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SUBLINEAR FUNCTIONALS AND KNOPP'S CORE THEOREM

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ABSTRACT. In this paper we are concerned with inequalities involving certain sublinear functionals on m, the space of real bounded sequences. Such inequalities being analogues of Knopp's Core theorem.

KEY WORDS AND PHRASES. Core theorem, Sublinear functionals, Infinite matrices, Almost convergence.

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1. INTRODUCTION.

Let m be the linear space of real bounded sequences with the usual supremum norm. We write

$$m_0 = \{x \in m : \sup_n |\sum_{k=0}^{n} x_k| < \infty\}$$

Let $\mathcal{A}$ be the sequence of infinite matrices $(A^i) = (a_{nk}(i))$. Given a sequence $x = (x_k)$ we write

$$A_n^i(x) = \sum_{k=0}^{\infty} a_{nk}(i)\, x_k \tag{1.1}$$

if it exists for each n and $i \geq 0$. We also write Ax for $\left(A_n^i(x)\right)_{i,n=0}^{\infty}$. The sequence $x = (x_k)$ is said to be summable to the value s by the method $(\mathcal{A})$ if

$$A_n^i(x) \to s \quad (n \to \infty \text{, uniformly in } i) \tag{1.2}$$

If (1.2) holds, then we write $x \to s(\mathcal{A})$.

If we define $(a_{nk}(i)$ by

$$a_{nk}(i) = \begin{cases} 1/n+1 & , \quad i \leq k \leq i+n \\ 0 & , \quad \text{otherwise} \end{cases}$$

then $(\mathcal{A})$ reduces to the method f (Lorentz [1]). In the case

$$a_{nk}(i) = \frac{1}{n+1} \sum_{r=i}^{i+n} a_{rk}$$

$(\mathcal{A})$ reduces to the almost summability method (King [2]). If $\mathcal{A} = A = (a_{nk})$, then we get the usual summability method (A).

The method $(\mathcal{A})$ is said to be conservative if $x \to s$ implies $x \to s'(\mathcal{A})$. If $(\mathcal{A})$ is conservative and $s = s'$, then $(\mathcal{A})$ is called regular.

It is well known, (Stieglitz [3]), that $(\mathcal{A})$ is regular if and only if the following conditions hold:

$$\sum_k |a_{nk}(i)| < \infty \quad , \qquad \text{(for all n, for all i)}. \tag{1.3}$$

and there exist an integer m such that

$$\sup_{i \geq 0, n \geq m} \sum_k |a_{nk}(i)| < \infty \tag{1.4}$$

$$\lim_n a_{nk}(i) = 0 , \qquad \text{uniformly in i}, \tag{1.5}$$

$$\lim_n \sum_k a_{nk}(i) = 1 , \qquad \text{uniformly in i}, \tag{1.6}$$

Throughout the paper we write

$$\|\mathcal{A}\| = \sup_{n,i} \sum_k |a_{nk}(i)| < \infty \tag{1.7}$$

to mean that, there exists a constant M such that

$$\sum_k |a_{nk}(i)| \leq M \qquad \text{(for all n, for all i)} \tag{1.8}$$

and the series

$$\sum_k a_{nk}(i) \tag{1.9}$$

convreges uniformly in i for each n.

If, for every bounded sequence x, $x \to s(\mathcal{A})$ then $(\mathcal{A})$ is said to be a Schur method

Throughout the paper we consider only real matrices and real bounded sequences.

In this paper we are concerned with inequalities involving certain sublinear functionals on m, the space of real bounded sequences. Such inequalities being analogues of Knopp's Core theorem That theorem determines a class of regular matrices for which

$$\limsup Ax \leq \limsup x$$

for all $x \in m$, see e.g Cooke [4], Maddox [5], Simons [6]. This result has also been extended to coregular matrices by Rhoades [7], Schaefer [8], and, Das [9].

