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TWO COUNTABLE HAUSDORFF ALMOST REGULAR SPACES EVERY CONTINUOUS MAP OF WHICH INTO EVERY URYSOHN SPACE IS CONSTANT

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ABSTRACT. We construct two countable, Hausdorff, almost regular spaces I(S), I(T) having the following properties: (1) Every continuous map of I(S) (resp. I(T)) into every Urysohn space is constant (hence, both spaces are connected). (2) For every point of I(S) (resp. of I(T)) and for every open neighbourhood U of this point there exists an open neighbourhood V of it such that V⊆U and every continuous map of V into every Urysohn space is constant (hence both spaces are locally connected). (3) The space I(S) is first countable and the space I(T) nowhere first countable. A consequence of the above is the construction of two countable, (connected) Hausdorff, almost regular spaces with a dispersion point and similar properties. Unfortunately, none of these spaces is Urysohn.

KEY WORDS AND PHRASES. Hausdorff, Urysohn, almost regular, connected, locally connected, dispersion point.
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1. INTRODUCTION.

Iliadis and Tzannes [1] posed the question whether for every countable Hausdorff space R there exists a countable Hausdorff (Urysohn, almost regular) space I(X) having the following properties:

(1) Every continuous map of I(X) into the given space R is constant

(2) For every point x of X and for every open neighbourhood U of x there exists an open neigbourhood V of x such that V⊆U and every continuous map of V into the given space R is constant. (Spaces having properties (1) and (2) are called in [1], R-monolithic and locally R-monolithic, respectively, and by their construction are connected and locally connected).

It is obvious that the above mentioned spaces I(S), I(T) answer partially this question (in case the countable space R is Urysohn) because both have properties (1) and (2) for every Urysohn space.

For countable spaces, first countable or nowhere first countable, connected, locally connected Hausdorff or Urysohn, almost regular or having a dispersion point, or with other properties see [1]-[37].

A space X is called 1) Urysohn, if for every two distinct points x, y of X there exist open neighbourhoods V, U of the points x, y such that $\bar{V}\cap\bar{U}=\emptyset$ 2) Almost regular if it has a dense subset of regular points. A point p of a space X is called a regular point (or, that X is regular at p) if for every open neighbourhood U of p there exists an open neighbourhood V of p such that $\bar{V}\subseteq U$.

A point p of a connected space X is called a dispersion point if the space $X\setminus\{p\}$ is totally disconnected.

Let X be a set and let $\{X_i: i\in I\}$ be a family of subsets of X with each X_i having a topology. Assume that for every $(i,j)\in I\times I$ both 1) The topologies of X_i, X_j, agree on $X_i\cap X_j$, 2) Each $X_i\cap X_j$ is open in X_i and in X_j. Then the weak topology in X induced by $\{X_i: i\in I\}$ is $\tau=\{U: U\cap X_i$ is open in X_i for every $i\in I\}$.

2. AN AUXILIARY SPACE.

The following space X which is due to Urysohn [35], will be used for the construction of two auxiliary spaces S,T which with the help of the embedding described in [1] will yield the required spaces.

On the set $X=\{a_{ij}, b_{ij}, c_i, a, b: i=1,2,\ldots, j=1,2,\ldots\}$ we define the following topology: Each a_{ij}, b_{ij}, $i=1,2,\ldots$, $j=1,2,\ldots$, is isolated. A basis of open neigbourhoods of the points c_i, $i=1,2,\ldots$, a,b are the sets

$$B(c_i)=\{V^n(c_i) = \bigcup_{j=n}^{\infty}\{a_{ij}, b_{ij}, c_i\}: n=1,2,\ldots\}$$

$$B(a)=\{V^n(a) = \bigcup_{j=1}^{\infty}\bigcup_{i=n}^{\infty}\{a_{ij}, a\} : n=1,2,\ldots\}$$

$$B(b)=\{V^n(b)= \bigcup_{j=1}^{\infty}\bigcup_{i=n}^{\infty}\{b_{ij}, b\} : n=1,2\ldots\}$$

The countable space X has the following properties:

(1) It is Hausdorff, almost regular (all points of X besides a,b are regular points).

(2) f(a)=f(b), for every continuous map f of X into every Urysohn space, (because the points a,b can not be separated by disjoint closed neighbourhoods).

