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REMARKS ON A FIXED-POINT THEOREM OF GERALD JUNGCK

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ABSTRACT. Jungck [1] obtained a fixed-point theorem for a pair of continuous self-mappings on a complete metric space. Recently, Barada K. Ray [2] extended the theorem of Jungck [1] for three self-mappings on a complete metric space. In the present paper we omit the continuity of the mapping used by Ray [2] and replace his four conditions by a single condition. Our results so obtained generalize and/or unify fixed-point theorems of Jungck [1], Ray [2], Rhoades [3], Ciric [4], Pal and Maiti [5], and Sharma and Yuel [6].

KEYWORDS AND PHRASES. Fixed Point Theorem, Continuous Self-Mappings, and Complete Metric Space.
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1. INTRODUCTION.

We quote two theorems:

Theorem 1. (Jungck [1]). If S and T are continuous mappings of a complete metric space (X,d) into itself such that

i) $S(X) \subset T(X)$,

ii) $ST = TS$, and

iii) $d(Sx,Sy) \leq \alpha\, d(Tx, Ty)$ for every pair of points $x,y \in X$ and for $\alpha \in [0,1)$, then

$$F_S = F_T = F_{S,T} = \{u\} \text{ for some } u \text{ in } X,$$

where $F_S = \{x \in X: x = Sx\}$, $F_T = \{x \in X: x = Tx\}$

and $F_{S,T} = \{x \in X: x = Sx = Tx\}$.

Theorem 2 (Ray [2]). Let T be a continuous mapping and T_1 and T_2 be any other two mappings of a complete metric space (X,d) into itself such that

i) $TT_i = T_iT$, $i = 1,2$,

ii) $\bigcup_{i=1}^{2} T_i(X) \subseteq T(X)$, and

iii) at least one of the following is satisfied for every pair of points x,y in X:

$$d(T_1x,\ T_2y) \leq \frac{\alpha d(Ty,\ T_2y)\ d(Tx,\ T_1x)}{1 + d(Tx,\ Ty)} + \beta\ d(Tx,Ty),$$

where $0 \leq \alpha,\ \beta,\ \alpha + \beta < 1,$ (1.1)

$$d(T_1x,\ T_2y) \leq \lambda \max \{d(Tx,Ty),\ 1/2[d(Tx,T_1x)+d(Ty,T_2y)],\ 1/2[d(Tx,T_2y)+d(Ty,T_1x)]\}$$

where $0 \leq \lambda < 1,$ (1.2)

$$d(T_1x,T_2y) \leq \mu \max \{d(Tx,Ty),\ d(Tx,T_1y),\ d(Ty,T_2y),\ d(Tx,T_2y),\ d(Ty,T_1x)\}$$

where $0 \leq \mu < 1/2,$ (1.3)

$$d(T_1x,T_2y) \leq \max \{|K_1d(Tx,Ty) - K_2d(Tx,T_1x)|,\ |K_1d(Tx,Ty) - K_2d(Ty,T_2y)]\}$$

where $-1 < K_2 \leq K_1 < K_2 + 1 < 2,\ K_1 < 1.$ (1.4)

Then $F_{T,\ T_1,T_2}$ is non-empty, where

$$F_{T,T_1,T_2} = \{x \in X:\ x = Tx = T_1x = T_2x\}$$

Furthermore, $F_{T_1} = F_{T_2} = F_{T,T_1T_2} = \{u\}$, for some u in X.

2. MAIN RESULTS.

Now we give our result.

THEOREM 2.1. Let (X,d) be a complete metric space. Let $T,T_1,T_2: X \to X$ satisfy (i), (ii) of Theorem 2 and (i) let the following conditions hold for every pair of points x,y in X:

$$d(T_1Tx,T_2Ty) \leq \mu \max \{d(x,T_1Tx),\ d(y,T_2Ty),\ d(y,T_1Tx),d(x,T_2Ty),\ [d(x,T_1Tx)+d(y,T_2Ty)],\ [\frac{\alpha[1+d(y,T_2Ty)]d(x,T_1Tx)}{1 + d(x,y)} + \beta[d(x,T_1Tx)+d(y,T_2Ty)]+ \nu[d(y,T_1Tx)+d(x,T_2Ty)] + \delta\ d(x,y)],\ |K_1d(x,y)-K_2d(x,T_1Tx)|,\ |K_1d(x,y)-K_2d(y,T_2Ty)|\}$$

where $0 \leq \mu \leq 1$, $\alpha,\beta,\nu, \delta \geq 0$, $\alpha + \beta + \nu + \delta < 1$, $2\nu + \delta < 1$,

$$0 \leq \frac{\mu(\beta + \nu + \delta)}{1-\mu(\alpha + \beta + \nu)} < 1, \; -1 < K_2 \leq K_1 < 1 + \mu K_2 < 2, \; K_1 < 1.$$

