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ON APPROXIMATION IN THE L_p-NORM BY HERMITE INTERPOLATION

MIN GUOHUA

Department of Mathematics
East China Institute of Technology
Nanjing, Jiangsu, 210014
People's Republic of China

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ABSTRACT. L_p-approximation by the Hermite interpolation based on the zeros of the Tchebycheff polynomials of the first kind is considered. The corresponding result of Varma and Prasad [1] is generalized and perfected.

KEY WORDS AND PHRASES. Approximation, Hermite Interpolation, L_p-Norm.
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1. INTRODUCTION.

Let $-1 < x_n < x_{n-1} < \cdots < 1$ be the zeros of $T_n(x) = \cos n\theta, (\cos\theta = x)$, the nth degree Tchebycheff polynomial of the first kind.

If $f \in C^1[-1,1]$, then it is known that a Hermite interpolation $H_n^*(f,x)$ of degree $\leq 2n-1$ which satisfies the conditions

$$H_n^*(f,x_k) = f(x_k) \text{ and } H_n^{*\prime}(f,x_k) = f'(x_k) \qquad k = 1, \cdots, n$$

is given by

$$H_n^*(f,x) = \sum_{k=1}^{n} f(x_k)h_k(x) + \sum_{k=1}^{n} f'(x_k)\sigma_k(x) \tag{1.1}$$

where

$$h_k(x) = (1 - xx_k)\left(\frac{T_n(x)}{n(x-x_k)}\right)^2 \geq 0, \quad \sum_{k=1}^{n} h_k(x) \equiv 1 \tag{1.2}$$

$$\sigma_k(x) = (x - x_k)l_k^2(x), \qquad l_k(x) = \frac{T_n(x)}{T_n'(x_k)(x-x_k)} \tag{1.3}$$

Concerning the polynomial $H_n^*(f,x)$, Varma and Prasad [1] proved the following:

THEOREM A. Let $f \in C^1[-1,1]$, then we have

$$\left(\int_{-1}^{1} \frac{1}{\sqrt{1-x^2}} \,|\, H_n^*(f,x) - f(x) \,|^2 dx\right)^{1/2} \leq c\, n^{-1}E_{2n-2}(f'), \tag{1.4}$$

where $E_{2n-2}(f')$is the best approximation to $f'(x)$ by polynomials of degree at most $2n-2$ and c is a positive absolute constant.

Naturally, one raises the problem that if there is similar result of (1.4) in $L_p(p > 0)$ norm. Here we give an affirmative answer for the above problem, we shall prove the following:

THEOREM 1. Let $f \in C^1[-1,1]$, then we have

$$\left(\int_{-1}^{1} \frac{1}{\sqrt{1-x^2}} \,|\, H_n^*(f,x) - f(x) \,|^p dx\right)^{1/p} \leq cn^{-1}E_{2n-2}(f') \tag{1.5}$$

Therefore the corresponding result of [1] is generalized and perfected.

2. LEMMAS AND THE PROOF OF THEOREM 1.

At first, we state and prove several lemmas.

LEMMA 1 (Féjer [2]). If

$$\sum_{k=1}^{n} l_k^2(x) \le 2\,, \tag{2.1}$$

therefore it follows that

$$\sum_{k=1}^{n} |\, l_k^2(x)\,|^{\,r} \le c \qquad r = 3,4,\cdots. \tag{2.2}$$

LEMMA 2. Let k be even and $y_1, y_2, \cdots, y_k$ be distinct integers between 1 and n, then we have $(k = 2m)$

$$\int_{-1}^{1} \frac{1}{\sqrt{1-x^2}}\, \sigma_{\gamma_1}(x)\sigma_{\gamma_2}(x)\cdots\sigma_{\gamma_k}(x)dx = 0 \tag{2.3}$$

PROOF. Since

$$\cos^{4m-1} n\theta = \frac{1}{2^{2(2m-1)}} \sum_{j=0}^{2m-1} \binom{4m}{j} \cos(4m-2j-1)n\theta$$

$$= \sum_{i \ge n}^{(4m-1)n} \mu_i \cos i\theta = \sum_{i \ge n}^{(4m-1)n} \mu_i\, T_i(x) \tag{2.4}$$

and

$$\frac{T_n(x)}{(x - x_{\gamma_1})\cdots(x - x_{\gamma_k})} = q_{n-2m}(x) \tag{2.5}$$

where $q_{n-2m}(x)$ is a polynomial of degree $\le n - 2m$.

On using these ideas together with orthogonality of Tchebycheff polynomials, we obtain

$$\int_{-1}^{1} \frac{1}{\sqrt{1-x^2}}\, \sigma_{\gamma_1}(x)\sigma_{\gamma_2}(x)\cdots\sigma_{\gamma_k}(x)dx$$

$$= \frac{1}{[T'_n(x_{\gamma_1})\cdots T'_n(x_{\gamma_k})]^2} \sum_{i \ge n}^{(4m-1)n} \int_{-1}^{1} \frac{1}{\sqrt{1-x^2}}\, T_i(x) q_{n-2m}(x)dx = 0$$

This proves Lemma 2.

