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RECURRENT POINTS AND DISCRETE POINTS FOR ELEMENTARY AMENABLE GROUPS

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ABSTRACT. Let βG be the Stone-Cech compactification of a group G, A^G the set of all almost periodic points in βG, $K^G = c\ell[\cup \{ \operatorname{supp}\mu_\varphi : \varphi \epsilon \mathrm{LIM}(G)\}]$ and R^G the set of all recurrent points in βG. In this paper we will study the relationships between K^G and R^G, and between A^G and R^G. We will show that for any infinite elementary amenable group $G, A^G \subsetneqq R^G$ and $R^G - K^G \neq \phi$.

Key words and Phrases: Topological group, recurrent point, elementary amenable groups.

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1. INTRODUCTION

Let G be a discrete group, $M(G)$ the Banach space of all bounded real valued functions on G with the supremum norm, and $M(G)^*$ the conjugate Banach space on $M(G)$. A group G is called left amenable if there exists a mean on $M(G)$ which is left invariant [4]. Denote by $\mathrm{LIM}(G)$ the set of left invariant means on $M(G)$. By using Riesz representation theorem, then there exists a bounded regular Borel measure μ_φ on βS such that

$$\varphi(f) = \int_{\beta S} \overline{f} d\, \mu_\varphi, \quad f \epsilon M(G).$$

For a subset H of a left amenable group G define the upper density as follows:

$$\overline{d}(H) = \sup \left\{ \varphi(\chi_H) : \varphi \epsilon \mathrm{LIM}(G) \right\} = \sup \left\{ \mu_\varphi(\overline{H}) : \varphi \epsilon \mathrm{LIM}(G) \right\}$$

where χ_H is the characteristic function of H.

Put $K^G = c\ell[\cup \{\operatorname{supp}\mu_\varphi : \varphi \epsilon \mathrm{LIM}(G)\}]$.

DEFINITIONS:

(1) For each $A \subset G$, let $\hat{A} = c\ell_{\beta G}A - G$. The set $\hat{A}$ is closed and open in $\hat{G} = \beta G - G$. Then a point $w \epsilon \hat{G}$ is said to be almost periodic if for every neighbourhood U of w there exist $A, K \subset G$ with K finite, $Aw \subset \mathsf{U}$, and $G = KA$. Denote by A^G the set of all almost periodic points in βG.

(2) $w \epsilon \beta G$ is recurrent if whenever U is a neighbourhood of w then the set $\{x \epsilon G : xw \epsilon \mathsf{U}$ is infinite. Let R^G be the set of all recurrent points in βG.

(3) $w \epsilon \beta G$ is discrete if its orbit $0(w)$ is discrete with respect to the subspace topology of βG. Let D^G be the set of all such points in βG.

(4) $w \epsilon \beta G$ is strongly discrete if there exists a neighbourhood U of w such that $x\mathsf{U} \cap y\mathsf{U} = \phi$ where $x, y \epsilon G, x \neq y$. The set of all strongly discrete points will be denoted by DS^G. Let $WR^G = \beta G - SD^G$, the set of all weak recurrent points of βG.

If G is a finite group, then $A^G = D^G = K^G = G = \beta G$ and $R^G = WR^G = \phi$. Therefore, we are only interested in infinite groups. For the remainder of this paper, G stands for an infinite group.

LEMMA 1.1. (1) $A^G \subset R^G \subset WR^G$, $SD^G \subset D^G$ and $D^G = \beta G - R^G$.

(2) If G is amenable, then $A^G \subset K^G \subset WR^G$. [4]

As mentioned in Day [3], the family of amenable groups is closed under the following four standard processes of constructing new groups from given ones: (a) subgroup, (b) factor group, (c) group extension, (d) expanding union (or direct limit). Denote by EG the family of elementary amenable groups. It is the smallest family of groups containing all finite groups and all abelian groups, and is closed under (a) - (d).

As pointed out by Chou [2], the groups in EG can be constructed from abelian groups and finite groups by applying processes (c) and (d) only. Moreover, every periodic group in EG is locally finite, and since it is known that every infinite locally finite group contains an infinite abelian subgroup (Robinson [5; p. 95]), we have the following:

LEMMA 1.2 (Chou). If G is an infinite group in EG, then G contains an infinite abelian subgroup.

2 · R_0 - SETS AND RECURRENT POINTS:

In this section we will show that $A^G \subsetneq R^G$ and $R^G - K^G \neq \phi$, for any infinite elementary amenable group G.

DEFINITION. A subset A of an amenable group G is called R_0-set if:

$(R_0 \cdot 1)$ $\overline{d}(A) = 0$, and

$(R_0 \cdot 2)$ $\overline{A} \cap R^G \neq \phi$.

PROPOSITION 2.1: If an amenable group G contains an R_0-set, then $A^G \subsetneq R^G$ and $R^G - K^G \neq \phi$.

