Internat. J. Math. & Math. Sci.
VOL. 15 NO. 4 (1992) 813-818

ON COMMUTATIVITY OF ONE-SIDED s-UNITAL RINGS

H.A.S. ABUJABAL
and
M.A. KHAN

Department of Mathematics
Faculty of Science
King Abdul Aziz University
P. O. Box 31464, Jeddah 21497
Saudi Arabia

(Received October 11, 1990 and in revised form December 18, 1990)

ABSTRACT. The following theorem is proved: Let $r = r(y) > 1$, s, and t be non-negative integers. If R is a left s-unital ring satisfies the polynomial identity $[xy - x^s y^r x^t, x] = 0$ for every $x, y \in R$, then R is commutative. The commutativity of a right s-unital ring satisfying the polynomial identity $[xy - y^r x^t, x] = 0$ for all $x, y \in R$, is also proved.

KEY WORDS AND PHRASES. Commutativity of rings, Left s-unital rings, Ring with unity, Nilpotent elements, Nil commutator ideal, Zero-divisors, Semi-prime rings.
1980 AMS SUBJECT CLASSIFICATION CODE. 16A70.

1. INTRODUCTION.

Throughout this paper, R will be an associative ring (may be without unity 1). $Z(R)$ will represent the center of R, $N(R)$ the set of all nilpotent elements in R, $N'(R)$ the set of all zero-divisors in R and $C(R)$ the commutator ideal of R. For any $x, y \in R$, $[x, y] = xy - yx$, the well-known lie product. By $GF(q)$, we mean the Galois field (finite field) with q elements, and $(GF(q))_2$ the ring of all 2×2 matrices over $GF(q)$.

A ring R is called left (resp. right) s-unital if $x \in Rx$ (resp. $x \in xR$) for each $x \in R$. Further, R is called s-unital if it is both left and right s-unital, that is, $x \in xR \cap Rx$ for each $x \in R$. If R is s-unital (resp. left or right s-unital), then for any finite subset F of R, there exists an element $e \in R$ such that $ex = xe = x$ (resp. $ex = x$ or $xe = x$) for all $x \in F$. Such an element e is called the pseudo (resp. pseudo left or pseudo right) identity of F in R.

In a recent paper, it was proved.

THEOREM HK ([1, Theorem]). Let R be a ring with unity 1. If there exist fixed positive integers $r > 1$, $s > 1$ such that $[xy - x^s y^r x^s, x] = 0$ for all $x, y \in R$, then R is commutative.

The objective of this paper is to generalize Theorem HK. Indeed, we consider the case that r is no longer fixed, depending on y for its value, and also R is left s-unital. Another commutativity theorem for right s-unital rings is also obtained.

2. PRELIMINARY.

In preparation for the proof of our results, we need the following well-known results.

LEMMA 1 ([2, Lemma 2]). Let R be a ring with unity 1, and let x and y be elements in R. If $kx^m[x, y] = 0$ and $k(x+1)^m[x, y] = 0$ for some integers $m \geq 1$ and $k \geq 1$, then necessarily $k[x, y] = 0$.

LEMMA 2 ([3, Lemma 3]). Let x and y be elements in a ring R. If $[x, [x, y]] = 0$, then $[x^k, y] = kx^{k-1}[x, y]$ for all integers $k \geq 1$.

LEMMA 3 ([4, Lemma]). Let R be a left (resp. right) s-unital ring. If for each pair of elements x and y in R, there exists a positive integer $m = m(x, y)$ and an element $e = e(x, y) \in R$ such that $x^m e = x^m$ and $y^m e = y^m$ (resp. $ex^m = x^m$ and $ey^m = y^m$), then R is an s-unital ring.

LEMMA 4 ([5, Lemma 3]). Let R be a ring with unity 1, and let x and y be elements in R. If $(1-y^k)x = 0$, then $(1-y^{km})x = 0$ for some integers $k > 0$ and $m > 0$.

