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ON Θ-C-OPEN SETS

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ABSTRACT. The properties of the collection of complements of θ-closures of sets in a topological space are investigated in this paper. A strong continuity condition is defined in terms of these sets. Some applications to H-closed spaces and Katetov spaces are given.

KEY WORDS AND PHRASES. θ-C-open, θ-C-continuous, super-continuous, H-closed space, Katetov space.
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1. PRELIMINARIES.

All spaces are topological spaces with no separation axioms assumed unless explicitly stated. Let A be a subset of a space X. The closure of A and the interior of A are denoted by Cl A and Int A, respectively. The set A is said to be regular open (regular closed) if A = Int Cl A (A = Cl Int A). The θ-closure (δ-closure) (Velicko [1]) of A is the set of all x in X such that every closed neighborhood (the interior of every closed neighborhood) of x intersects A nontrivially. The θ-closure and the δ-closure of A are denoted by Cl_θ A and Cl_δ A, respectively. The set A is called θ-closed (δ-closed) if A = Cl_θ A (A = Cl_δ A). A set A is said to be θ-open (δ-open) if its complement is θ-closed (δ-closed). For a given space X both the collection of all θ-open sets and the collection of all δ-open sets form topologies. The collection of δ-open sets is usually referred to as the semi-regular topology.

DEFINITION 1. Arya and Gupta [2]. A function f: X $\to$ Y is said to be completely continuous if for each open subset V of Y, $f^{-1}(V)$ is regular open in X.

DEFINITION 2. Munshi and Basson [3]. A function f: X $\to$ Y is said to be super-continuous if for each x $\in$ X and each open neighborhood V of f(x), there exists an open neighborhood U of of x for which $f(\text{Int Cl } U) \subseteq V$.

DEFINITION 3. Long and Herrington [4]. A function $f: X \to Y$ is said to be strongly θ-continuous if for each $x \in X$ and each open neighborhood V of f(x), there exists an open neighborhood U of x for which $f(\text{Cl } U) \subseteq V$.

DEFINITION 4. Porter and Tikoo [5]. A space X is said to be H-closed if X is a closed subset in every space containing X as a subspace.

DEFINITION 5. Porter and Tikoo [5]. A space is said to be Katětov if it has a coarser minimal H-closed topology or equivalently a coarser H-closed topology.

2. θ-C-OPEN SETS

We define a subset U of a space X be θ-C-open provided there exists a subset A of X for which $X - U = \text{Cl}_\theta A$. We call a set θ-C-closed if its complement is θ-C-open or equivalently if there is a subset A of X such that the set equals $\text{Cl}_\theta A$.

THEOREM 1. If U is θ-open, then U is θ-C-open.

PROOF. Since U is θ-open, X - U is θ-closed. Hence $X - U = \text{Cl}_\theta(X - U)$.

THEOREM 2. If U is open, then Int Cl U is θ-C-open.

PROOF. $\text{Int Cl } U = X - \text{Cl}(X - \text{Cl } U)$. Since $X - \text{Cl } U$ is open, $\text{Cl}(X - \text{Cl } U) = \text{Cl}_\theta(X - \text{Cl } U)$ (Velicko [1]). Hence $X - \text{Int Cl } U = \text{Cl}_\theta(X - \text{Cl } U)$.

COROLLARY. If U is regular open, then U is θ-C-open.

Since the real numbers with the usual topology contain θ-open sets that are not regular open, it follows that the real numbers contain θ-C-open sets that are not regular open.

THEOREM 3. Regular openness is equivalent to θ-C openness if and only if $\text{Cl}_\theta A$ is regular closed for every set A.

PROOF. Let X be a space. Assume regular openness is equivalent to θ-C-openness and let $A \subseteq X$. Then $X - \text{Cl}_\theta A$ is regular open. Thus $X - \text{Cl}_\theta A = \text{Int Cl}(X - \text{Cl}_\theta A) = \text{Int}(X - \text{Int Cl}_\theta A) = X - \text{Cl Int Cl}_\theta A$. Therefore $\text{Cl}_\theta A = \text{Cl Int Cl}_\theta A$ which implies that $\text{Cl}_\theta A$ is regular closed.

Assume $\text{Cl}_\theta A$ is regular closed for every set A. Suppose U is θ-C-open and let $A \subseteq X$ such that $U = X - \text{Cl}_\theta A$. Then $\text{Int Cl } U = \text{Int Cl}(X - \text{Cl}_\theta A) = \text{Int}(X - \text{Int Cl}_\theta A) = X - \text{Cl Int Cl}_\theta A = X - \text{Cl}_\theta A = U$. Therefore U is regular open and hence regular openness is equivalent to θ-C-openness.

