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ON DUAL INTEGRAL EQUATIONS ARISING IN PROBLEMS OF BENDING OF ANISOTROPIC PLATES

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ABSTRACT. In this paper we consider dual integral equations, which arise in boundary value problems of bending of anisotropic plates. The function involved in these equations is a linear combination of elementary function, which turns out to be a particular case of a class of Fourier kernels, [2]. The method used here for solving the equations is some what similar to the method used for solving dual integral equations of Titchmarsh type, [1].

KEY WORDS AND PHRASES. Dual integral equations, Bessel functions, Erdelyi–Kober operators, Parseval theorem, Mellin transforms.
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1. INTRODUCTION

In this note we consider dual integral equations

$$\int_0^\infty \phi(t)h_1(xt)dt = f(x), \quad 0<x<1$$
$$\int_0^\infty \phi(t)h_2(xt)dt = g(x), \quad x>1, \tag{1.1}$$

where ϕ is the unknown function and

$$h_1(x) = e^{-x} \mp \cos x \pm \sin x$$
$$h_2(x) = x^{\nu} h_1(x), \quad -1 \le \nu \le 1.$$

The method we employ for solving the system (1.1) is similar to a method developed by Nasim & Sneddon, [1]. This procedure has been very effective for solving dual integral equations when the functions h_1 and h_2 involve Bessel functions J_ν, Y_ν and K_ν. As in almost all the papers concerned with dual integral equations, here also, the solutions are not derived in a rigorous fashion. But that is not to say that the analysis in this paper cannot be made rigorous by imposing appropriate conditions on the functions involved.

It is worth noting that in our case the functions h_1 are a special case of Fourier kernels defined by us elsewhere, [2]. Also we wish to point out that the functions $h_1(x)$

satisfy the differential equation

$$\frac{d^4}{dx^4} h(\lambda x) = \lambda^4 h(\lambda x)$$

and equations (1.1) would therefore arise in many situations where we wanted to solve a partial Differential Equation involving the operator $\frac{\partial^4}{\partial x^4}$. A particular such case e.g., is the bending of an anisotropic plate where the lateral deflection ω satisfies the equation

$$\frac{\partial^4\omega}{\partial x^4} + 2k \frac{\partial^4\omega}{\partial x^2\partial y^2} + \frac{\partial^4\omega}{\partial y^4} = 0.$$

The case k=0 is of some importance for such plates, [3]. If we wanted to solve the resulting equation (k=0) in x>0, y>0 with the plate clamped along x=0 and having discontinuous boundary conditions along y=0, we would quickly arrive at equations (1.1). For, in such a situation (with k=0), we seek solution $\omega(x,y)$ in the form

$$\omega(x,y) = \int_0^\infty \frac{f(\lambda)}{\lambda} \left[e^{-\lambda y/\sqrt{2}} \sin\frac{\lambda y}{\sqrt{2}}\right]\left[e^{-\lambda x} - \cos \lambda x + \sin \lambda x\right] d\lambda,$$

where $f(\lambda)$ is to be determined. This form is seen to satisfy the Partial Differential Equation, the condition of clampness along x=0 (i.e. $\omega = \frac{\partial\omega}{\partial x} = 0$, on x=0, y>0) and the condition that $\omega=0$ on y=0 in x>0. If now, the plate is bent by bending moments of of magnitude ln(x) in 0<x<1 (on = 0) and is clamped along x>1, (on y=0), then $f(\lambda)$ must satisfy

$$\int_0^\infty \lambda f(\lambda)(e^{-\lambda x} - \cos \lambda x + \sin \lambda x)dx = \mu \ln(x) \quad \text{in } 0<x<1$$

and

$$\int_0^\infty f(\lambda)(e^{-\lambda x} - \cos \lambda x + \sin \lambda x)dx = 0 \quad \text{in } x>1$$

where μ is an appropriate (material) constant, [3]. These equations are a particular case of equations(1.1). We propose to solve such equations in this note.

2. PRELIMINARIES.

We shall need the following known definitions and results. The results from the Mellin Transform theory can be found in Titchmarsh [4].

