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ON ULTRACONNECTED SPACES

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ABSTRACT. In this paper, we study some properties of ultraconnected spaces and show that ultraconnected $T_{\frac{1}{2}}$ spaces are maximal ultraconnected and minimal $T_{\frac{1}{2}}$. We also introduce the notion of F-connected spaces, topological spaces which are both hyperconnected and ultraconnected and characterize compact maximal F-connected topologies on a set.

KEY WORDS AND PHRASES. Ultraconnected, hyperconnected, Semi-topological, generalized closed.
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1. INTRODUCTION.

A topological space is ultraconnected if the intersection of any two nonempty closed sets is nonempty (Steen and Seebach [1]). Each topology τ on a set X may be associated with a pre-order relation $\rho(\tau)$ on X, defined by $(a,b) \in \rho(\tau)$ if every open set containing b contains a. In 1978 Andima and Thron [2] defined a topological space (X,τ) to be upward directed if any two elements in $(X,\rho(\tau))$ have an upper bound, and it can easily be seen that the notion of upward directed and that of ultraconnected are equivalent.

Let (X,R) be a pre-ordered set. Define $\overline{\{x\}} = \{y \in X \mid x R y\}$ and $\{x\} = \{y \in X \mid y R x\}$, for each $x \in X$. $\mu(R)$, the point closure topology of R, is the smallest topology in which all sets $\overline{\{x\}}$, $x \in X$, are closed and V(R), the kernel topology of R, is the topology with basis $\{\hat{\{x\}} \mid x \in X\}$. A topology τ on X induces a pre-order R as described above iff $\mu(R) \subset \tau \subset V(R)$ [2].

2. ULTRACONNECTED SPACES.

In [2], it is proved that a topological space (X,τ) is maximal upward directed iff $(X,\rho(\tau))$ is a partially ordered set of length 1, with a greatest element and

$\tau = V(\rho(\tau))$. If (X,R) is a partially ordered set of length 1, with a greatest element, say a, then $V(R) = P(X\setminus\{a\}) \cup \{X\}$. Thus the maximal ultraconnected topologies on a set X are precisely $P(X\setminus\{a\}) \cup \{X\}$, where $a \in X$.

DEFINITION 2.1. A topological space is $T_{\frac{1}{2}}$ if each singleton subset is either open or closed (Levine [3]).

REMARK 2.1. Any $T_{\frac{1}{2}}$ space is T_o and Dunham [4] characterized the minimal $T_{\frac{1}{2}}$ topologies on a set X as those of the form $\{0 \subset X \mid 0 \subset A \text{ or } A \subset 0 \text{ and } 0' \text{finite}\}$, for some proper subset A of X. (When X is finite with more than one element, A must also be nonempty.) Obviously, any maximal ultraconnected space is minimal $T_{\frac{1}{2}}$.

THEOREM 2.1. Any ultraconnected $T_{\frac{1}{2}}$ space is maximal ultraconnected and minimal $T_{\frac{1}{2}}$.

PROOF. Let (X,τ) be an ultraconnected $T_{\frac{1}{2}}$ space. Since (X,τ) is $T_{\frac{1}{2}}$ the induced order $\rho(\tau)$ is a partial order. Suppose there exist $x,y,z \in X$ such that $x\, \rho(\tau) y$ and $y\, \rho(\tau) z$. If $\{y\}$ is open, then $x\, \rho(\tau) y \implies x \in \{y\}$; i.e., $x = y$. On the other hand, if $\{y\}$ is closed, then $y\, \rho(\tau) z \implies z \in \overline{\{y\}} = \{y\}$; i.e., $z = y$. Since the singletons are either open or closed, it is evident that the length of $(X, \rho(\tau))$ is at most 1.