PROOF: If A is an R_0-set in G, then by $(R_0 \cdot 2)$ there exists $w \epsilon \overline{A} \cap R^G$ since $\overline{d}(A) = 0$, $w \notin K^G$. The proof is completed.

R_0-sets were first studied by Chou for the case $G = Z$ where he showed that Z contains such sets (see Chou [1], P. 60, example 2). In this section we will show that every finite elementary amenable group contains R_0-sets.

THEOREM 2.2: Let $\{A_k\}$ be a sequence of subsets of a group G, and $\{g_k\}$ be a sequence of different elements of G such that

1. $A_1 \supset A_2 \supset A_3 \supset \dots$, and
2. $g_n A_{n+1} \subset A_n$ for all n. Then $\bigcap_{n=1}^{\infty} \overline{A}_n$ contains a recurrent point.

PROOF. Let π be the family of sequences of closed subsets of $\hat{G}$. Let π be defined as follows: a sequence of closed subsets $\{F_n\}$ of $\hat{G}$ belongs to π if for each $n \epsilon N$,

(i) $F_n \subset \hat{A}_n$,

(ii) $F_{n+1} \subset F_n$,

(iii) $g_n F_{n+1} \subset F_n$, and

(iv) $F_n \neq \phi$.

Note that π is non-empty since $\left\{\hat{A}_n\right\} \epsilon \pi$. π can be ordered in the following natural way: $\{F_n\} \leq \{F'_n\}$ if and only if $F_n \subset F'_n$ for each n. It is easy to check that each chain in π has a lower bound. Then using Zorn's lemma, π has a minimal element $\{K_n\}$. Let $w \epsilon \bigcap_{n=1}^{\infty} K_n$. We claim that w is a recurrent point, by showing that if $\hat{A}$ is an open neighbourhood of w, then there exists infinitely many $g \epsilon G$ such that $gw \epsilon A$.

Let $V = \bigcup_{g \epsilon G} g\hat{A}$. Consider the sequence $\{K_n - V\}$. It satisfies (i), (ii), and using the fact that for $g \epsilon G, gV = V$, one sees that $\{K_n - V\}$ satisfies (iii). Since $K_n - V \subsetneqq K_n$ and $\{K_n\}$ is minimal in π, $\{K_n - V\} \notin \pi$, therefore, $\{K_n - V\}$ does not satisfy (iv), i.e., there exists n_0 such that $K_{n_0} - V = \phi$ which implies $K_{n_0} \subseteq V = \bigcup_{g \epsilon G} g\hat{A}$. Since K_{n_0} is compact, there exists $a_1, a_2, \ldots, a_m \epsilon G$ such that

$$K_{n_0} \subset \bigcup_{i=1}^{m} a_i \hat{A}. \tag{$*$}$$

For each $n \geq n_0, g_n w \epsilon g_n K_{n+1} \subset K_n \subset K_{n_0}$, thus by $(*) g_n w \epsilon a_i \overline{A}$ for some $1 \leq i \leq m$. Then clearly, there exists $i_0, i_0 \leq m$ such that $g_n w \epsilon a_{i_0} \overline{A}$ for infinitely many n. Thus the set $\left\{ g \epsilon G : g w \epsilon \hat{A} \right\}$ is infinite. Therefore $w \epsilon R^G$.

REMARK: When $G = Z$, the above theorem is contained in Chou [1]. The idea of the proof of the above theorem follows from [1, Proposition 3.1].

LEMMA 2.3 (Chou [1]): The additive group of integers Z contains an R_0-set.

For prime number p, let

$$Z(p^\infty) = \{m/p^n : 0 \leq m < p^n; m \epsilon Z, n = 0, n = 1, 2, \ldots\}$$

be the subgroup of Q/Z generated by

$$\left\{ \frac{1}{p^n} : n = 1, 2, \ldots \right\}.$$

LEMMA 2.4: For each prime number $p, A(p^\infty)$ contains an R_0-set.

PROOF: Let $G = A(p^\infty)$. Then G is a subgroup of Q/Z. Note that G can be written as $G = \bigcup_{n=1}^{\infty} H_n$ where $H_1 \subset H_2 \subset \ldots$ and each

$$H_n = \left\{ 0, \frac{1}{p^n}, \frac{2}{p^n}, \ldots, p^n - \frac{1}{p^n} \right\}.$$

is a cyclic group of order p^n. For convenience, we will write $H_n = \langle \frac{1}{p^n} \rangle$ (with the usual addition in Q (mod Z)). Then following Chou's construction for Z [1], we define a sequence of subsets E_n in G by induction as follows:

$$E_1 = \left\{ \frac{1}{p} \right\}, E_{n+1} = E_n \cup (E_n + \frac{1}{p^{(n+1)^2}})$$