We define the Mellin Transform and inverse Mellin Transform, under appropriate conditions, respectively as:

$$\begin{aligned} M[f(x); s] &= \int_0^\infty f(x)\, x^{s-1} dx \\ &= f^*(s), \text{ where } s=\sigma+i\tau,\ a<\sigma<b,\ -\infty<\tau<\infty, \text{ and} \\ M^{-1}[f^*(s);x] &= \frac{1}{2\pi i} \int_{\sigma-i\infty}^{\sigma+i\infty} f^*(s) x^{-s} ds \\ &= f(x). \end{aligned}$$

Throughout this note, we shall denote the Mellin Transform of a function f by f^*.

A function k is said to be a Fourier Kernel when for

$$\int_0^\infty k(xt)\, f(t)\, dt = g(x)$$

and

$$\int_0^\infty k(xt)\, g(t) dt = f(x),$$

for some suitable functions f & g. Furthermore, if $k^*(s)$ denotes the Mellin transform of $k(x)$, then

$$k^*(s)\, k^*(1-s) = 1$$

on some line of the complex s-plane.

The Erdelyi-Kober [5] operators are given by,

$$I_{\eta,\alpha}(0,x;\sigma)\, f = \frac{\sigma}{\Gamma(\alpha)}\, x^{-\sigma(\eta+\alpha)} \int_0^x (x^\sigma - t^\sigma)^{\alpha-1}\, t^{\sigma(\eta+1)-1} f(t)dt, \quad \alpha>0$$

and

$$K_{\eta,\alpha}(x,\infty:\sigma)f = \frac{\sigma}{\Gamma(\alpha)}\, x^{\sigma\eta} \int_x^\infty (t^\sigma - x^\sigma)^{\alpha-1}\, t^{\sigma(1-\alpha-\eta)-1} f(t)dt, \quad \alpha>0.$$

It is an easy matter to see that

$$M\left[I_{\eta,\alpha}(0,x:\sigma)\ f(x):\ s\right] = \frac{\Gamma(1+\eta-s/\sigma)}{\Gamma(1+\eta+\alpha-s/\sigma)}\, f^*(s)$$

and

$$M\left[K_{\eta,\alpha}(x,\infty;\sigma)f(x);\ s\right] = \frac{\Gamma(\eta+s/\sigma)}{\Gamma(\eta+\alpha+s/\sigma)}\, f^*(s).$$

The inversion form of those equations give us the following two results.

Lemma 1.
$$M^{-1}\left[\frac{\Gamma(1+\eta-\frac{s}{\sigma})}{\Gamma(1+\eta+\alpha-\frac{s}{\sigma})}\, f^*(s);\ x\right] = I_{\eta,\alpha}(0,x;\ \sigma)M^{-1}[f^*(s);\ x] = I_{\eta,\alpha}(0,x;\ \sigma)f$$

Lemma 2.
$$M^{-1}\left[\frac{\Gamma(\eta+\frac{s}{\sigma})}{\Gamma(\eta+\alpha+\frac{s}{\sigma})}\, f^*(s);\ x\right] = K_{\eta,\alpha}(x,\infty;\ \sigma)M^{-1}[f^*(s);\ x] = K_{\eta,\alpha}(x,\infty;\ \sigma)f$$

Note that $I_{\eta,0} = K_{\eta,0} = I$, the identity operator.

Now an important result from the theory of Mellin Transform.

Lemma 3. The Parseval Theorem, [4]

$$M^{-1}[f^*(s)g^*(s);\ x] = \int_0^\infty f(x)g(\tfrac{x}{t})\tfrac{1}{t}\, dt = fog,$$

the convolution of f and g.