Before stating the theorems to be proved, we introduce some further notation.

$$\ell(x) = \liminf x_n \, ; \; L(x) = \limsup x_n \, , \; \|x\| = \sup |x_n|$$

$$\ell^*(x) = \liminf_n \sup_i \frac{1}{n+1} \sum_{r=i}^{i+n} x_r$$

$$L^*(x) = \limsup_n \sup_i \frac{1}{n+1} \sum_{r=i}^{i+n} x_r$$

$$W^*(x) = \inf_{z \in m_0} L^*(x+z)$$

If f, g are any two of the above functionals, we shall write $fA \le gB$ to denote that, for every bounded sequence x, the transforms Ax and Bx are defined and bounded and $f(Ax) \le g(Bx)$.

2. THE MAIN RESULTS.

We write, for $x \in m$,

$$Q_A(x) = \limsup_n \sup_i \sum_k a_{nk}(i)\, x_k$$

and

$$q_A(x) = \liminf_n \sup_i \sum_k a_{nk}(i)\, x_k$$

With this notation we have

THEOREM.1. Let $\|\mathcal{A}\| < \infty$. Then

$$Q_A \le L \tag{2.1}$$

if and only if $(\mathcal{A})$ is regular and

$$\sum_k \left| a_{nk}(i) \right| \to 1 \qquad (n \to \infty \text{ , uniformly in } i) \tag{2.2}$$

PROOF. Necessity. Let $x = (x_k)$ be a convergent sequence. Then $\ell(x) = L(x) = \lim x$. By (2.1), we have

$$\ell(x) \le -Q_A(-x) \le Q_A(x) \le L(x).$$

Hence we get that $Q_A(x) = q_A(x) = \lim x$. So $(\mathcal{A})$ is regular.

Since $(\mathcal{A})$ is regular, the requirement of Lemma 2, (Das [9]), is satisfied. Hence there exists $y \in m$ such that $\|y\| \le 1$ and

$$Q_A(y) = \limsup_n \sup_i \sum_k \left| a_{nk}(i) \right| .$$

Hence, taking $x = e = (1,1,\ldots)$, we have

$$1 = q_A(e) \le \liminf_n \sup_i \sum_k \left| a_{nk}(i) \right|$$
$$\le \limsup_n \sup_i \sum_k \left| a_{nk}(i) \right| = Q_A(y) \le L(y) \le \|y\| \le 1$$

which proves the necessity of (2.2)

Sufficiency. We define, for any real λ, $\lambda^+ = \max(\lambda,0)$, $\lambda^- = \max(-\lambda,0)$. Then $|\lambda| = \lambda^+ + \lambda^-$ and $\lambda = \lambda^+ - \lambda^-$. Hence

$$\sum_k a_{nk}(i)\, x_k = \sum_{k<m} a_{nk}(i)\, x_k + \sum_{k\ge m} \left(a^+_{nk}(i)\right) x_k - \sum_{k\ge m} \left(a^-_{nk}(i)\right) x_k$$

So we have

$$\sum_k a_{nk}(i)\, x_k \le \|x\| \sum_{k<m} |a_{nk}(i)| + (\sup_{k\ge m} x_k) \sum_{k\ge m} |a_{nk}(i)| + \|x\| \sum_{k\ge m} \left(|a_{nk}(i)| - a_{nk}(i)\right)$$

By hypothesis, we get that $Q_A(x) \le L(x)$.

REMARK. We could use Theorem 2, (Das [9]), to get the sufficiency.

COROLLARY.2. We have on m,

$$\mathfrak{l} \le \mathfrak{l}^* \le L^* \le L.$$

PROOF. In theorem 1, it is enough to take

$$a_{nk}(i) = \begin{cases} 1/n+1 & , \quad i \le k \le i+n \\ 0 & , \quad \text{otherwise} \end{cases}$$

We deduce at once from Corollary 2 that if a sequence x is convergent to s, then it is almost convergent to s which is a well-known result.