If $\{x\}$ is open and $y\, \rho(\tau) x$, then $y = x$ and hence x is minimal in $(X, \rho(\tau))$. Similarly if $\{x\}$ is closed, then x is maximal in $(X, \rho(\tau))$. Since (X,τ) is ultraconnected any two minimal elements have an upper bound and there exists only one maximal element which will be the greatest element in $(X, \rho(\tau))$. Moreover, if x is minimal in $(X, \rho(\tau))$, then $\{x\}$ is open and not closed. Hence $\tau = V(\rho(\tau))$. Thus (X,τ) is maximal ultraconnected, and by the above remark it is minimal $T_{\frac{1}{2}}$ too.

NOTE 2.1. Though every maximal ultraconnected space is minimal $T_{\frac{1}{2}}$, there are minmal $T_{\frac{1}{2}}$ spaces which are not even ultraconnected. However, every minimal $T_{\frac{1}{2}}$ space is connected [4].

Let X be a set with 3 or more elements and $\phi \neq A \subset X$ such that $|X\setminus A| > 2$. Then $\tau = \{0 \subset X \mid 0 \subset A \text{ or } A \subset 0 \text{ and } 0' \text{ finite}\}$ is a minimal $T_{\frac{1}{2}}$ topology, which is not ultraconnected. For if $x,y \in X\setminus A$, then $\{x\}$ and $\{y\}$ are closed subsets of (X,τ) with empty intersection.

DEFINITION 2.2. A subset of a topological space is called ultraconnected if it is ultraconnected as a subspace.

REMARK 2.2. We will call two subsets A and B of a topological space (X, τ) equivalent $(A \equiv B)$ if every open set containing A contains B and conversely. $A^* = \cap\{0 \in \tau \mid 0 \supset A\}$ is the largest subset of X equivalent to A. Note that, if $A \subset B \subset C$ and $A \equiv C$, then $A \equiv B$ and $B \equiv C$.

THEOREM 2.2. Let A and B be subsets of a topological space (X,τ) and $A \equiv B$. Then A is ultraconnected iff B is ultraconnected.

PROOF. Suppose A is ultraconnected, but B is not. Then there exist two nonempty disjoint closed sets C_1, C_2 in B. Let $C_i = D_i \quad B$; $i = 1,2$; D_i closed in (X, τ).

$$C_1 \cap C_2 \neq \phi \implies D_1 \cap D_2 \cap B = \phi \implies B \subset D_1' \cup D_2'$$

Since $A \equiv B$, $A \subset D_1' \cup D_2'$ and hence $D_1 \cap D_2 \cap A = \phi$. But $D_1 \cap A \neq \phi$, for otherwise $A \subset D_1' \implies B \subset D_1' \implies C_1 = D_1 \cap B = \phi$. Similarly $D_2 \cap A \neq \phi$. Since $D_1 \cap A$, $D_2 \cap A$ are nonempty disjoining closed sets in A, we get a contradiction. Hence the result.

DEFINITION 2.3. A subset A of a topological space is generalized closed if $\bar{A} \subset 0$ and 0 whenever $A \subset 0$ and 0 is closed [3].

COROLLARY 2.1. If A is a generalized closed subset of (X, τ), then A is ultraconnected iff $\bar{A}$ is ultraconnected.

PROOF. In view of Theorem 2.2, it is sufficient to show that $A \equiv \bar{A}$. Since A is generalized closed, if $A \subset 0 \in \tau$, then $\bar{A} \subset 0$. The other implication is trivial.

COROLLARY 2.2. If A and B are subsets of a space (X, τ) such that $A \subset B \subset A^*$, then A is ultraconnected iff B is ultraconnected.

PROOF. Since $A \subset B \subset A^*$ and $A \equiv A^*$, it follows that $A \equiv B$ (see the previous remark). Thus the conclusion is an immediate consequence of Theorem 2.2.

DEFINITION 2.4. A subset A of a space X is called semi-open if there exists an open set 0 such that $0 \subset A \subset \bar{0}$ (Levine [5]). A semi-homeomorphism is a bijection under which both images and inverse images of sem-open sets are semi-open. A topological property invariant under semi-homeomorphisms is called a semi-topological property by Crossley and Hildebrahd [6].