THEOREM K ([6, Theorem]). Let f be a polynomial in n non-commuting indeterminates $x_1, x_2, \ldots, x_n$ with relatively prime integral coefficients. Then the following are equivalent:

(1) For any ring R satisfying the polynomial identity $f = 0$, $C(R)$ is a nil ideal.

(2) For every prime p, $(GF(p))_2$ fails to satisfy $f = 0$.

(3) Every semi-prime ring R satisfying $f = 0$ is commutative.

THEOREM H ([7, Theorem 21]). Let R be a ring, and let $n = n(x) > 1$ be an integer depending on x. Suppose that $x^n - x \in Z(R)$ for all $x \in R$. Then R is commutative.

3. RESULTS.

The main result of this paper is the following:

THEOREM 1. Let R be a left s-unital ring, and let $r = r(y) > 1$, s, and t be non-negative integers. Suppose that R satisfies the polynomial identity

$$[xy - x^s y^r x^t, x] = 0 \text{ for all } x, y \in R. \tag{3.1}$$

Then R is commutative.

LEMMA 5. Let R be a ring, and let $r = r(x,y) > 1$, $s = s(x,y)$, and $t = t(x,y)$ be non-negative integers. Suppose that R satisfies the polynomial identity (3.1). Then $C(R)$ is nil. Further, if R has unity 1, then

$$C(R) \subseteq N(R) \subseteq Z(R). \tag{3.2}$$

PROOF. Let $x = e_{22} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$, and $y = e_{21} + e_{22} = \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}$. Then x and y fail to satisfy the polynomial identity (3.1) in $(GF(p))_2$, for a prime p. Therefore, $C(R) \subseteq N(R)$ by Theorem K.

Now, we notice that the polynomial identity (3.1) can be written in the form

$$x[x,y] = x^s[x,y^r]x^t \text{ for all } x, y \in R. \tag{3.3}$$

Let $u \in N(R)$, and $x \in R$. Then there exist integers $r_1 = r(x,u) > 1$, $s_1 = s(x,u) \geq 0$, and $t_1 = t(x,u) \geq 0$ such that

$$x[x,u] = x^{s_1}[x,u^{r_1}]x^{t_1} \text{ for all } x \in R. \tag{3.4}$$

If we choose $r_2 = r(x,u^{r_1}) > 1$, $s_2 = s(x,u^{r_1}) \geq 0$, and $t_2 = t(x,u^{r_1}) \geq 0$, then (3.3) becomes $x[x,u^{r_1}] = x^{s_2}[x,(u^{r_1})^{r_2}]x^{t_2}$. Hence $x^2[x,u] = x^{s_1+s_2}[x,u^{r_1 r_2}]x^{t_1+t_2}$. Thus for any positive integer q,

$$\begin{aligned} x^q[x,u] &= x^{q-1}x^{s_1}[x,u^{r_1}]x^{t_1} \\ &= x^{q-2}x^{s_1+s_2}[x,u^{r_1 r_2}]x^{t_1+t_2} \\ &= \cdots \\ &= x^{s_1+s_2+\cdots+s_q}[x,u^{r_1 r_2 \cdots r_q}]x^{t_1+t_2+\cdots+t_q}. \end{aligned}$$

But u is nilpotent, $u^{r_1 r_2 \cdots r_q} = 0$ for sufficiently large q. So $x^q[x,u] = 0$ and by Lemma 1, we get $[x,u] = 0$ for all $x \in R$. Therefore, $N(R) \subseteq Z(R)$ and hence (3.2) holds.

LEMMA 6. Let R be a left s-unital ring, and let $r = r(x,y) > 1$, $s = s(x,y) \geq 0$, and $t = t(x,y) \geq 0$. Suppose that R satisfies the polynomial identity (3.1). Then R is an s-unital ring.

