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SOLUTIONS OF AN ORDINARY DIFFERENTIAL EQUATION AS A CLASS OF FOURIER KERNELS

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ABSTRACT. While there are various methods of generating Fourier kernels mentioned in the literature it is not so well recognized that Fourier kernels can be obtained as solutions of differential equations. In this note we define a class of Fourier kernels, which are solutions of a k – fold Bessel equation.

KEY WORDS AND NEW PHRASES. Bessel equation, the Bessel functions J_ν, Y_ν, K_ν of order ν, Fourier kernels, Hankel kernel, Mellin transform.

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INTRODUCTION

While there are various methods of generating Fourier kernels mentioned in the literature [1,2,3], it is not so well recognized that Fourier kernels can be obtained as solutions of differential equations. In this note we generate Fourier kernels as solutions of ordinary differential equations and report some kernels which have not been reported in the literature before.

Consider the differential equation

$$D^{2k}f = (-1)^k f, \quad 0 < x < \infty. \tag{1.1}$$

where

$$D^{2k} = \frac{d^{2k}}{dx^{2k}}, \quad k = 1,2,3,\ldots .$$

Note that if $f(x)$ is a solution of this differential equation then $u(x) = f(\lambda x)$ satisfies

$$D^{2k}u = (-\lambda^2)^k u, \quad 0 < x < \infty. \tag{1.2}$$

Now for suitable $u(x)$ and $v(x)$, defined over $0 < x < \infty$, we may write

$$\int_0^\infty u(x)(D^{2k}v(x))dx = (u(x)v^{2k-1}(x) - u^1(x)v^{2k-2}(x) + \cdots + u^{k-1}(x)v^k(x))\Big|_0^\infty + (-1)^k\int_0^\infty u^k(x)v^k(x)dx \tag{1.3}$$

where $u^k(x)$, denotes the k–th derivative of $u(x)$.

Equation (1.3) implies that if we disregard the contributions of the terms on the right hand side as $x \to \infty$, the operator D^{2k} is symmetrical provided $u(x)$ (and $v(x)$), satisfies the following conditions:

Conditions A.

(1) k of the constants $u(0), u^1(0),\dots,u^{2k-1}(0)$ are zero.

(2) The k constants $u^{i_1}(0), u^{i_2}(0),\dots,u^{i_k}(0)$ which are zero are such that no two $i_1,i_2,\dots,i_k$ add up to 2k–1.

It is interesting to note that these are exactly the conditions imposed on the constants by Guinand [1], though his arguments are quite different. For example, if we consider the equation

$$D^6u + u = 0,$$

and look for the solutions of this equation which are bounded at infinity and which satisfy the conditions (A) at zero, we arrive at the same functions as those given by Guinand [1]. The possible combinations of conditions (A) in this case are:

$u = u^{(1)} = u^{(2)} = 0$; $u = u^{(1)} = u^{(3)} = 0$; $u = u^{(2)} = u^{(4)} = 0$; $u = u^{(3)} = u^{(4)} = 0$;

$u^{(1)} = u^{(2)} = u^{(5)} = 0$; $u^{(1)} = u^{(3)} = u^{(5)} = 0$; $u^{(2)} = u^{(4)} = u^{(5)} = 0$ and $u^{(3)} = u^{(4)} = u^{(5)} = 0$.

These are precisely the combinations cited by Guinand [1] for the case k = 3. Similarly for other values of k.

2. We now consider a more general equation

$$\left[D^2 - \frac{\nu^2 - 1/4}{x^2}\right]^k u = (-1)^k u, \tag{2.1}$$

the k–fold Bessel equation.