Next we give a formal description of the method we shall employ in solving dual integral equations (1.1) for arbitrary functions h_1 and h_2. These dual equations (1.1), are equivalent to

$$\int_0^t m_1(\tfrac{t}{x})\, \tfrac{1}{x}\, dx \int_0^\infty \phi(u)h_1(ux)du = \psi_1(t), \quad 0<t<1$$
$$\int_t^\infty m_2(\tfrac{t}{x})\tfrac{1}{x}\, dx \int_0^\infty \phi(u)h_2(ux)du = \psi_2(t), \quad t>1. \tag{2.1}$$

Here the functions ψ_1 and ψ_2 are defined in terms of the functions f and g and (as yet) arbitrary functions m_1 and m_2 by the equations.

$$\psi_1(t) = \int_0^t f(x)\, m_1(\tfrac{t}{x})\tfrac{1}{x}\, dx \tag{2.2}$$

and

$$\psi_2(t) = \int_t^\infty g(x) m_2(\tfrac{t}{x})\tfrac{1}{x}\, dx. \tag{2.3}$$

Suppose now we find functions m_1 and m_2 such that

$$\int_1^\infty m_1(y) h_1(\tfrac{z}{y})\tfrac{1}{y}\, dy = \int_0^1 m_2(y) h_2(\tfrac{z}{y})\tfrac{1}{y}\, dy = k(z), \text{ say} \tag{2.4}$$

then the unknown function ϕ is the solution of the single integral equation

$$\int_0^\infty \phi(u) k(ut)\, du = \psi(t) \tag{2.5}$$

where

$$\psi(t) = \psi_1(t)\, H(1-t) + \psi_2(t)\, H(t-1), \tag{2.6}$$

H(t) being the Heaviside function. To determine the arbitrary functions m_1 and m_2, we exploit the theory of Mellin Transforms.

Now, due to the result of lemma 3, we can write the equation (2.4) as

$$\overset{*}{m}_1(s)\overset{*}{h}_1(s) = \overset{*}{m}_2(s)\overset{*}{h}_2(s) = \overset{*}{k}(s), \tag{2.7}$$

whence

$$\frac{\overset{*}{m}_1(s)}{\overset{*}{m}_2(s)} = \frac{\overset{*}{h}_2(s)}{\overset{*}{h}_1(s)}$$

It is then possible to decompose $\overset{*}{h}_2(s) \big/ \overset{*}{h}_1(s)$ in such a way so that $\overset{*}{m}_1(s)$ and $\overset{*}{m}_2(s)$ are appropriately determined and eventually the functions m_1 and m_2 and hence ψ_1 and ψ_2 are then known. Next, to find ϕ, from (2.5), we have due to lemma 3,

$$\phi^*(1-s) k^*(s) = \psi^*(s),$$

i.e.,

$$\phi^*(s) = \frac{\psi^*(1-s)}{k^*(1-s)} = \psi^*(1-s) h^*(s), \text{ say}$$

Then the last equation is equivalent to

$$\phi(x) = \int_0^\infty \psi(t) h(xt) dt,$$

giving us the required solution of the system (1.1), where

$$h(x) = M^{-1}\left[\frac{1}{k^*(1-s)};\ x\right].$$

3. THE DUAL INTEGRAL EQUATIONS I.

We consider now

$$\begin{aligned} \int_0^\infty \phi(t) h_1(xt) dt &= f(x),\ 0<x<1 \\ \int_0^\infty \phi(t) h_2(xt) dt &= g(x),\ x>1 \end{aligned} \tag{3.1}$$

where

$$h_1(x) = e^{-x} - \cos x + \sin x,$$

and

$$h_2(x) = x^{\nu}(e^{-x} - \cos x + \sin x), \quad -1 \le \nu \le 1,$$

the functions f and g are known and ϕ is to be determined. Now, for an appropriate complex s, the Mellin transforms of h_1 and h_2 are, respectively, [6, Chapter 6],

$$\begin{aligned} h_1^*(s) &= 2\sqrt{2}\,\Gamma(s) \sin \tfrac{1}{4}\pi s \sin\tfrac{1}{4}\pi(s+1) \\ &= \pi^{\frac{1}{2}}\, 2^{2s-1} \frac{\Gamma(\frac{1}{2} + \frac{s}{4})\Gamma(\frac{3}{4} + \frac{s}{4})}{\Gamma(1 - \frac{s}{4})\Gamma(\frac{3}{4} - \frac{s}{4})} \\ h_2^*(s) &= h_1^*(s+\nu) \end{aligned}$$