Then F_{T,T_1T_2} is non-empty, where

$$F_{T,T_1,T_2} = \{x \in X: x = Tx = T_1x = T_2x\}$$

Furthermore, $F_{T_1} = F_{T_2} = F_{T,T_1,T_2} = \{u\}$, for some u in X.

PROOF. Let $x_o \in X$, define

$$x_{2n+1} = T_1x_{2n}, \; n = 0, 1,2\ldots$$

$$x_{2n} = T_2x_{2n-1}, \; n = 1,2,3\ldots$$

Then, using Theorem 2.1, (i), we have

$$d(x_{2n+1}, x_{2n}) \leq K\, d(x_{2n}, x_{2n-1})$$

where $K = \max \{\mu, \frac{\mu}{1-\mu}, \frac{\mu(\beta + \nu + \delta)}{1-\mu(\alpha + \beta + \nu)}, r\}$

$$\text{where } r = \begin{cases} \mu \max \{K_1 - K_2, \dfrac{K_1}{1 + \mu K_2}\}, & K_1 > 0, \\ \mu \max \{K_1 - K_2, \dfrac{-K_1}{1 - \mu K_2}\}, & K_1 < 0. \end{cases}$$

$\therefore \{x_n\}$ is a Cauchy sequence. Since X is complete there exists $u \in X$ such that $x_n \to u$ as $n \to \infty$.

Now,

$$d(T_1Tu, X_{2n}) = d(T_1Tu,T_2Tx_{2n-1}).$$

Then using Theorem 2.1 (i) and allowing $n \to \infty$ such that $x_{2n} \to u$, $x_{2n-1} \to u$ etc, we have $u = T_1Tu$. Hence $u = T_1Tu = TT_1u$ using Theorem 2 (i). Further, $d(x_{2n+1},T_2Tu) = d(T_1Tx_{2n},T_2Tu)$. Again using Theorem 2 (i) and allowing $n \to \infty$ such that $x_{2n} \to u$, $x_{2n+1} \to u$ etc, we have $u = T_2Tu$. Hence $u = T_2Tu = TT_2u$.
Now, let v denote any common fixed point of T_1T and T_2T. From Theorem 2.1 (i), it is easy to see that $u = v$ since $2\nu + \delta < 1$. For proving $u = Tu$ we have

$$d(Tu,u)=d(TT_1Tu,T_2Tu) = d(T_1TTu,T_2Tu)$$

which yields $Tu = u$ using Theorem 2.1 (i). Hence $u = T_1Tu = T_1u$. Similarly, $u = T_2Tu = T_2u$. Hence, $u = Tu = T_1u = T_2u$ which shows that F_T,F_{T_1},F_{T_2} are non-empty. Then we can see that $F_{T_1} = F_{T_2} = F_{T,T_1T_2} = \{u\}$ for some u in X. This completes the proof.

EXAMPLE. Let $X = [0,1]$ with Euclidean metric d. Let $Tx = x$, $0 \leq x < 1$, $Tx = 1/2$, $x = 1$, $T_1x = \frac{x}{4}$, $0 \leq x < 1$, $T_1x = \frac{1}{8}$, $x = 1$, $T_2x = \frac{x}{8}$, $0 \leq x < 1$, $T_2x = \frac{1}{16}$, $x = 1$. Here T, T_1, T_2, are all discountinuous at $x = 1$ and have a unique common fixed point $x = 0$. Take $x = \frac{1}{2}$, $y = \frac{1}{4}$. Obviously all the conditions (i), (ii) of Theorem 2 and (i) of Theorem 2.1 hold true. Hence the result.

REMARKS. (1) Contractive Definition 20 of Rhoades [3] is a special case of condition (i) of Theorem 2.1. (2) Theorem 1 of Circi [4], Theorem 1 of Pal and Maiti [5], and Theorem 4 of Sharma and Yuel [6] are special cases of Theorem 2.1.

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