PROOF. Let $x, y \in R$. Then there is an element $e = e(x,y) \in R$ such that $ex = x$, and $ey = y$. Further, there exist integers $r = r(x,e) > 1$, $s = s(x,e) \geq 0$, and $t = t(x,e) \geq 0$ such that $x^{s+t+1}e^r = [xe - x^s e^r x^t, x] + x^{s+t+1} = x^{s+t+1}$. Also if $r' = r'(y,e) > 1$, $s' = s'(y,e) \geq 0$, and $t' = t'(y,e) \geq 0$, then we have $y^{s'+t'+1}e^{r'} = y^{s'+t'+1}$. Hence, $x^{s+t+s'+t'+2}e^r = x^{s+t+s'+t'+2}$, and $y^{s+t+s'+t'+2}e^{r'} = y^{s+t+s'+t'+2}$. Thus we obtain $x^{s+t+s'+t'+2}e^{rr'} = x^{s+t+s'+t'+2}\underbrace{e^r e^r \cdots e^r}_{r'-\text{times}}$. So $x^{s+t+s'+t'+2}e^{rr'} = x^{s+t+s'+t'+2}$, and similarly $y^{s+t+s'+t'+2}e^{rr'} = y^{s+t+s'+t'+2}$. Therefore, R is s-unital by Lemma 3.

LEMMA 7. Let R be a ring with unity 1, and let $r = r(y) > 1$, s, and t be non-negative integers. Suppose that R satisfies the polynomial identity (3.1). Then R is commutative.

PROOF. If $s = t = 0$, then (3.1) becomes $x[x,y] = [x,y^r]$. Replace x by $x+1$ in the last identity to get $[x,y] = 0$ for all $x,y \in R$. Therefore, R is commutative. Next, if $s = 0$, and $t = 1$ (resp. $s = 1$, and $t = 0$), then $x[x,y] = [x,y^r]x$ (resp. $x[x,y] = x[x,y^r]$). Replacing x by $x+1$ gives $[x,y^r - y] = 0$ for all $x,y \in R$, that is, $y^{r(y)} - y \in Z(R)$, $r(y) > 1$ for all $y \in R$. Thus R is commutative by Theorem H.

Now, assume that $s > 1$ or $t > 1$. Consider the positive integer $k = p^{s+t+1} - p^2$, where p is a positive integer larger than 1. Then by (3.3), we get for all $x,y \in R$,

$$kx[x,y] = p^{s+t+1}x^s[x,y^r]x^t - p^2x[x,y] = (px)^s[(px),y^r](px)^t - (px)[(px),y] = 0.$$

Replace x by $x+1$ to obtain $k[x,y] = 0$ for all $x,y \in R$. In view of Lemma 5, $C(R) \subseteq Z(R)$, and hence $[x^k,y] = kx^{k-1}[x,y] = 0$. So

$$x^k \in Z(R) \text{ for all } x \in R. \tag{3.5}$$

If $r_1 = r(y)$, then (3.3) becomes

$$x[x,y] = x^s[x,y^{r_1}]x^t \text{ for all } x,y \in R. \tag{3.3$'$}$$

Let $r_2 = r(y^{r_1})$. Then replace y by y^{r_2} in (3.3)$'$ to get

$$x[x,y^{r_2}] = x^s[x,(y^{r_2})^{r_1}]x^t \text{ for all } x,y \in R.$$

Thus

$$x[x,y^{r_2}] = x^s[x,(y^{r_1})^{r_2}]x^t \text{ for all } x,y \in R. \tag{3.6}$$

Since $C(R) \subseteq Z(R)$ by Lemma 5,

$$\begin{aligned} x[x,y^{r_2}] = [x,y^{r_2}]x &= r_2y^{r_2-1}[x,y]x \\ &= r_2y^{r_2-1}x[x,y] = r_2y^{r_2-1}x^s[x,y^{r_1}]x^t \\ &= r_2y^{r_2-1}[x,y^{r_1}]x^{s+t}. \end{aligned}$$

Also by using (3.3)$'$, we have

$$\begin{aligned} x^s[x,(y^{r_1})^{r_2}]x^t &= [x,(y^{r_1})^{r_2}]x^{s+t} \\ &= r_2(y^{r_1})^{r_2-1}[x,y^{r_1}]x^{s+t} \\ &= r_2y^{r_1(r_2-1)}[x,y^{r_1}]x^{s+t}. \end{aligned}$$