To prove Theorem 1 in the general case, we again follow the method of Erdös and Feldheim [3], it is enough to prove for even values of p only. To illustrate the method we limit for the case $p = 4$. For arbitrary fixed even p the proof is similar. Let $s_{2n-1}(x)$ be the polynomial of best approximation to $f(x)$ by the polynomials of degree $\le 2n-1$. One can easily see that for $-1 \le x \le 1$:

$$H_n^*(f,x) - f(x) = H_n^*(f - s_{2n-1}, x) + s_{2n-1}(x) - f(x) \tag{2.6}$$

One notes that

$$|\, a + b\,|^{\,p} \le c(p)(\,|\, a\,|^{\,p} + |\, b\,|^{\,p}) \tag{2.7}$$

where $c(p)$ is a constant of dependent of p only.

$$\int_{-1}^{1} \frac{1}{\sqrt{1-x^2}}\, |\, H_n^*(f,x) - f(x)\,|^{4} dx$$

$$\le c\left[\int_{-1}^{1} \frac{1}{\sqrt{1-x^2}} H_n^{*4}(f - s_{2n-1}, (x)dx + \int_{-1}^{1} \frac{1}{\sqrt{1-x^2}} (s_{2n-1}(x) - f(x))^4 dx\right]$$

$$\le c(I_1 + I_2) \tag{2.8}$$

From the definition of $s_{2n-1}(x)$ we have

$$|s_{2n-1}(x)-f(x)| \leq E_{2n-1}(f) \tag{2.9}$$

where $E_{2n-1}(f)$ is the best approximation of $f(x)$.
From (2.9) we have

$$I_2 \leq \pi E^4_{2n-1}(f) \tag{2.10}$$

On using (1.1) and (2.7) we have

$$I_1 \leq c\left\{\int_{-1}^{1} \frac{1}{\sqrt{1-x^2}}\left[\sum_{k=1}^{n} (f(x_k)-s_{2n-1}(x_k))h_k(x)\right]^4 dx\right.$$

$$+\int_{-1}^{1} \frac{1}{\sqrt{1-x^2}}\left[\sum_{k=1}^{n} (f'(x_k)-s'_{2n-1}(x_k))\sigma_k(x)\right]^4 dx$$

$$= c(I_3+I_4) \tag{2.11}$$

Now from (1.2) and (2.9) it follows that

$$I_3 \leq \int_{-1}^{1} \frac{1}{\sqrt{1-x^2}}\left[\sum_{k=1}^{n} (f(x_k)-s_{2n-1}(x_k))h_k(x)\right]^4 dx$$

$$\leq \pi E^4_{2n-1}(f) \tag{2.12}$$

Let $\Delta_k = f'(x_k)-s'_{2n-1}(x_k) \quad k=1,\cdots,n$

One notes that

$$\left[\sum_{k=1}^{n} \Delta_k\sigma_k(x)\right]^4 = \sum_{k\neq 1}^{n} \Delta^4_k\sigma^4_k(x) + \sum\sum_{k\neq j} \Delta^3_k\Delta_j(x)\sigma^3_k(x)\sigma_j(x)$$

$$+\sum\sum\sum_{k\neq j\neq i} \Delta^2_k\Delta_j\Delta_i\sigma^2_k(x)\sigma_j(x)\sigma_i(x) + \sum\sum\sum\sum_{k\neq j\neq i\neq s} \Delta_k\Delta_j\Delta_i\Delta_s\sigma_k(x)\sigma_j(x)\sigma_i(x)\sigma_s(x)$$

$$+\sum\sum_{k\neq j} \Delta^2_k\Delta^2_j\sigma^2_j(x)\sigma^2_j(x) = L_1(x)+L_2(x)+L_3(x)+L_4(x)+L_5(x) \tag{2.13}$$

One notes also that

$$|T_n(x)| \leq 1 \tag{2.14}$$

$$1-x^2_k|\Delta_k| \leq 40E_{2n-2}(f') \quad \text{(See [4])} \tag{2.15}$$

and

$$\int_{-1}^{1} \frac{1}{\sqrt{1-x^2}} l_k(x)l_j(x)dx = \begin{cases} 0 & k\neq j \\ \frac{\pi}{n} & k=j \end{cases} \tag{2.16}$$