Then

$$\frac{m_2^*(s)}{m_1^*(s)} = \frac{h_1^*(s)}{h_2^*(s)} = \frac{\Gamma(\frac{1}{2} + \frac{s}{4})\Gamma(\frac{3}{4} + \frac{s}{4})\Gamma(1 - \frac{\nu}{4} - \frac{s}{4})\Gamma(\frac{3}{4} - \frac{\nu}{4} - \frac{s}{4})}{2^{2\nu}\Gamma(1 - \frac{s}{4})\Gamma(\frac{3}{4} - \frac{s}{4})\Gamma(\frac{1}{2} + \frac{\nu}{4} + \frac{s}{4})\Gamma(\frac{3}{4} + \frac{\nu}{4} + \frac{s}{4})}$$

whence,

$$m_1^*(s) = 2^{2\nu} \frac{\Gamma(1 - \frac{s}{4})\Gamma(\frac{3}{4} - \frac{s}{4})}{\Gamma(1 - \frac{\nu}{4} - \frac{s}{4})\Gamma(\frac{3}{4} - \frac{\nu}{4} - \frac{s}{4})}$$

and

$$m_2^*(s) = \frac{\Gamma(\frac{1}{2} + \frac{s}{4})\Gamma(\frac{3}{4} + \frac{s}{4})}{\Gamma(\frac{1}{2} + \frac{\nu}{4} + \frac{s}{4})\Gamma(\frac{3}{4} + \frac{\nu}{4} + \frac{s}{4})}$$

From the equation (2.2),

$$\begin{aligned} \psi_1(t) &= \int_0^t f(x) m_1(\tfrac{t}{x})\tfrac{1}{x}\, dx \\ &= \int_0^{\infty} f(x) \overline{m}_1(\tfrac{t}{x})\tfrac{1}{x}\, dx \end{aligned}$$

where $\overline{m}_1(x) = m_1(x)\, H(x-1)$, H being the Heaviside function. Due to lemma 3, we have for $-1<\nu<0$,

$$\begin{aligned} \psi_1(t) &= M^{-1}[m_1^*(s) f^*(s);\, t] \\ &= M^{-1}\Big[2^{2\nu} \frac{\Gamma(1 - \frac{s}{4})\Gamma(\frac{3}{4} - \frac{s}{4})}{\Gamma(1 - \frac{\nu}{4} - \frac{s}{4})\Gamma(\frac{3}{4} - \frac{\nu}{4} - \frac{s}{4})} f^*(s);\, t\Big] \\ &= 2^{2\nu}\, I_{0,-\frac{\nu}{4}}(0,t:4)\; I_{-\frac{1}{4},-\frac{\nu}{4}}(0,t;4) f, \end{aligned} \tag{3.2}$$

by repeated use of lemma 1.

But if $0<\nu<1$, then

$$\psi_1(t) = M^{-1}\Big[2^{2\nu} \frac{\Gamma(\frac{3}{4} - \frac{s}{4})\Gamma(1 - \frac{s}{4})}{\Gamma(1 - \frac{\nu}{4} - \frac{s}{4})\Gamma(\frac{3}{4} - \frac{\nu}{4} - \frac{s}{4})} f^*(s);\, t\Big]$$

$$= 2^{2\nu} I_{-\frac{1}{4}, \frac{1}{4}-\frac{\nu}{4}}(0,t;\ 4)M^{-1}\left[\frac{\Gamma(1-\frac{s}{4})}{\Gamma(\frac{3}{4}-\frac{\nu}{4}-\frac{s}{4})} f^*(s);\ t\right]$$

$$= 2^{2\nu-2} I_{-\frac{1}{4}, \frac{1}{4}-\frac{\nu}{4}}(0,t;\ 4)M^{-1}\left[\frac{\Gamma(1-\frac{s}{4})}{\Gamma(\frac{7}{4}-\frac{\nu}{4}-\frac{s}{4})} (3-\nu-s)f^*(s);\ t\right]$$

$$= 2^{2\nu-2} I_{-\frac{1}{4}, \frac{1}{4}-\frac{\nu}{4}}(0,t;4)\ I_{0, \frac{3}{4}-\frac{\nu}{4}}(0,t;4)M^{-1}\left[(3-\nu-s)f^*(s);\ t\right]$$

$$= 2^{2\nu-2} I_{-\frac{1}{4}, \frac{1}{4}-\frac{\nu}{4}}(0,t;\ 4)\ I_{0, \frac{3}{4}-\frac{\nu}{4}}(0,t;\ 4)F, \tag{3.2a}$$

where $F(x) = x^{\nu-2} \frac{d}{dx}\left[x^{3-\nu}f(x)\right]$.