Let

$$u = \sqrt{x}\, K_\nu(\theta x),$$

where θ is a constant and K_ν is the MacDonald function. Then

$$\left[D^2 - \frac{\nu^2 - 1/4}{x^2}\right]u = \theta^2 u,$$

and hence

$$\left[D^2 - \frac{\nu^2 - 1/4}{x^2}\right]^k u = \theta^{2k}u, \quad k = 1,2,3,\dots .$$

Now if we set $\theta = i\, e^{im\pi/k}$, $\quad 0 \le m \le 2k-1$,

then the function

$$u(x) = \sqrt{x}\, K_\nu(i\, e^{im\pi/k} x), \qquad 0 \le m \le 2k-1$$

satisfies the differential equation (2.1) above. The general solution of (2.1) is given by

$$U_{\nu,k}(x) = \sum_{m=0}^{2k-1} B_m \sqrt{x}\, K_\nu(i\, e^{im\pi/k} x) \tag{2.2}$$

Now we look for the solutions which satisfy the following conditions:

Conditions B

(1) $U_{\nu,k}$ is bounded as $x \to \infty$. (B_1)

(2) $U_{\nu,k}(0) = U^{(1)}_{\nu,k}(0) = U^{(2)}_{\nu,k}(0) = \cdots = U^{(k-1)}_{\nu,k}(0) = 0.$ (B_2)

(3) $\int_0^\infty \frac{1}{x^{2k}} U_{\nu,k}(\lambda x) U_{\nu,k}(\mu x)dx$ exists (B_3)

We shall show that the function $U_{\nu,k}(x)$, which is the general solution of the differential equation (2.1), will generate some interesting Fourier kernels for some particular values of k, provided $U_{\nu,k}(x)$ satisfies conditions B.

In our analysis below, we shall make use of the fact that if $U_{\nu,k}(x)$ is a Fourier kernel then

$$U^*_{\nu,k}(s)U^*_{\nu,k}(1-s) = 1, \; s = \sigma+i\tau, \; -\infty < \tau < \infty, \tag{2.3}$$

where $U^*_{\nu,k}(s)$ denotes the Mellin transform of $U_{\nu,k}(x)$, [4]. First we derive some known results as special cases of the function $U_{\nu,k}(x)$.

Let $k = 1$. The general solution (2.2) which is bounded as $x \to \infty$ is given by

$$\begin{aligned} U_{\nu,1}(x) &= B_0\sqrt{x}\; K_0(ix) + B_1\sqrt{x}\; K_1(ix) \\ &= \sqrt{x}[A\; J_\nu(x) + B\; Y_\nu(x)], \end{aligned}$$

using the fact that, [5,p.78]

$$K_\nu(iz) = -\tfrac{1}{2}\pi e^{-\frac{1}{2}\nu\pi i}[i\; J_\nu(z) + Y_\nu(z)] \tag{2.4}$$

The conditions B are satisfied for $\nu > 0$, provided $B = 0$. The constant A may now be obtained from the consideration that if $U_{\nu,1}(x)$ is a Fourier kernel and $U^*_{\nu,1}(s)$ is its Mellin transform then

$$U^*_{\nu,1}(s)\; U^*_{\nu,1}(1-s) = 1$$

In this case this condition gives $A = 1$, so that

$$U_{\nu,1}(x) = \sqrt{x}\; J_\nu(x), \qquad \nu > 0,$$

is a Fourier kernel, the well known Hankel kernel. Now once the kernel is obtained from these conditions, we may extend the values for ν (for which it is a Fourier kernal) from other considerations. We may, for example, consider all values of ν for which the Mellin transform exists at $s = \frac{1}{2}$. In this case this gives $U_{\nu,1}(x)$ as a Fourier kernel for $\nu > -1$.