Let X_n, $n=1,2,\ldots$, be disjoint copies of X and let a_n, b_n be the copies of a,b, respectively in the space X_n.

For every $n=1,2,\ldots$, we attach the space X_n to the space X_{n+1} identifying the point b_n with a_{n+1}. We set $a_1=x_0$, $b_n=a_{n+1}=x_n$, $n=1,2,\ldots$, and on the space

$$Y=\{x_0, x_1, \ldots, x_n \ldots\} \cup \bigcup_{n=1}^{\infty}(X_n\setminus\{a_n, b_n\})$$

we add one more point p.

On the set $Z=Y\cup\{p\}$ we define the basis of open neighbourhoods of the point p to be the sets

$$U_n(p)=\{x_i : i\geq n\}\cup \bigcup_{i=n+1}^{\infty}(X_n\setminus(a_n,b_n))\bigcup U(x_n)$$

$n=1,2,\ldots$, where $U(x_n)$ is an open set of x_n in X_n. Since

$$\overline{U_{n+1}(p)}=\{x_i : i\geq n+1\}\cup \bigcup_{i=n+2}^{\infty}(X_{n+1}\setminus(a_{n+1},b_{n+1}))\bigcup \overline{U(x_{n+1})}$$

it follows that $\overline{U_{n+1}(p)} \subseteq U_n(p)$, that is, Z is regular at the point p.

Now consider two disjoint copies Z^-,Z^+ of Z and let X_n^- , X_n^+ be the copies of X_n in Z^-, Z^+, respectively. Let x_o^-, x_o^+ and p^-,p^+ be the copies of x_o and p in Z^-, Z^+, respectively. We attach the copy Z^- to Z^+ identifying x_o^- with x_o^+. We set $x_o^-=x_o^+=0$ and we consider the space

$S=(Z^-\setminus\{x_o^-\})\bigcup\{0\}\bigcup(Z^+\setminus\{x_o^+\})$ which has the following properties:

(1) It is Hausdorff almost regular (all points of S besides $\{x_n^-,0,x_n^+ : n=1,2,\ldots,\}$ are regular points).

(2) $f(p^-)=f(p^+)$, for every continuous map f of S into every Urysohn space.(To prove this observe that since S is not Urysohn at every pair (x_{n+1}^-,x_n^-), $(x_1^-,0)$, $(0,x_1^+)$, (x_n^+,x_{n+1}^+), $n=1,2,\ldots$, it follows that $f(x_{n+1}^-)=f(x_n^-)=f(0)=f(x_n^+)=f(x_{n+1}^+)$, for every $n=1,2,\ldots$, and hence $f(p^-)=f(p^+)$ for every continuous map f of S into every Urysohn space).

3. MAIN RESULTS.

PROPOSITION 3.1. There exists a countable, first countable, Hausdorff, almost regular space I(S) having the following properties:

(1) Every continuous map of I(S) into every Urysohn space is constant (hence I(S) is connected).

(2) For every point s of I(S) and for every open neighbourhood U of s, there exists an open neighbourhood V of s, such that $V\subseteq U$ and every continuous map of V into every Urysohn space is constant (hence I(S) is locally connected).

PROOF. Let S be the space consrtucted above. We set $J=S\setminus\{p^-,p^+\}$, $\Lambda_o=S\times S\setminus\Delta(S)$ $\Delta(S)=\{(x,y)\in S\times S : x=y\}$ and we construct,as in [1],first the space $I^1(S,\Lambda_o)=S\bigcup\bigcup_{\lambda\in\Lambda_o}J^{\lambda}$, then , by induction,the space $I^n(S,\Lambda_{n-1})=I^1(I^{n-1}(S,\Lambda_{n-2}), \Lambda_{n-1})$ and finally, the space

$$I(S)=\bigcup_{n=1}^{\infty}I^n(S,\Lambda_{n-1}).$$

That I(S) is Hausdorff almost regular is proved as in [1,Lemma 2].

That it is first countable follows by the fact that S is first countable (all spaces X,Y,Z are first countable) and by relation (7) [1,Theorem 1] which in this case becomes $\chi(I(S))=\max\{\chi(S), \aleph_o\}=\aleph_o$.