Next, from equation (2.3) we have

$$\psi_2(t) = \int_t^\infty g(x)\ m_2\left[\frac{t}{x}\right] \frac{1}{x}\, dx$$

$$= \int_0^\infty g(x)\ \overline{m}_2\left[\frac{t}{x}\right] \frac{1}{x}\, dx,$$

where $\overline{m}_2(x) = m_2(x)H(1-x)$. Then by lemmas 2 and 3, for $0<\nu<1$,

$$\psi_2(t) = M^{-1}\left[\frac{\Gamma(\frac{1}{2}+\frac{s}{4})\ \Gamma(\frac{3}{4}+\frac{s}{4})}{\Gamma(\frac{1}{2}+\frac{\nu}{4}+\frac{s}{4})\Gamma(\frac{3}{4}+\frac{\nu}{4}+\frac{s}{4})} g^*(s);t\right]$$

$$= K_{\frac{1}{2}, \frac{\nu}{4}}(t,\infty;\ 4)K_{\frac{3}{4}, \frac{\nu}{4}}(t,\infty;\ 4)g \tag{3.3}$$

But if $-1<\nu<0$, then

$$\psi_2(t) = K_{\frac{1}{2}, \frac{1}{4}+\frac{\nu}{4}}(t,\infty;\ 4)M^{-1}\left[\frac{\Gamma(\frac{3}{4}+\frac{s}{4})}{\Gamma(\frac{1}{2}+\frac{\nu}{4}+\frac{s}{4})} g^*(s);\ t\right]$$

$$= \frac{1}{4}K_{\frac{1}{2}, \frac{1}{4}+\frac{\nu}{4}}(t,\infty;\ 4)M^{-1}\left[\frac{\Gamma(\frac{3}{4}+\frac{s}{4})}{\Gamma(\frac{3}{2}+\frac{\nu}{4}+\frac{s}{4})} (2+\nu+s)g^*(s);\ t\right]$$

$$= \frac{1}{4}K_{\frac{1}{2}, \frac{1}{4}+\frac{\nu}{4}}(t,\infty;\ 4)K_{\frac{3}{4}, \frac{3}{4}+\frac{\nu}{4}}(t,\infty;\ 4)M^{-1}[(2+\nu+s)g^*(s);\ t]$$

$$= \frac{1}{4}K_{\frac{1}{2}, \frac{1}{4}+\frac{\nu}{4}}(t,\infty;\ 4)K_{\frac{3}{4}, \frac{3}{4}+\frac{\nu}{4}}(t,\infty;\ 4)G, \tag{3.3a}$$

where $G(x) = -\ x^{\nu+3} \frac{d}{dx}\left(x^{-\nu-2}g(x)\right)$.

Hence the function

$$\psi(t) = \psi_1(t)H(1-t) + \psi_2(t)H(t-1)$$

is completely known, with ψ_1 and ψ_2 as defined above by the equations (3.2) and (3.3).

Further, from (2.7),

$$k^*(s) = m_1^*(s)h_1^*(s) = m_2^*(s)h_2^*(s)$$
$$= \pi^{\frac{1}{2}} 2^{2s+2\nu-1} \frac{\Gamma(\frac{1}{2}+\frac{s}{4})\Gamma(\frac{3}{4}+\frac{s}{4})}{\Gamma(1-\frac{\nu}{4}-\frac{s}{4})\Gamma(\frac{3}{4}-\frac{\nu}{4}-\frac{s}{4})}$$