Another known kernel can be obtained by setting $k = 2$. In this case the general solution of equation

$$\left[D^2 - \frac{\nu^2 - 1/4}{x^2}\right]^2 u = u$$

is given by

$$U_{\nu,2}(x) = \sum_{m=0}^{3} B_m\; \sqrt{x}\; K_\nu(i\; e^{i\frac{m\pi}{2}}\; x) \tag{2.5}$$

For this solution to be bounded as $x \to \infty$, $B_1 = 0$ and using (2.4), we may write (2.5) in a more familiar form as

$$U_{\nu,2}(x) = \sqrt{x}[AJ_\nu(x) + BY_\nu(x) + CK_\nu(x)], \tag{2.6}$$

where J_ν, Y_ν and K_ν are the usual Bessel functions of order ν, and A,B,C are suitable constants. Now we equate the coefficients of x^ν and $x^{-\nu}$ of the bracketed expression in (2.6) above to zero, so that $U_{\nu,2}(x)$ satisfies conditions B for $0 < \nu < 1$. This gives,

$$A + B\cot\nu\pi - \frac{\pi}{2}\, C\; \mathrm{cosec}\; \nu\pi = 0$$

$$-B\; \mathrm{cosec}\; \nu\pi + \frac{\pi}{2}\, C\; \mathrm{cosec}\; \nu\pi = 0.$$

Solving the above system we obtain

$$A = \frac{\pi}{2}\, C\tan\frac{1}{2}\nu\pi \text{ and } B = \frac{\pi}{2}\, C$$

Hence for $0 < \nu < 1$,

$$U_{\nu,2}(x) = \frac{\pi}{2} C \sec \tfrac{1}{2}\nu\pi \sqrt{x}\Big[(\sin \tfrac{1}{2}\nu\pi) J_\nu(x) + (\cos \tfrac{1}{2}\nu\pi)(Y_\nu(x)+\tfrac{2}{\pi} K_\nu(x)\Big]$$

To evaluate the unknown C, we make use of the functional equation

$$U^*_{\nu,2}(s)\, U^*_{\nu,2}(1-s) = 1. \tag{2.7}$$

which $U_{\nu,2}(x)$ satisfies if it is to be a Fourier kernel.

Now from the Mellin transforms of J_ν, Y_ν and K_ν, [6] the Mellin transform of $U_{\nu,2}(x)$ is given by,

$$U^*_{\nu,2}(s) = C\, 2^{s+\frac{1}{2}} \sec \tfrac{1}{2}\nu\pi\Big[\pi\frac{\Gamma(\frac{1}{2}\nu+\frac{1}{2}s+\frac{1}{4})}{\Gamma(\frac{1}{2}\nu-\frac{1}{2}s+\frac{3}{4})} \sin \tfrac{1}{2}\nu\pi - \Gamma(\tfrac{1}{2}s+\tfrac{1}{2}\nu+\tfrac{1}{4}) *$$

$$*\Gamma(\tfrac{1}{2}s - \tfrac{1}{2}\nu+\tfrac{1}{4})(\cos \tfrac{\pi}{2}(s-\nu+\tfrac{1}{2}) - 1)\Big]$$

$$= C\pi 2^{s-3/2} \frac{\Gamma(\frac{1}{2}\nu + \frac{1}{2}s+\frac{1}{4})\sin \frac{\pi}{4}(s + \nu+\frac{1}{2})}{\Gamma(\frac{1}{2}\nu - \frac{1}{2}s+\frac{3}{4})\cos \frac{\pi}{4}(s - \nu+\frac{1}{2})} \sec \tfrac{1}{2}\nu\pi.$$

It is an easy matter to see that (2.7) is satisfied if

$$C = \frac{2}{\pi} \cos \tfrac{1}{2} \nu\pi ,$$

and we have,

$$U_{\nu,2}(x) = \sqrt{x}[\sin\tfrac{1}{2}\nu\pi\, J_\nu(x) + \cos \tfrac{1}{2}\nu\pi(Y_\nu(x) + \tfrac{2}{\pi} K_\nu(x)].$$

This function is the Fourier kernel given by Nasim [3]. It is to be noted that this kernel once found is seen to be a Fourier kernel for $-3 < \nu < 3$.