Thus (3.6) gives $r_2(1-y^{(r_1-1)(r_2-1)})y^{r_2-1}[x,y^{r_1}]x^{s+t} = 0$. The usual argument of replacing x by $x+1$ and using Lemma 1, yields $r_2(1-y^{(r_1-1)(r_2-1)})y^{r_2-1}[x,y^{r_1}] = 0$. Then Lemma 4 gives

$$r_2(1 - y^{k(r_1-1)(r_2-1)})y^{r_2-1}[x,y^{r_1}] = 0 \text{ for all } x,y \in R. \tag{3.7}$$

It is well-known that R is isomorphic to a subdirect sum of subdirectly irreducible rings R_i ($i \in I$, the index set), each of which as a homomorphic image of R satisfies the property placed on R. Thus R itself can be assumed to be subdirectly irreducible ring. Let S be the intersection of all its non-zero ideals of R. So $S \neq (0)$. Thus $Sd = 0$ for all central zero-divisors d (see [8]).

Let $a \in N'(R)$. Then by (3.5), $a^{k(r_1-1)(r_2-1)} \in N'(R) \cap Z(R)$, and $Sa^{k(r_1-1)(r_2-1)} = 0$. By using (3.7), we get $r_2(1 - a^{k(r_1-1)(r_2-1)})a^{r_2-1}[x,a^{r_1}] = 0$ for all $x \in R$.

If $r_2a^{r_2-1}[x,a^{r_1}] \neq 0$, then $1 - a^{k(r_1-1)(r_2-1)} \in N'(R)$, and so

$$0 = S(1 - a^{k(r_1-1)(r_2-1)}) = S - Sa^{k(r_1-1)(r_2-1)} = S$$

which is a contradiction to the fact $S \neq (0)$. Thus $r_2 a^{r_2-1}[x, a^{r_1}] = 0$ for all $x \in R$. From (3.3),and using Lemma 2 repeatedly we obtain for $r_1 = r(a)$, and $r_2 = r(a^{r_1})$,

$$\begin{aligned} x^2[x,a] &= x^s x[x, a^{r_1}]x^t \\ &= x^{2s}[x, (a^{r_1})^{r_2}]x^{2t} \\ &= [x, (a^{r_1})^{r_2}]x^{2s+2t} \\ &= r_2 a^{r_1(r_2-1)}[x, a^{r_1}]x^{2s+2t} \\ &= r_2 a^{(r_1-1)(r_2-1)} a^{r_2-1}[x, a^{r_1}]x^{2s+2t} \\ &= 0. \end{aligned}$$

Replace x by $x+1$, and apply Lemma 1 to obtain $[x,a] = 0$ for all $x \in R$. Hence $N'(R) \subseteq Z(R)$.

Now, if $x \in R$, then $x^k \in Z(R)$, and $x^{kr} \in Z(R)$, where $r = r(y)$ for any $y \in R$. By (3.3), we get $(x^k - x^{kr})x[x,y] = x^k x[x,y] - x^{kr}x[x,y] = x[x,(x^k y)] - x^s[x,(x^k y)^r]x^t = 0$. Thus

$$(x - x^{kr-k+1})x^k[x,y] = 0 \text{ for all } x, y \in R. \tag{3.8}$$

If R is not commutative, then by Theorem H, there exists an element $x \in R$ such that $x - x^n \notin Z(R)$, where $n = kr - k + 1$. This also reveals $x \notin Z(R)$. Thus neither x nor $x - x^n$ is a zero-divisor, and so $(x - x^n)x^k \notin N'(R)$. Hence (3.8) forces that $[x,y] = 0$ for all $x, y \in R$. Thus $x \in Z(R)$ which is a contradiction. Therefore, R is commutative.

EXAMPLE 1. Lemma 7 is false for rings without unity 1. In fact, any nilpotent ring of index ≤ 4 and nil ring of index 2 will easily satisfy the polynomial identity (3.1), but such a ring need not be commutative (see [9]).