Then, [7 Sec. 20.5],

$$h(x) = M^{-1}\left[\frac{1}{k^*(1-s)} ; x\right]$$
$$= M^{-1}\left[\pi^{-\frac{1}{2}} 2^{2s-2\nu-1} \frac{\Gamma(\frac{3}{4}-\frac{\nu}{4}+\frac{s}{4})\Gamma(\frac{1}{2}-\frac{\nu}{4}+\frac{s}{4})}{\Gamma(\frac{3}{4}-\frac{s}{4})\Gamma(1-\frac{s}{4})} ; x\right]$$
$$= \pi^{-\frac{1}{2}} 2^{1-2\nu} G_{04}^{20}\left[(\tfrac{x}{4})^4 \middle| \tfrac{3}{4}-\tfrac{\nu}{4}, \tfrac{1}{2}-\tfrac{\nu}{4}, \tfrac{1}{4}, 0\right]. \tag{3.4}$$

Finally the solution of the dual equations (3.1), can now be written as

$$\phi(u) = \int_0^\infty \psi(t)h(ut)dt, \quad -1 < \nu < 1.$$

Note that the solution is also valid for $\nu = \pm 1$.

Next we list two special cases, which are of some interest.

Let $\nu = 1$, $g(x) = 0$ and $f(x) = x^\alpha$, $\alpha > -2$. Then

$$\psi_2(t) = 0$$

and from (3.2a),

$$\psi_1(t) = I_{0,\frac{1}{2}}(0,t;\ 4)F$$

where $F(x) = \frac{1}{x}\frac{d}{dx}(x^{2+\alpha}) = (2+\alpha)x^\alpha$.

That is,

$$\psi_1(t) = (2+\alpha)\, I_{0,\frac{1}{2}}(0,t;\ 4)x^\alpha$$
$$= 2\pi^{-\frac{1}{2}}(2+\alpha)\, t^{-2} \int_0^t (t^4 - x^4)^{-\frac{1}{2}} x^{3+\alpha}dx$$
$$= 4\frac{\Gamma(1+\frac{\alpha}{4})}{\Gamma(\frac{1}{2}+\frac{\alpha}{4})} x^\alpha .$$

The solution of the integral equations

$$\int_0^\infty \phi(t)h_1(xt)dt = x^\alpha, \quad 0<x<1$$
$$\int_0^\infty t\phi(t)\, h_2(xt)dt = 0, \quad x>1 \tag{3.5}$$

is then given by

$$\phi(x) = 4\frac{\Gamma(1+\frac{\alpha}{4})}{\Gamma(\frac{1}{2}+\frac{\alpha}{4})}\int_0^1 h(xt)t^\alpha dt$$

where from (3.4),

$$h(x) = \tfrac{1}{2}\pi^{-\frac{1}{2}} G_{04}^{20}\left[(\tfrac{x}{4})^4 \middle| \tfrac{1}{2}, \tfrac{1}{4}, \tfrac{1}{4}, 0\right].$$

Evaluating the above integral, we obtain, [7 Sec. 20.5], for $\alpha > -2$,

$$\phi(x) = \frac{\Gamma(1+\frac{\alpha}{4})}{\Gamma(\frac{1}{2}+\frac{\alpha}{4})} G^{21}_{15}\left[(\tfrac{x}{4})^4 \middle| \begin{matrix} \frac{3}{4} - \frac{\alpha}{4} \\ \frac{1}{2}, \frac{1}{4}, \frac{1}{4}, 0, -\frac{1}{4} - \frac{\alpha}{4} \end{matrix}\right]$$

as the solution of the system (3.5).

Let $\nu = -1$, $f(x) = 0$ and $g(x) = x^\beta$, $\beta < -\frac{1}{2}$.

Then $$\psi_1(t) = 0$$
and from (3.3a),

$$\psi_2(t) = \tfrac{1}{4} K_{\frac{3}{4},\frac{1}{2}}(t,\infty;\ 4)G,$$

where $$G(x) = -x^2 \frac{d}{dx}(x^{\beta-1})$$
$$= (1-\beta)x^\beta$$

Therefore,

$$\begin{aligned}\psi_2(t) &= \tfrac{1}{4}(1-\beta)\ K_{\frac{3}{4},\frac{1}{2}}(t,\infty:4)x^\beta \\ &= \pi^{-\frac{1}{2}}(1-\beta)t^3\int_t^\infty (x^4 - t^4)^{-\frac{1}{4}}\ x^{\beta-2}\ dx \\ &= 2\ \frac{\Gamma(\frac{3}{4}-\frac{\beta}{4})}{\Gamma(\frac{1}{4}-\frac{\beta}{4})}\ t^\beta.\end{aligned}$$