Next we derive new and more complex Fourier kernels, using a similar procedure. Now set k = 3, and in this case the general solution of the differential equation

$$\Big[D^2 - \frac{\nu^2-\frac{1}{4}}{x^2}\Big]^3 u = -u,$$

is given by

$$U_{\nu,3}(x) = \sum_{m=0}^{5} B_m\sqrt{x}\, K_\nu(i\, e^{im\pi/3}x).$$

In order that the solution be bounded as $x \to \infty$, $B_1 = B_2 = 0$ and again using (2.4), we write, more conveniently,

$$U_{\nu,2}(x) = \sqrt{x}\Big[A\, J_\nu(x)+BY_\nu(x)+C(J_\nu(e^{i\pi/3}x) + iY_\nu(e^{i\pi/3}x)) + \overline{C}(J_\nu(e^{-i\pi/3}x) - iY_\nu(e^{-i\pi/3}x))\Big] \tag{3.1}$$

Here $\overline{C}$ denotes the complex conjugate of C. We now equate the coefficients of $x^\nu, x^{-\nu}$ and $x^{-\nu+2}$ of the bracketed expression in (3.1) to zero and observe that in doing so, we satisfy the conditions B for $0 < \nu < 1$. This gives us the system

$$\begin{aligned} A + B \cos \nu\pi + C(e^{i\pi\nu/3} + i (\cot \nu\pi)e^{i\pi\nu/3}) & \\ + \overline{C}(e^{-i\pi\nu/3} - i (\cot \nu\pi)e^{-i\pi\nu/3}) &= 0 \\ \operatorname{cosec} \nu\pi(B + iCe^{-i\pi\nu/3} - i\overline{C}e^{i\pi\nu/3}) &= 0 \\ \operatorname{cosec} \nu\pi(B + iCe^{i\pi(2-\nu)/3} - i\overline{C}e^{-i\pi(2-\nu)/3}) &= 0 \end{aligned}$$

Solving the above system, we obtain

$$A = \alpha\, (\operatorname{cosec} \nu\pi)(2 \sin\tfrac{\pi}{3}(\tfrac{1}{2} - \nu) - \cos \nu\pi),\ B = \alpha;\ C = \alpha e^{i\frac{\pi}{3}(\nu+\frac{1}{2})},$$ where

α, is a constant. Hence, for $0 < \nu < 1$, (3.1) gives,

$$U_{\nu,3}(x) = \alpha\sqrt{x}[\operatorname{cosec}\nu\pi\cdot(2\sin\tfrac{\pi}{3}(\tfrac{1}{2}-\nu) - \cos\nu\pi)J_\nu(x) + Y_\nu(x)$$
$$+ e^{i\frac{\pi}{3}(\nu+\frac{1}{2})}(J_\nu(e^{i\pi/3}x) + iY_\nu(e^{i\pi/3}x)) + e^{-i\frac{\pi}{3}(\nu+\frac{1}{2})}(J_\nu(e^{-i\pi/3}x) - iY_\nu(e^{-i\pi/3}x)] \qquad (3.2)$$

α may again be calculated from the consideration that if $U_{\nu,3}(x)$ is a Fourier kernel then it satisfies the functional equation

$$U^*_{\nu,3}(s)U^*_{\nu,3}(1-s) = 1. \qquad (3.3)$$

From the Mellin transforms of $\sqrt{x}\,J_\nu(x)$ and $\sqrt{x}\,Y_\nu(x)$, [6] and after considerable simplification, we find that,