In order to prove Properties (1) and (2), observe that by the property of the space S (that $f(p^-)=f(p^+)$, for every continuous map of S into every Urysohn space) and by the definition of topology on $I^n(S,\Lambda_{n-1})$ it follows that 1) Every continuous map of $I^n(S,\Lambda_{n-1})$ into every Urysohn space is constant on $I^{n-1}(S,\Lambda_{n-2})$ and 2) For every point s of $I^{n-1}(S,\Lambda_{n-2})$ and for every open neighbourhood U of s in $I^n(S,\Lambda_{n-1})$, there exists an open neighbourhood V of s in $I^n(S,\Lambda_{n-1})$ such that $V\subseteq U$ and every continuous map of V into every Urysohn space, is constant on $V\cap I^{n-1}(S,\Lambda_{n-2})$. Finally, by the definition of topology on I(S) it follows that 1) Every continuous map of I(S) into every Urysohn space is constant (hence I(S) is connected because the set of real-numbers with the usual topology is a Urysohn space) and 2) For every point s of I(S) and for every open neighbourhood U of s there exists an open neighboorhood V of s such that $V\subseteq U$ and every continuous map of V into every Urysohn space is constant (hence I(S) is locally connected).

REMARK 3.1. If we consider as initial space the space X of Section 1 then the resulting space I(X) will be a countable, first countable, Hausdorff, anti-Urysohn space having Properties (1) and (2), (a space S is called anti-Urysohn if for every $x,y\in S$ and for every open neighbourhoods U,V of x,y respectively, $\bar{U}\cap\bar{V}\neq\emptyset$).

REMARK 3.2. If on the set $I^1(S,\Lambda_0)$ we define the topology to be the weak topology induced by the spaces J^λ, $\lambda\in\Lambda_0$ and S, then the space $I^1(S,\Lambda_0)$ is not first countable at every point of S. Hence the set $I^n(S,\Lambda_{n-1})$ with the weak topology induced by the spaces J^λ, $\lambda\in\Lambda_{n-1}$ and $I^{n-1}(S,\Lambda_{n-2})$ is not first countable at every point of $I^{n-1}(S,\Lambda_{n-2})$. Therefore if on the set $I(S)=\bigcup_{n=1}^{\infty} I^n(S,\Lambda_{n-1})$ we define the topology to be the weak topology induced by the spaces $I^n(S,\Lambda_{n-1})$, $n=1,2,\ldots$, then the space I(S) will be nowhere first countable. It is easy to prove that while I(S) is Hausdorff almost regular having Property (1), it is not locally connected (hence does not have Property (2)).

PROPOSITION 3.2. There exists a countable, nowhere first countable, Hausdorff, almost regular space having Properties (1) and (2) of Proposition 3.1.

PROOF. We consider the space $M=\mathbb{N}\bigcup\{p\}$, $p\in\beta\mathbb{N}\setminus\mathbb{N}$, where $\mathbb{N}$ is the set of natural numbers and $\beta\mathbb{N}$ is the Stone-Cech compactification of $\mathbb{N}$. The space M is countable regular and not first countable at the point p.

Let M_1,M_2 be two disjoint copies of M and let p_1,p_2 be the copies of p in M_1,M_2, respectively. We attach the copies M_1,M_2 to the space S attaching the point p_1 to p^- and the point p_2 to p^+. We consider the space $T=S\bigcup M_1'\bigcup M_2'$, where $M_i'=M_i\setminus\{p_i\}$, i=1,2.
Obviously, the space T is Hausdorff almost regular, not first countable at the points p^-,p^+ and $f(p^-)=f(p^+)$, for every continuous map f into every Urysohn space.

The space I(T) constructed as in Proposition 3.1 is the required space.

COROLLARY 3.1. There exists a countable, first countable, (or nowhere first countable) Hausdor f f, almost regular space having the following properties:

(1) Every continuous map of it into every Urysohn space is constant

(hence it is connected).

(2) It has a dispersion point.

(3) For every open neighbourhood U of the dispersion point there exists an open neighbourhood V of it such that every continuous map of V into every Urysohn space is constant (hence it is locally connected only at the dispersion point).

PROOF. First we observe that both spaces S and T of Section 2 and Proposition 3.3, respectively are totally disconnected. Hence, we can apply [1, Theorem 2] using as initial space the space S, for the construction of countable first countable and the space T, for the construction of the countable nowhere first countable space. The other properties of both spaces are proved as in Proposition 3.1.

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