Hence the solution of dual equation

$$\int_0^\infty \phi(t)\ h_1(xt)dt = 0, \quad 0<x<1$$
$$\int_0^\infty (xt)^{-1}\phi(t)h_1(xt)dt = x^\beta, \quad 1<x<\infty \tag{3.6}$$

is given by

$$\phi(x) = 2\ \frac{\Gamma(\frac{3}{4}-\frac{\beta}{4})}{\Gamma(\frac{1}{4}-\frac{\beta}{4})}\int_1^\infty h(xt)t^\beta\ dt.$$

Now from (3.4),

$$h(x) = 8\pi^{-\frac{1}{2}}\ G^{20}_{04}\left[(\tfrac{x}{4})^4 \middle| 1, \tfrac{3}{4}, \tfrac{1}{4}, 0\right],$$

and on evaluating the above integral, we obtain,

$$\phi(x) = 16\ \pi^{-\frac{1}{2}}\ \frac{\Gamma(\frac{3}{4}-\frac{\beta}{4})}{\Gamma(\frac{1}{4}-\frac{\beta}{4})}\ G^{30}_{15}\left[(\tfrac{x}{4})^4 \middle| \begin{matrix} \frac{3}{4} - \frac{\beta}{4} \\ 1, \frac{3}{4}, -\frac{1}{4} - \frac{\beta}{4}, \frac{1}{4}, 0 \end{matrix}\right], \qquad \beta < -\frac{1}{2},$$

as the solution of the system (3.6).

4. DUAL INTEGRAL EQUATIONS II.

Next we consider the system (1.1) with
$$h_1(x) = e^x + \cos x - \sin x$$

$$h_2(x) = x^{\nu} h_1(x), \ -1\leq\nu\leq 1.$$

Now, [6],

$$\overset{*}{h}_1(s) = 2\sqrt{2}\, \Gamma(s) \sin \tfrac{1}{4} \pi(2+s) \sin \tfrac{1}{4}\pi(1-s)$$

$$= \pi^{\frac{1}{2}} 2^{2s-5\ 2} \frac{\Gamma(\frac{3}{4})\Gamma(\frac{1}{4} + \frac{s}{4})}{\Gamma(\frac{1}{4} - \frac{s}{4})\Gamma(\frac{1}{2} - \frac{s}{4})}$$

and $\overset{*}{h}_2(s) = \overset{*}{h}_1(s+\nu)$.

The equation (2.7) is satisfied, if we set

$$\overset{*}{m}_1(s) = 2^{2\nu} \frac{\Gamma(\frac{1}{4} - \frac{s}{4})\Gamma(\frac{1}{2} - \frac{s}{4})}{\Gamma(\frac{1}{4} - \frac{\nu}{4} - \frac{s}{4})\Gamma(\frac{1}{2} - \frac{\nu}{4} - \frac{s}{4})},$$

$$\overset{*}{m}_2(s) = \frac{\Gamma(\frac{s}{4})\Gamma(\frac{1}{4} + \frac{s}{4})}{\Gamma(\frac{\nu}{4} + \frac{s}{4})\Gamma(\frac{1}{4} + \frac{\nu}{4} + \frac{s}{4})}$$

and

$$\overset{*}{k}(s) = \sqrt{\pi}\ 2^{2s+2\nu-5\ 2} \frac{\Gamma(\frac{s}{4})\Gamma(\frac{1}{4} + \frac{s}{4})}{\Gamma(\frac{\nu}{4} - \frac{s}{4})\Gamma(\frac{1}{4} - \frac{\nu}{4} - \frac{s}{4})}.$$