$$U^*_{\nu,3}(s) = \alpha\, 2^{s-\frac{1}{2}}\frac{\Gamma(\frac{1}{2}\nu + \frac{1}{2}s+\frac{1}{4})}{\Gamma(\frac{1}{2}\nu - \frac{1}{2}s+\frac{3}{4})}[\cot\tfrac{\pi}{2}(\tfrac{1}{3}\nu - \tfrac{1}{3}s+\tfrac{1}{2})$$
$$+ 2\operatorname{cosec}\nu\pi\cdot(\sin\tfrac{1}{3}\pi(\tfrac{1}{2}-\nu) - \cos\nu\pi)]$$
$$= \alpha\, 2^{s-\frac{1}{2}}\frac{\Gamma(\frac{1}{2}\nu + \frac{1}{2}s+\frac{1}{4})}{\Gamma(\frac{1}{2}\nu - \frac{1}{2}s+\frac{3}{4})}\frac{\sin\frac{\pi}{6}(\nu + s+\frac{1}{2})(2\sin\frac{\pi}{6}(1+4\nu) - 1)}{\sin\frac{\pi}{6}(\nu - s+\frac{3}{2})\sin\nu\pi}$$

And equation (3.3) is satisfied provided

$$\alpha = \left|\frac{\sin\nu\pi}{2\sin\frac{\pi}{6}(1+4\nu) - 1}\right|,$$

and the resulting function $U_{\nu,3}(x)$ given by (3.2) is now a Fourier kernel. Once again one can extend its range to $-3 < \nu < 5$. And finally, we consider the case $k = 4$. The general solution of the equation (2.1) is, in this case,

$$U_{\nu,4}(x) = \sum_{m=0}^{7} B_m\sqrt{x}\,K_\nu(ie^{i\frac{m\pi}{4}}x)$$

In order that this be bounded as $x \to \infty$, we may again write the solution as

$$U_{\nu,4}(x) = \sqrt{x}[AJ_\nu(x) + BY_\nu(x) + CK_\nu(x) + D(J_\nu(e^{i\pi/4}x) + iY_\nu(e^{i\pi/4}x))$$
$$+ \bar{D}(J_\nu(e^{-i\pi/4}x) - i\,Y_\nu(e^{-i\pi/4}x))] \qquad (3.4)$$

Equating the coefficients of the terms $x^\nu, x^{\nu+2}, x^{-\nu}$ and $x^{-\nu+2}$ in the bracketed expression in (3.4) to zero respectively, we obtain the following system:

$$A + B\cot\nu\pi - C\tfrac{\pi}{2}\operatorname{cosec}\nu\pi + De^{i\nu\frac{\pi}{4}}(1+i\cot\nu\pi) + \bar{D}e^{-i\nu\frac{\pi}{4}}(1-i\cot\nu\pi) = 0$$

$$A + B\cot\nu\pi + C\tfrac{\pi}{2}\operatorname{cosec}\nu\pi + De^{i\frac{\pi}{4}(\nu+2)}(1+i\cot\nu\pi) + \bar{D}e^{-i\frac{\pi}{4}(\nu+2)}(1-i\cot\nu\pi) = 0$$

$$\operatorname{cosec}\nu\pi(B - C\tfrac{\pi}{2} + iDe^{-i\nu\frac{\pi}{4}} - i\bar{D}e^{i\nu\frac{\pi}{4}}) = 0$$

$$\operatorname{cosec}\nu\pi(B + C\tfrac{\pi}{2} + iDe^{i\frac{\pi}{4}(-\nu+2)} - i\bar{D}e^{i\frac{\pi}{4}(-\nu+2)}) = 0.$$

This gives

$$A = \alpha\sqrt{2}\operatorname{cosec}\nu\pi\,(\sin\nu\tfrac{\pi}{4})(1 + \cos\nu\pi);\; B = -\alpha\sqrt{2}\sin\nu\tfrac{\pi}{4};$$

$$C = -\alpha\frac{2\sqrt{2}}{\pi}\cos\nu\pi;\quad D = i\alpha e^{i\frac{\pi}{4}(2\nu+1)}$$

where α is some constant. With these values of the coefficients, $U_{\nu,4}(x)$ satisfies the conditions B for $0 < \nu < 1$. The unknown α can now be calculated from the fact that