We note that, by considering Theorem 1, one may get necessary and sufficient conditions for $L^*A \le L$ and $LA \le L$.

In the next theorem we consider the inequality $LA \le L^*$.

THEOREM.3. $LA \le L^*$ if and only if A is strongly regular and

$$\sum_k |a_{nk}| \to 1 \qquad (n \to \infty) \tag{2.3}$$

PROOF. Recall that a matrix A is called strongly regular if it maps all almost convergent sequences into the convergent sequences and lim Ax = f - lim x.

We first prove the necessity. It is easy to see that $\mathfrak{l}^* \le \mathfrak{l}A \le LA \le L^*$. If x is almost convergent then f - lim x = $\mathfrak{l}^*(x) = L^*(x)$. Hence, by the hypothesis, $\mathfrak{l}(Ax) = L(Ax) =$ f - lim x. So A is strongly regular. Using the fact that $L^* \le L$ (see Corollary 2) and that $LA \le L^*$, we get that $LA \le L$. Now the necessity of (2.3) follows from Knopp's Core theorem (see, e.g. Maddox [5]).

We note in passing that a matrix A is strongly regular, Lorentz [1], if and only if it is regular and that

$$\sum_k |a_{nk} - a_{n,k+1}| \to 0 \qquad (n \to \infty) \tag{2.4}$$

Sufficiency. Given $\varepsilon > 0$, we can find a positive integer p such that for $x \in m$ and for all $k \ge 0$,

$$\frac{1}{p+1} \sum_{r=k}^{k+p} x_r < L^*(x) + \varepsilon \tag{2.5}$$

(We fix p throughout the analysis).

As in Lorentz's proof (see [1]; Th. 7) one can show that

$$\begin{aligned} \sum_{k=0}^{\infty} a_{nk} x_k = & \sum_{k=0}^{\infty} a_{nk} \frac{1}{p+1} \sum_{r=k}^{k+p} x_r \\ & - \sum_{k=p}^{\infty} \left(\frac{a_{nk} + \dots + a_{n,k-p}}{p+1} - a_{nk} \right) \\ & + \sum_{k=0}^{p-1} a_{nk} x_k \\ & + \sum_{k=0}^{p-1} \left(\frac{a_{nk} + \dots + a_{n,k-p+1}}{p+1} \right) x_k \end{aligned} \tag{2.6}$$

Since $x \in m$, it follows from the regularity of A that the third and fourth sigmas in (2.6) tend to zero as $n \to \infty$. If we write

$$F_{np} = -\sum_{k=p}^{\infty}\left(\frac{a_{nk} + \dots + a_{n,k-p}}{p+1} - a_{nk}\right)x_k$$

then

$$|F_{np}| \le \frac{1}{p+1}\sum_{k=p}^{\infty}|a_{nk} + \dots + a_{n,k-p} - (p+1)a_{nk}|\,|x_k|$$

$$\le \frac{\|x\|}{p+1}\sum_{r=0}^{p}\sum_{k=p}^{\infty}|a_{n,k-r} - a_{nk}|$$

$$\le \frac{\|x\|}{p+1}\sum_{r=0}^{p} r \sum_{k=0}^{\infty}|a_{nk} - a_{n,k+1}|$$

$$\le \frac{p}{2}\|x\|\sum_{k=0}^{\infty}|a_{nk} - a_{n,k+1}| \tag{2.7}$$

Since A is strongly regular, (2.4) holds. Thus the expression in (2.7) tends to zero as $n \to \infty$. Hence we find that

$$L(Ax) \le \limsup_n \sum_{k=0}^{\infty} a_{nk}\left(\frac{x_k + \dots + x_{k+p}}{p+1}\right)$$

$$\le \limsup_n \sum_k \left(a_{nk}^{+}\right)\left(\frac{x_k + \dots + x_{k+p}}{p+1}\right)$$

$$- \limsup_n \sum_k \left(a_{nk}^{-}\right)\left(\frac{x_k + \dots + x_{k+p}}{p+1}\right)$$

By (2.5), we have

$$L(Ax) \le (L^*(x) + \varepsilon)\limsup_n \sum_k |a_{nk}| + \|x\| \limsup_n \sum_k \left(|a_{nk}| - a_{nk}\right)$$

Using the regularity of A and (2.3) we get that

$$L(Ax) \le L^*(x) + \varepsilon .$$

Since ε is arbitrary, sufficiency follows.