Let $0<\nu<1$, then as before

$$\psi_2(t) = \int_t^{\infty} g(x)\ m_2(\tfrac{t}{x})\tfrac{1}{x}\ dx$$

$$= M^{-1}[\overset{*}{m}_2(s)\overset{*}{g}(s);\ t]$$

$$= M^{-1}\left[\frac{\Gamma(\frac{1}{4} + \frac{s}{4})\Gamma(\frac{s}{4})}{\Gamma(\frac{1}{4} + \frac{\nu}{4} + \frac{s}{4})\Gamma(\frac{1}{2}\frac{\nu}{4} + \frac{s}{4})}\ g^*(s);\ t\right]$$

$$= K_{\frac{1}{4},\frac{\nu}{4}}(t,\infty;\ 4)\ K_{0,\frac{\nu}{4}}(t,\infty;\ 4)g$$

Also,

$$\psi_1(t) = \int_0^t f(x) m_1(\tfrac{t}{x})\tfrac{1}{x}\ dx$$

$$= M^{-1}[\overset{*}{m}_1(s)\overset{*}{f}(s);\ t]$$

$$= M^{-1}\left[2^{2\nu-2} \frac{\Gamma(\frac{1}{4} - \frac{s}{4})\Gamma(\frac{1}{2} - \frac{s}{4})}{\Gamma(\frac{1}{2} - \frac{\nu}{4} - \frac{s}{4})\Gamma(\frac{5}{4} - \frac{\nu}{4} - \frac{s}{4})}\ (1-\nu-s)f^*(s);\ t\right]$$

$$= I_{-\frac{3}{4},\frac{1}{4}-\frac{\nu}{4}}(0,x;\ 4) I_{-\frac{1}{2},\frac{3}{4}-\frac{\nu}{4}}(0,x:4)F$$

where $F(x) = 2^{2\nu-2}x^{\nu} \frac{d}{dx}(x^{1-\nu}f(x))$.

Now let $-1<\nu<0$. Then, as above

$$\psi_1(t) = M^{-1}\left[2^{2\nu} \frac{\Gamma(\frac{1}{4} - \frac{s}{4})\Gamma(\frac{1}{2} - \frac{s}{4})}{\Gamma(\frac{1}{4} - \frac{\nu}{4} - \frac{s}{4})\Gamma(\frac{1}{2} - \frac{\nu}{4} - \frac{s}{4})}\ f^*(s);\ t\right]$$

$$= 2^{2\nu} I_{-\frac{3}{4}, -\frac{\nu}{4}}(0,x; 4)\, I_{-\frac{1}{2}, -\frac{\nu}{4}}(0,x; 4) f$$

and

$$\psi_2(t) = M^{-1}\left[\frac{\Gamma(\frac{s}{4})\Gamma(\frac{1}{4}+\frac{s}{4})}{4\Gamma(\frac{1}{4}+\frac{\nu}{4}+\frac{s}{4})\Gamma(1+\frac{\nu}{4}+\frac{s}{4})}(\nu+s)g^*(s);\ t\right]$$

$$= K_{0,\frac{1}{4}+\frac{\nu}{4}}(x,\infty,4)\, K_{\frac{1}{4},\frac{3}{4}+\frac{\nu}{4}}(x,\infty; 4) G,$$

where $G(x) = -\frac{1}{4} x^{\nu+1} \frac{d}{dx}(x^{-\nu} g(x))$.

Hence the solution of the system is given by

$$\phi(x) = \int_0^\infty \psi(t) h(xt) dt,$$

where

$$\psi(t) = \psi_1(t)H(1-t) + \psi_2(t)H(t-1),$$

and

$$h(x) = M^{-1}\left[\frac{1}{k^*(1-s)};\ x\right]$$

$$= \pi^{-\frac{1}{2}}\, 2^{\frac{9}{2}-2\nu}\, G^{20}_{04}\left[\left(\frac{x}{4}\right)^4 \middle|\ -\frac{\nu}{4}\frac{1}{4}, -\frac{\nu}{4}\frac{3}{4}\frac{1}{2}\right].$$

Special cases when $\nu = \pm 1$ can easily be derived from the general solution.

We also note that the solution of equations (1.1) where

$$h_1(x) = \sqrt{x}\,(Y_0(x) \pm \frac{2}{\pi} K_0(x))$$

and

$$h_2(x) = x^\nu h_1(x), \quad -1 \le \nu \le 1$$

can also be obtained along similar lines. It is well known that the functions $h_1(x)$ are Fourier Kernels, [2].

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