$$U^*_{\nu,4}(s)\ U^*_{\nu,4}(1-s) = 1. \tag{3.5}$$

After much tedious work, one finds that

$$U^*_{\nu,4}(s) = \alpha \frac{2^s\Gamma(\frac{1}{2}\nu + \frac{1}{2}s + \frac{1}{4})\operatorname{cosec}\ \nu\pi}{\Gamma(\frac{1}{2}\nu - \frac{1}{2}s + \frac{3}{2})\sin\frac{\pi}{2}(s - \nu + \frac{1}{2})}[2\sin\frac{\nu\pi}{2}\cos\frac{\nu\pi}{2}\sin\frac{\pi}{2}(s + \frac{1}{2})$$

$$- \cos\frac{\nu\pi}{4}\sin\nu\pi + \sqrt{2}\sin\nu\pi\cos\frac{\pi}{4}(s + \frac{3}{2})]$$

$$= \alpha \frac{2^s\Gamma(\frac{1}{2}\nu + \frac{1}{2}s + \frac{1}{4})\operatorname{cosec}\ \nu\pi}{\Gamma(\frac{1}{2}\nu - \frac{1}{2}s + \frac{3}{4})\sin\frac{\pi}{2}(s - \nu + \frac{1}{2})} \cdot \sin\frac{\nu\pi}{4}\cos\frac{\nu\pi}{2}\sin\frac{\pi}{8}(s+\nu+\frac{1}{2}) *$$

$$* \sin\frac{\pi}{8}(s-\nu + \frac{1}{2})\cos\frac{\pi}{8}(s+\nu - \frac{3}{2})\cos\frac{\pi}{8}(s-\nu - \frac{3}{2}).$$

and (3.5) is satisfied if

$$\alpha = \sqrt{2}\left|\cos\frac{\nu\pi}{4}\right| .$$

Thus $U_{\nu,4}(x)$ is now established as a Fourier kernel for $0 < \nu < 1$, with $A = \frac{1}{2}\sec\frac{1}{2}\nu\pi(1+\cos\nu\pi)$; $B = -\sin\frac{\nu\pi}{2}$; $C = -\frac{4}{\pi}\cos^2\frac{\nu\pi}{4}$ and $D = \sqrt{2}\cos\frac{\nu\pi}{4} * e^{i\frac{\pi}{4}(2\nu+3)}$. Again $U_{\nu,4}(x)$ is seen to be a Fourier kernel for $-5 < \nu < 5$, as well. We believe that the kernels $U_{\nu,3}(x)$ and $U_{\nu,4}(x)$ are new and are being reported for the first time in literature.

We now note some special cases of the kernels $U_{\nu,3}(x)$ and $U_{\nu,4}(x)$, which are combinations of exponentials, sine and cosine functions.

Examples are:

$$U_{\frac{1}{2},3}(x) = \sqrt{\frac{2}{\pi}}\,(2\,e^{-\frac{\sqrt{3}}{2}x}\sin(\tfrac{1}{2}x + \tfrac{\pi}{6}) - \cos x)$$

$$U_{-\frac{1}{2},3}(x) = \sqrt{\frac{2}{\pi}}\,(e^{-\frac{\sqrt{3}}{2}x}\cos(\tfrac{1}{2}x - \tfrac{\pi}{6}) - \cos(x + \tfrac{\pi}{6}))$$

$$U_{\frac{3}{2},3}(x) = \sqrt{\frac{2}{\pi}}\,[e^{-\frac{\sqrt{3}}{2}x}(\tfrac{1}{x}\sin(\tfrac{x}{2} + \tfrac{\pi}{6}) + \sin\tfrac{x}{2}) + \tfrac{1}{x}\sin(x - \tfrac{\pi}{6}) - \cos(x + \tfrac{\pi}{6})]$$

$$U_{-\frac{3}{2},3}(x) = \sqrt{\frac{2}{\pi}}\,[e^{-\frac{\sqrt{3}}{2}x}(\tfrac{1}{x}\cos(\tfrac{\pi}{6} - \tfrac{x}{2}) - \cos\tfrac{x}{2}) - \tfrac{1}{x}\cos(x + \tfrac{\pi}{6}) - \sin(x + \tfrac{\pi}{6})]$$