REMARK 2.3. Ultraconnectedness is not semitopological. Let $X = \{a,b,c\}$. $\tau_1 = \{\phi, \{a\}, \{a,b\}, X\}$ and $\tau_2 = \{\phi, \{a\}, \{a,b\}, \{a,c\}, X\}$. Now (X, τ_1) is ultraconnected, but (X, τ_2) is not, while τ_1 and τ_2 yield the same collection of semi-open sets and hence are semi-homeomorphic.

3. F-CONNECTED SPACES.

A topological space in which the intersection of any two nonempty open sets is nonempty is called hyperconnected [1]. We define a topological space to be F-connected if it is both hyperconnected and ultraconnected.

REMARK 3.1. In the above remark (X, τ_1) is F-connected while (X, τ_2) is not. Hence F-connectedness is not a semi-topological property. Neither the join nor the product of two F-connected topologies on a set are F-connected. Let $\tau_1 = \{\phi, A, X\}$ and $\tau_2 = \{\phi, B, X\}$ where $A \cap B = \phi$. Then $\tau_1 \vee \tau_2$ and $\tau_1 \times \tau_2$ are not F-connected but τ_1 and τ_2 are F-connected.

THEOREM 3.1. Every subspace of a topological space (X, τ) is F-connected iff τ is nested.

PROOF. Necessity: Assume τ is not nested. Then there exist $A, B \subset X$ such that $A \not\subset B$ and $B \not\subset A$. Choose $x \in A \backslash B$ and $y \in B \backslash A$. Then the subspace $\{x,y\}$ has the discrete topology which is obviously not F-connected.

Sufficiency: Let τ be nested and $A \subset X$. Let 0_1, 0_2 be nonempty open sets in A. Then there exist B_1, $B_2 \in \tau$ such that $0_1 = A \cap B_1$ and $0_2 = A \cap B_2$. Since τ is nested, $B_1 \subset B_2$ or $B_2 \subset B_1$. Assume $B_1 \subset B_2$. Then $0_1 \subset 0_2$ and hence $0_1 \cap 0_2 \neq \phi$. Similarly, the intersection of any two nonempty closed sets in A is also nonempty. Thus A is F-connected.

THEOREM 3.2. If **U** is an ultrafilter on $X \setminus \{a\}$, for some $a \in X$, then $\tau = \{\phi, X\} \cup$ **U** is a maximal F-connected topology on X.

PROOF. Obviously, (X, τ) is F-connected. Suppose (X, τ_1) is F-connected and $\tau_1 > \tau$. Let $A \in \tau_1 \setminus \tau$. Since $A \in \tau_1 \setminus \tau$, $a \notin A$. For if $a \in A$, then $\{a\} \cap (X \setminus A) = \phi$, a contradiction since (X, τ_1) is ultraconnected. Now $a \notin A$ and $A \notin$ **U** implies $(X \setminus \{a\}) \setminus A \in$. Thus A and $(X \setminus \{a\}) \setminus A$ are two nonempty disjoint open sets in (X, τ_1), a contradiction. Hence the result.

THEOREM 3.3. Any compact, maximal F-connected topology on a set X is of the form $\tau_a = \{\phi, X\} \cup \mathbf{U}_a$, where $\mathbf{U}_a$ is an ultrafilter on $X \setminus \{a\}$, for some $a \in X$.

PROOF. Let (X, τ) be compact and maximal F-connected. Since the family of all the nonempty closed sets has finite intersection property and (x, τ) is compact, it has nonempty intersection. Choose $a \in \cap \{C \subset X | C$ is closed and nonempty $\}$. Thus the proper open sets are subsets of $X \setminus \{a\}$ and they form a filter base F. Let $\mathbf{U}_a$ be an ultrafilter on $X \setminus \{a\}$ containing F. Then $\tau \subset \{\phi, X\} \cup \mathbf{U}_a = \tau_a$. Since (X, τ) is maximal F-connected in view of Thoerem 3.2, $\tau = \tau_a$.

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