Therefore from (1.3), (2.2) and (2.14)-(2.16) we have

$$\int_{-1}^{1} \frac{1}{\sqrt{1-x^2}} L_1(x)dx = \sum_{k=1}^{n} \frac{\Delta^4_k}{T'^4_n(x_k)} \int_{-1}^{1} \frac{1}{\sqrt{1-x^2}} T^4_n(x)l^4_k(x)dx$$

$$\leq \frac{2}{n^4}\sum_{k=1}^{n}\left(\sqrt{1-x^2_k}|\Delta_k|\right)^4 \int_{-1}^{1} \frac{1}{\sqrt{1-x^2}} l^2_k(x)dx \leq c\frac{E^4_{2n-2}(f')}{n^4} \tag{2.17}$$

One notes that

$$L_2(x) = \left(\sum_{k=1}^{n} \Delta^3_k\sigma^3_k(x)\right)\left(\sum_{k=1}^{n} \Delta_k\sigma_k(x)\right) - L_1(x)$$

Using (1.3), (2.2), (2.14-(2.17) and the Cauchy inequality we have

$$\int_{-1}^{1} \frac{1}{\sqrt{1-x^2}} \mid L_2(x) \mid dx$$
$$\leq \int_{-1}^{1} \frac{1}{\sqrt{1-x^2}} \sum_{k=1}^{n} \frac{\left(\sqrt{1-x_k^2} \mid \Delta_k \mid\right)^3}{n^3} \mid T_n^3(x) \mid\mid l_k^3(x) \mid$$
$$\mid \sum_{k=1}^{n} \frac{(-1)^{k-1}\sqrt{1-x_k^2}\,\Delta_k}{n} T_n(x) l_k(x) \mid dx + c\frac{E_{2n-2}^4(f')}{n^4}$$
$$\leq c\frac{E_{2n-2}^3(f')}{n^4}$$
$$\int_{-1}^{1} \frac{1}{\sqrt{1-x^2}} \mid \sum_{k=1}^{n} (-1)^{k-1}\sqrt{1-x_k^2}\,\Delta_k l_k(x) \mid dx + c\frac{E_{2n-2}^4(f')}{n^4}$$
$$\leq \frac{E_{2n-2}^3(f')}{n^4}\left(\int_{-1}^{1} \frac{1}{\sqrt{1-x^2}}\left[\sum_{k=1}^{n} (-1)^{k-1}\sqrt{1-x_k^2}\;\Delta_k l_k(x)\right]^2 dx\right)^{1/2} + c\frac{E_{2n-2}^4(f')}{n^4}$$
$$\leq c\frac{E_{2n-2}^4(f')}{n^4} \tag{2.18}$$

From (1.3), (2.1), (2.15) and the estimation of $\int_{-1}^{1} \frac{1}{\sqrt{1-x^2}} \mid L_2(x) \mid dx$, we note that

$$L_3(x) = \left(\sum_{k=1}^{n} \Delta_k^2 \sigma_k^2(x)\right)\left[\left(\sum_{k=1}^{n} \Delta_k \sigma_k(x)\right)^2 - \sum_{j=1}^{n} \Delta_j^2 \sigma_j^2(x)\right] - L_2(x)$$

Thus we have also that

$$\int_{-1}^{1} \frac{1}{\sqrt{1-x^2}} \mid L_3(x) \mid dx \leq c\frac{E_{2n-2}^4(f')}{n^4} \tag{2.19}$$

Using Lemma 2 we have

$$\int_{-1}^{1} \frac{1}{\sqrt{1-x^2}} L_4(x) dx = 0 \tag{2.20}$$

One notes that

$$L_5(x) = \left(\sum_{k=1}^{n} \Delta_k^2 \sigma_k^2(x)\right)^2 - L_1(x)$$

and similar to estimation of $\int_{-1}^{1} \frac{1}{\sqrt{1-x^2}} \mid L_3(x) \mid dx$ we have

$$\int_{-1}^{1} \frac{1}{\sqrt{1-x^2}} \mid L_5(x) \mid dx \leq c\frac{E_{2n-2}^4(f')}{n^4} \tag{2.21}$$

From (2.17) - (2.21) we have

$$I_4 \leq c\frac{E_{2n-2}^4(f')}{n^4} \tag{2.22}$$

Combining (2.11), (2.12) and (2.22) we obtain

$$\int_{-1}^{1} \frac{1}{\sqrt{1-x^2}} \mid H_n^*(f,x) - f(x) \mid^4 dx \leq c\frac{E_{2n-2}^4(f')}{n^4} \tag{2.23}$$

This proves Theorem 1.

3. REMARKS.

1. Concerning quasi-Hermite interpolation [5] based on the zeros of Tchebycheff polynomial of the second kind, there is similar result in Theorem 1.

2. For almost-Hermite interpolation [6] based on the zeros of $(1-x)J_n^{(1/2,\,-1/2)}(x)$ (or $(1+x)J_n^{(-1/2,1/2)}(x)$) (where $J_n^{(\alpha,\beta)}(x)$ be the Jacobi polynomial), there is similar result of Theorem 1 also.

Here we omit the details.

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