$$U_{\frac{1}{2},4}(x) = \frac{1}{\sqrt{\pi}}\,[\sin x + \cos x - 2\sqrt{2}\,(\cos^2\tfrac{\pi}{8})\,e^{-x} - 4\cos\tfrac{\pi}{8}\sin(\tfrac{x}{2} - \tfrac{\pi}{8})\,e^{\frac{x}{\sqrt{2}}}]$$

$$U_{-\frac{1}{2},4}(x) = \frac{1}{\sqrt{\pi}}[\cos x - \sin x - 2\sqrt{2}\,(\cos^2\tfrac{\pi}{8})\,e^{-x} - 4\cos\tfrac{\pi}{8}\sin(\tfrac{x}{2} - \tfrac{\pi}{8})\,e^{-\frac{x}{\sqrt{2}}}]$$

$$U_{\frac{3}{2},4}(x) = \frac{1}{\sqrt{\pi}}\Big[\cos x + \sin x + \frac{1}{x}(\cos x - \sin x) - 2\sqrt{2}\sin^2\frac{\pi}{8}\cdot\Big(1 + \frac{1}{x}\Big)e^{-x}$$

$$-4\sin\frac{\pi}{8}\cdot e^{-\frac{x}{\sqrt{2}}}\Big(\frac{1}{x}\sin\Big(\frac{x}{\sqrt{2}} + \frac{\pi}{8}\Big) + \sin\Big(\frac{x}{\sqrt{2}} - \frac{\pi}{8}\Big)\Big)\Big]$$

$$U_{-\frac{3}{2},4}(x) = \frac{1}{\sqrt{\pi}}\Big[\sin x - \cos x + \frac{1}{x}(\cos x + \sin x) - 2\sqrt{2}\Big(\sin^2\frac{\pi}{8}\Big)\Big(1 + \frac{1}{x}\Big)e^{-x}$$

$$-4\sin\frac{\pi}{8}\,e^{-\frac{x}{\sqrt{2}}}\Big(\sin\Big(\frac{x}{\sqrt{2}} + \frac{\pi}{8}\Big) - \frac{1}{x}\sin\Big(\frac{x}{\sqrt{2}} - \frac{\pi}{8}\Big)\Big)\Big].$$

It is interesting to note that for $k = 3$, for example, while conditions B are satisfied only when $0 < \nu < 2$, other conditions which are various cases of conditions A, may be satisfied for other values of ν. Thus, for instance; it is easy to verify that, with $U_{\nu,3}$ as calculated above:

$$U_{-\frac{1}{2},3}(0) = U^{(1)}_{-\frac{1}{2},3}(0) = U^{(3)}_{-\frac{1}{2},3}(0) = 0;\quad U_{-\frac{3}{2},3}(0) = U^{(2)}_{-\frac{3}{2},3}(0) = U^{(4)}_{-\frac{3}{2},3}(0) = 0;$$

$$U^{(1)}_{-\frac{5}{2},3}(0) = U^{(3)}_{-\frac{5}{2},3}(0) = U^{(5)}_{-\frac{5}{2},3}(0) = 0;\quad U_{\frac{5}{2},3}(0) = U^{(1)}_{\frac{5}{2},3}(0) = U^{(3)}_{\frac{5}{2},3} = 0;$$

$$U_{\frac{7}{2},3}(0) = U^{(2)}_{\frac{7}{2},3}(0) = U^{(4)}_{\frac{7}{2},3}(0) = 0;\quad U^{(1)}_{\frac{9}{2},3}(0) = U^{(3)}_{\frac{9}{2},3} = U^{(5)}_{\frac{9}{2},3}(0) = 0.$$

All these are examples of conditions A at $x = 0$.

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