Indeed, let D_k be the ring of all $k \times k$ matrices over a division ring D, and let

$$A_k = \{ (a_{ij}) \in D_k \mid a_{ij} = 0, \ j \geq i \}.$$

Then A_k is a non-commutative nilpotent ring of index k. for any positive integer $k > 2$. Obviously A_3 satisfies (3.1) and A_3 is not commutative (see [10]).

EXAMPLE 2. Let F be a field. Define an algebra A over F with basis $\{a, b, c\}$ where $ab = c$ and all other products are zero. A is nilpotent of index 3, satisfies (3.1) and A is not commutative.

COROLLARY 1 ([11, Theorem]). Let R be a ring with unity 1 such that there exist fixed integers $r > 1$, and $t \geq 1$ satisfying the polynomial identity $[xy - y^r x^t, x] = 0$ for all $x, y \in R$. Then R is commutative.

COROLLARY 2 ([12, Theorem]). Let R be a ring with unity 1 in which $[xy - x^s y^r, x] = 0$ for all $x, y \in R$ and fixed integers $r > 1$, $s \geq 1$. Then R is commutative.

PROOF OF THEOREM 1. By Lemma 6, R is an s-unital ring. Hence in view of Proposition 1 of [13], we can assume that R has unity 1, and satisfies (3.1). Hence R is commutative by Lemma 7.

REMARK 1. The example of Grassman algebras rules out the possibility that $r = 1$ in Theorem 1.

COROLLARY 3 ([5, Theorem]). Let R be a left s-unital ring, and let $r > 1$, and $t \geq 1$ be fixed non-negative integers. If R satisfies the polynomial identity $[xy - y^r x^t, x] = 0$ for all $x, y \in R$, then R is commutative.

REMARK 2. Corollary 1 is also true for right s-unital rings.

If $s = t = r = n > 1$, then we have the following:

COROLLARY 4. Let R be a left s-unital ring, and let $n > 1$ be a fixed integer. If R satisfies $[xy - x^n y^n x^n, x] = 0$ for all $x, y \in R$, then R is commutative.

REMARK 3. One might conjecture a possible generalization of Theorem 1 when R is right s-unital. Some extra conditions are required to established the commutativity.

The following example shows that there is a non-commutative right s-unital ring satisfying the polynomial identity (3.1).

EXAMPLE 3. Let $R = \left\{ a = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, b = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, c = \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}, d = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \right\}$ be a subring of $(GF(2))_2$. It is easy to check that R is a right s-unital ring satisfying the polynomial identity (3.1) for $r > 1$, $s > 1$, and $t > 1$. Also, R is not a left s-unital ring. However, R is a non-commutative ring (see [14, Examples]).

If $s = 0$ in Theorem 1, then Theorem 1 is also valid for right s-unital ring.

THEOREM 2. Let $r = r(y) > 1$, and t be non-negative integers. If R is a right s-unital ring satisfies the polynomial identity

$$[xy - y^r x^t, x] = 0 \text{ for all } x, y \in R, \tag{3.9}$$

then R is commutative.

LEMMA 8. Let $r = r(x,y) > 1$, and $t = (x,y)$ be fixed non-negative integers. If R is a right s-unital ring satisfies the polynomial identity (3.9), then R is s-unital.

PROOF. Since R is right s-unital, then for any $x, y \in R$ there exists an element $e = e(x,y) \in R$ such that $xe = x$ and $ye = y$. Let $r = r(x,y) > 1$, $t = t(x,y) \geq 0$, $r' = r'(x,y) > 1$, and $t' = t'(x,y) \geq 0$. Replace y by e in (3.9) and follow the argument of Lemma 6 to obtain $e^{rr'}x^{t+t'+2} = x^{t+t'+2}$, and $e^{rr'}y^{t+t'+2} = y^{t+t'+2}$. By Lemma 3, R is s-unital.

PROOF OF THEOREM 2. By Lemma 8, R is s-unital. Hence, we can assume that R has unity 1 (see [13, Proposition 1]). Therefore, R is commutative by Lemma 7.

ACKNOWLEDGMENT. We would like to express our gratitude and indebtedness to Professor M. S. Khan (Sultan Qaboos University, Muscat, Oman) for his helpful suggestions and comments.

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