THEOREM 4. If U is θ-C-open, then U is a union of regular open sets (that is δ-open).

PROOF. Let U be θ-C-open. Let $x \in U$. Since U is θ-C-open, there exists a set $A \subseteq X$ such that $U = X - \text{Cl}_\theta A$. Because $x \notin \text{Cl}_\theta A$, there exists an open set W for which $x \in W$ and $(\text{Cl } W) \cap A = \emptyset$. Hence $x \in \text{Int Cl } W \subseteq X - \text{Cl}_\theta A = U$. Thus U is a union of regular open sets.

It follows from Theorem 4 and the corollary to Theorem 2 that the θ-C-open sets form a basis for the semi-regular topology.

THEOREM 5. The intersection of two θ-C-open sets is θ-C-open.

PROOF. Let U and V be θ-C-open sets. There exist sets A and B such that $U = X - Cl_\theta A$ and $V = X - Cl_\theta B$. Then $U \cap V = (X - Cl_\theta A) \cap (X - Cl_\theta B) = X - (Cl_\theta(A) \cup Cl_\theta(B)) = X - Cl_\theta(A \cup B)$.

The following example shows that the union of two θ-C-open sets need not be θ-C-open. It follows that the θ-C-open sets do not form a topology and hence θ-C-openness is not equivalent to either δ-openness or θ-openness.

EXAMPLE 1. Let $X = \{a, b, c\}$ and $\mathfrak{T} = \{X, \emptyset, \{a\}, \{c\}, \{a, c\}\}$. The θ-C-open sets of X are X, $\emptyset$, {a}, and {c}.

3. θ-C-CONTINUITY.

We define a function $f: X \to Y$ to be θ-C-continuous if for each open subset V of Y, $f^{-1}(V)$ is θ-C-open in X. Since θ-C-open sets are open, obviously θ-C-continuity implies continuity. Since by Theorem 2 regular openness implies θ-C-openness, complete continuity implies θ-C-continuity. The identity mapping on the real numbers with the usual topology is θ-C-continuous but not completely continuous.

THEOREM 6. (Munshi and Basson [3]) A function $f: X \to Y$ is super-continuous if and only if the inverse image of each open set in Y is δ-open in X.

By Theorem 4 every θ-C-open set is δ-open. Hence θ-C-continuity implies super-continuity. The identity mapping on the space in Example 1 is super-continuous but not θ-C-continuous.

It also follows from Theorem 4 that the corresponding "local" or "pointwise" version of θ-C-continuity is equivalent to super-continuity.

THEOREM 7. A function $f: X \to Y$ is super-continuous if and only if for each $x \in X$ and each open neighborhood V of f(x), there exists a θ-C-open set $U \subseteq X$ for which $x \in U$ and $f(U) \subseteq V$.

THEOREM 8. (Long and Herrington [4]). A function $f: X \to Y$ is strongly θ-continuous if and only if the inverse image of each open set in Y is θ-open in X.

From Theorem 1 θ-openness implies θ-C-openness. Hence strong θ-continuity implies θ-C-continuity.

EXAMPLE 2. Let $X = \{a. b, c\}$, $\mathfrak{T}_1 = \{X, \emptyset, \{a\}, \{c\}, \{a, c\}\}$ and $\mathfrak{T}_2 = \{X, \emptyset, \{a\}\}$. The identity mapping $(X, \mathfrak{T}_1) \to (X, \mathfrak{T}_2)$ is θ-C-continuous but not strongly θ-continuous.

Based upon the above theorems and remarks, we have the following implications, none of which are reversible.

$$\left.\begin{array}{l}\text{complete continuity}\\ \text{strong } \theta\text{-continuity}\end{array}\right\} \Rightarrow \theta\text{-C-continuity} \Rightarrow \text{super-continuity}$$

THEOREM 9. If $f: X \to Y$ is θ-C-continuous and $Cl_\theta A$ is regular closed for every subset A of X, then f is completely continuous.

PROOF. By Theorem 3 θ-C-openness is equivalent to regular openness.

THEOREM 10. If f: X $\to$ Y is θ-C-continuous and for every subset A of X, Cl_θ A is θ-closed, then f is strongly θ-continuous.

PROOF. Follows from Theorem 8.

The following theorems and examples illustrate some of the basic properties of θ-C-continuous functions.