THEOREM.4. $L^*A \le L^*$ if and only if A is F-regular and

$$\limsup_n \sup_i \sum_k \left|\frac{1}{n+1}\sum_{r=i}^{i+n} a_{rk}\right| = 1 \tag{2.8}$$

PROOF. Recall that A is called F-regular if it maps F, the class of all almost convergent sequences, into itself and f-lim Ax = f-lim x. Corollary to Theorem 4 in [10] gives the necessary and sufficient conditions for A to be F-regular.

We now come to the proof of necessity.

One can easily show that

$$\ell^*(x) \le \ell^*(Ax) \le L^*(Ax) \le L^*(x).$$

If $x \in F$, then $\ell^*(x) = L^*(x) = f - \lim x$. Hence $\ell^*(Ax) = L^*(Ax) = f - \lim x$. So A is F-regular.

To get the necessity of (2.8), we define $(b_{nk}(i))$ by

$$b_{nk}(i) = \frac{1}{n+1} \sum_{r=i}^{i+n} a_{rk}$$

Observe now that the conditions of Lemma 2, Das [9], are satisfied. So we must have a bounded sequence y such that $\|y\| \le 1$ and

$$Q_B(y) = \limsup_n \sup_i \sum_k |b_{nk}(i)| \tag{2.9}$$

Hence by (2.9) and F-regularity of A, we get

$$1 \le \liminf_n \sup_i \sum_k \left| \frac{1}{n+1} \sum_{r=i}^{i+n} a_{rk} \right|$$

$$\le \limsup_n \sup_i \sum_k \left| \frac{1}{n+1} \sum_{r=i}^{i+n} a_{rk} \right|$$

$$= \limsup_n \sup_i \sum_k \left(\frac{1}{n+1} \sum_{r=i}^{i+n} a_{rk} \right) y_k \le L^*(y) \le \|y\| \le 1$$

which proves (2.8).

Sufficiency. We first note that

$$L^*(Ax) = \limsup_n \sup_i \frac{1}{n+1} \sum_{r=i}^{i+n} A_r(x)$$

$$= \limsup_n \sup_i \sum_k \left(\frac{1}{n+1} \sum_{r=i}^{i+n} a_{rk} \right) x_k$$

If we set

$$b_{nk}(i) = \frac{1}{n+1} \sum_{r=i}^{i+n} a_{rk}$$

then (2.6) with a_{nk} relaced by $b_{nk}(i)$ holds. Since $x \in m$ and A is F-regular, Corollary to Theorem 4 in [10] yields that the second and third sigmas with a_{nk} replaced by $b_{nk}(i)$, tend to zero as $n \to \infty$, uniformly in i. On the other hand $|F_{np}|$, with a_{nk} replaced by $b_{nk}(i)$ is not greater than

$$\frac{p}{2} \|x\| \sum_k |b_{nk}(i) - b_{n,k+1}(i)| = \frac{p}{2} \|x\| \sum_k \left| \frac{1}{n+1} \sum_{r=i}^{i+n} (a_{rk} - a_{r,k+1}) \right|$$

