of H(X). Then (G,T_g) is a STG.

PROOF. Note $T_{co} = T(\tilde{K},\tilde{O})$ where $\tilde{K} = \{ K \in P(X) : K \text{ is compact}\}$ and $\tilde{O} = T$. Define $T_h = T(\tilde{C},\tilde{U})$ where $\tilde{C} = \{C \in P(X) : C \text{ is closed}\}$ and $\tilde{U} = \{U \in T: (X\backslash U \text{ is compact}\}$. Then $T_g = T_{co} \vee T_h$. Since $\tilde{K},\tilde{O},\tilde{C}$, and $\tilde{U}$ are TD, we immediately obtain from Theorem 5 that T_g is a STG.

THEOREM 6. Let (X,T) be a topological space and let G be a subgroup of H(X). Let $\tilde{U},\tilde{V}$ be collections of subsets of X. Define, for $U \in \tilde{U}$ and $V \in \tilde{V}$, the set $B(U,V) = \{f \in G: f(U) \cap V \neq \phi\}$ and let $S(B(\tilde{U},\tilde{V})) = \{B(U,V) : U \in \tilde{U} \text{ and } V \in \tilde{V}\}$. If, $S(B(\tilde{U},\tilde{V}))$ is a subbasis for a topology, $T(B(\tilde{U},\tilde{V}))$, on G, and if $\tilde{U}$ and $\tilde{V}$ are TD then $(G,T(B(\tilde{U},\tilde{V}))$ is a STG.

PROOF. Let B(U,V) be a subbasic open set in G and let $f \in G$. Assume that $g \in m_f^{-1}(B\ (U,V))$. By definition of B(U,V), we know that this means that $fog(U) \cap V \neq \phi$. So, $g(U) \cap {}^{f-1}(V) = \phi$. Hence, $g \in B(U,f^{-1}(V)) \subseteq m_f B(U,V))$. Now assume $g \in {}_f m^{-1}(B(U,V))$, then $gof(U) \cap V \neq 0$. Thus, $g \in B(f(U),V) \subseteq m_f^{-1}(B(U,V))$.

From this we have:

COROLLARY 6.1 Let (X,T) be a topological space and let G be a subgroup of H(X). Then (G,T_B) is a STG.

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