THEOREM 11. If f: X $\to$ Y is θ-C-continuous and g: Y $\to$ Z is continuous, then $g \circ f$: X $\to$ Z is θ-C-continuous.

The proof is routine.

COROLLARY. The composition of two θ-C-continuous functions is θ-C-continuous.

THEOREM 12. Let f_α: X $\to$ Y_α be a function for each α in A and let f: X $\to$ $\Pi\, Y_\alpha$ be given by $f(x) = (f_\alpha(x))$. If f is θ-C-continuous, then f_α is θ-C-continuous for each α in A.

PROOF. For each $\alpha \in A$ denote the projection onto Y_α by p_α. Then $f_\alpha = p_\alpha \circ f$ is θ-C-continuous by Theorem 11.

The proof of the next theorem follows from Theorem 12.

THEOREM 13. Let f: X $\to$ Y be a function and let g: X $\to$ X $\times$ Y given by g(x) = (x, f(x)) be its graph function. If g is θ-C-continuous, then f is θ-C-continuous.

The following example shows that the converse of Theorem 13 does not hold.

EXAMPLE 3. Let X = {a, b, c} and $\mathfrak{T}$ = {X, $\emptyset$, {a}, {c}, {a, c}}. Define f: X $\to$ X by f(a) = f(b) = f(c) = a. Then f is θ-C-continuous, but its graph function is not θ-C-continuous since $g^{-1}(\{(a, a), (c, a)\})$ = {a, c} which is not θ-C-open.

The proof of the following theorem is straightforward and is omitted.

THEOREM 14. A function f: X $\to$ Y is θ-C-continuous if and only if for each closed subset F of Y, there exists a subset A of X for which $f^{-1}(F) = Cl_\theta$ A.

THEOREM 15. If the functions f, g: X $\to$ Y are θ-C-continuous and Y is Hausdorff, then the set A = {x : f(x) $\neq$ g(x)} is a union of θ-C-open sets.

PROOF. Let $x \in A$. Since f(x) $\neq$ g(x) and Y is Hausdorff, there exist disjoint open sets V and W containing f(x) and g(x), respectively. Then $f^{-1}(V)$ and $g^{-1}(W)$ are θ-C-open. By Theorem 5 $f^{-1}(V) \cap g^{-1}(W)$ is θ-C-open. Obviously $x \in f^{-1}(V) \cap g^{-1}(W) \subseteq A$.

COROLLARY. If the functions f, g: X $\to$ Y are θ-C-continuous, then the set B = {x : f(x) = g(x)} is δ-closed.

PROOF. By Theorem 15 X - B is a union of θ-C-open sets and by Theorem 4 each θ-C-open set is a union of regular open sets.

For a function f: X $\to$ Y the graph of f, denoted by G(f), is the subset {(x, f(x)) : x $\in$ X} of the product space X $\times$ Y.

THEOREM 16. If f: X $\to$ Y is θ-C-continuous and Y is Hausdorff, then X $\times$ Y - G(f) is a union of sets of the form A $\times$ B where A is θ-C-open and B is open.

PROOF. Let $(x, y) \in X \times Y - G(f)$. There exist disjoint open sets V and W for which $f(x) \in V$ and $y \in W$. Then $f^{-1}(V)$ is θ-C-open and $(x, y) \in f^{-1}(V) \times W \subseteq X \times Y - G(f)$.

The following theorem is easily proved.

THEOREM 17. If $f: X \to Y$ is a θ-C-continuous injection and Y is Hausdorff, then points in X can be separated by θ-C-open sets.

4. APPLICATIONS TO H-CLOSED SPACES AND KATETOV SPACES

In this section all spaces are assumed to be Hausdorff.

Since H-closed spaces and Katetov spaces are related to the θ-closures of sets, there are natural relationships between these spaces and θ-C-open sets and θ-C-continuity. The following results are required.

THEOREM 18. (Porter and Tikoo [5]). If X is an H-closed space and $A \subseteq X$, then $Cl_\theta A$ is Katetov.

THEOREM 19. (Porter and Tikoo [5]). An H-closed space in which every closed set is the θ-closure of some set is compact.

The next result follows immediately from Theorems 14 and 18.

Theorem 20. If X is H-closed and $f: X \to Y$ is θ-C-continuous, then for each closed subset F of Y, $f^{-1}(F)$ is a Katetov subspace of X.

As a consequence of Theorem 19 we have the following result.

Theorem 21. If X is H-closed and every open set is θ-C-open, then X is compact.

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