Since A is F-regular, the last sigma tends to zero as $n \to \infty$, uniformly in i. Hence we have, by (2.5), that

$$L^*(Ax) \le \limsup_n \sup_i \sum_k \left(\frac{1}{n+1} \sum_{r=i}^{i+n} a_{rk} \right) \left(\frac{x_k + \ldots + x_{k+p}}{p+1} \right)$$

$$\le (L^*(x)+\varepsilon)\limsup_n \sup_i \sum_k \left|\frac{1}{n+1}\sum_{r=i}^{i+n} a_{rk}\right|$$

$$+ \|x\| \limsup_n \sup_i \sum_k \left(\left|\frac{1}{n+1}\sum_{r=i}^{i+n} a_{rk}\right| - \frac{1}{n+1}\sum_{r=i}^{i+n} a_{rk}\right)$$

Using (2.8) and the fact that A is F-regular, we get

$$L^*(Ax) \le L^*(x) + \varepsilon.$$

Since ε is arbitrary, the required conclusion follows.

We now give another ingequality sharper than that of Theorem 4. (See Theorem 6 below). It is also an analogue of Theorem 3 given by Devi [11]. We first need to prove a Lemma.

LEMMA.5. Let $(B^{(i)})$ be a sequence of infinite matrices such that (1.3) and (1.5), with $a_{nk}(i)$ replaced by $b_{nk}(i)$, hold. Then, for every $z \in m_0$, we have $\mathfrak{B}\ z = \mathfrak{D}\ y$, where

$$\mathfrak{D} = (d_{nk}(i)) = (b_{nk}(i) - b_{n,k+1}(i)),$$

and

$$y = (y_n) = \left(\sum_{k=1}^{n} z_k\right) \in m .$$

If, further

$$\lim_n \sum_k |d_{nk}(i)| = 0 , \qquad \text{uniformly in } i,$$

then $y \to O(\mathfrak{D})$ and $z \to O(\mathfrak{B})$.

PROOF. The first assertion follows from Abel's partial summation. The second one is a consequence of the Result (3.2.1) given by Duran [10].

We are now in a position to give the inequality mentioned above.

THEOREM.6. $L^*A \le W^*$ if and only if A is F-regular and (2.8) holds.

Before proving the theorem we note that W^* is well-defined (see Devi [11]).

We now come to the proof.

Suppose that $L^*A \le W^*$. Since $W^* \le L^*$, it follows from Theorem 4 that A is F-regular and that (2.8) holds.

Conversely suppose that A is F-regular and (2.8) holds. By Theorem 4, we get

$$u(x) = \inf_{z \in m_0} L^*(A(x+z)) \le W^*(x) \tag{2.10}$$

On the other hand

$$L^*(A(x+z)) = \limsup_n \sup_i \sum_k \frac{1}{n+1}\sum_{r=i}^{i+n} a_{rk}(x_k + z_k) \tag{2.11}$$

Now write

$$b_{nk}(i) = \frac{1}{n+1}\sum_{r=i}^{i+n} a_{rk} .$$

Since A is F-regular, the requirement of Lemma 5 is satisfied. Hence we have that $z \to 0(\mathfrak{B})$. So, we get

$$u(x) \geq \inf_{z \in m_0} \left\{ L^*(Ax) + \mathfrak{L}^*(Az) \right\} = L^*(Ax) \qquad (2.12)$$

Hence the required conclusion follows from (2.10) and (2.11).

The following theorem is a generalization of Result VII given by Kuttner and Maddox [12].

THEOREM.7. Let $\|\mathfrak{A}\| < \infty$ and $\|\mathfrak{B}\| < \infty$. Then $Q_A(x) \leq q_B(x)$ if only if $(\mathfrak{B})$ is a Schur method and

$$\sum_k |a_{nk}(i) - b_{nk}(i)| \to 0 \quad (n \to \infty \text{, uniformly in } i)$$

PROOF. Since the proof uses the technique that Kuttner and Maddox used, [12], we omit the details.

We conclude the paper with the following remark: Since no Schur method is regular, Theorem 7 includes the result that $Q_A(x) \leq q_B(x)$ is impossible when $(\mathfrak{B})$ is a regular method. For example,

$$Q_A(x) \leq \mathfrak{L}(x) \qquad \text{(for every } x \in m\text{)},$$

is impossible.

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