Then

$$|E_n| = 2^{n-1}.$$

Let

$$\begin{aligned}E = \bigcup_{n=1}^{\infty} E_n = \Big\{ &\frac{1}{p}; \frac{1}{p} + \frac{1}{p4}; \frac{1}{p} + \frac{1}{p9}, \frac{1}{p} + \frac{1}{p4} + \frac{1}{p9}; \frac{1}{p} \\ &+ \frac{1}{p16}, \frac{1}{p} + \frac{1}{p4} \\ &+ \frac{1}{p16}, \frac{1}{p} + \frac{1}{p9} + \frac{1}{p16}, \frac{1}{p} + \frac{1}{p4} + \frac{1}{p9} + \frac{1}{p16}, \frac{1}{p} \\ &+ \frac{1}{p25}, \frac{1}{p} \\ &+ \frac{1}{p4} + \frac{1}{p25}, \frac{1}{p} + \frac{1}{p9} + \frac{1}{p25}, \dots \Big\}.\end{aligned}$$

Consider the F-sequence $\{F_n\}$ in G where $F_n = H_{n^2}$.

Then

$$|F_n| = p^{n^2}$$

and

$$\overline{d}(E) = \ell \lim_n \frac{|E_n|}{|F_n|} = \ell \lim_n \frac{2^{n-1}}{p^{n^2}} = 0.$$

Thus E satisfies $(R_0 \cdot 1)$.

We may choose a sequence of infinite subsets C_n in G such that

$$\begin{aligned}C_1 &= \Big\{ \frac{1}{p}, \frac{1}{p} + \frac{1}{p9}, \frac{1}{p} + \frac{1}{p16}, \frac{1}{p} + \frac{1}{p9} + \frac{1}{p16}, \frac{1}{p} + \frac{1}{p25}, \Big\} \\ &= \frac{1}{p} + \frac{1}{p9} + \frac{1}{p25}, \frac{1}{p} + \frac{1}{p16} + \frac{1}{p25}, \frac{1}{p} + \frac{1}{p9} + \frac{1}{p16} + \frac{1}{p25}, \frac{1}{p} + \frac{1}{p36}, \dots \Big\}. \\ C_2 &= \Big\{ \frac{1}{p}, \frac{1}{p} + \frac{1}{p16}, \frac{1}{p} + \frac{1}{p25}, \frac{1}{p} + \frac{1}{p16} + \frac{1}{p25}, \frac{1}{p} + \frac{1}{p36}, \dots \Big\} \\ C_3 &= \Big\{ \frac{1}{p}, \frac{1}{p} + \frac{1}{p25}, \frac{1}{p} + \frac{1}{p36}, \dots \Big\}, \text{ and so on.}\end{aligned}$$

So

$$E \supset C_1 \supset C_2 \supset \dots$$

and

$$C_{n+1} + p^{-(n+1)^2} \subset C_n \text{ for all } n.$$

Then by theorem 2.2, $\overline{E}$ contains a recurrent point, and hence E satisfies $(R_0 \cdot 2)$.

REMARK: If X is a discrete set and Y is a subset of X, then we will consider βY as a subset of βX. Indeed βY is the closure of Y in βX.

LEMMA 2.5: If an amenable group G contains an infinite subgroup H of infinite index, then H is an R_0-set in G.

PROOF: Since $[G : H] = \infty$, the number of disjoint cosets $\{tH\}$ is infinite. For any $\varphi\epsilon\mathrm{LIM}(G)$, since $\varphi(\chi_{_H}) = \varphi(\chi_{_{tH}})$ if $t_1H, t_2H, \dots, t_nH$ are distinct cosets, then $\varphi(\chi_{t_1H\cup t_2H.\ \cup t_nH}) = \varphi(\chi_{t_1H}) + \dots + \varphi(\chi_{t_nH}) = n\varphi(\chi_{t_nH}) < 1$. Since n is arbitrary, $\varphi(\chi_{_H}) = 0$ so $\overline{d}(H) = 0$. Thus H satisfies $(R_0 \cdot 1)$. To see that H satisfies $(R_0 \cdot 2)$ in G, note that if $w\epsilon R^H$ and if U is a neighbourhood of w in βG, then $\mathsf{U} \cap \overline{H}$ is also a neighbourhood of w and $X' = \left\{h\epsilon H : hw\epsilon\, \mathsf{U} \cap \overline{H}\right\}$ is infinite. Therefore $\{g\epsilon G : gw\epsilon\mathsf{U}\}$ $(\supset X)$ is infinite. Thus $w\epsilon R^G$. Since $\phi \neq A^H \subset R^H, R^G \cap \overline{H} \neq \phi$. Thus H satisfies $(R_0 \cdot 2)$ in G.

COROLLARY 2.6: If an amenable group G is a direct product of infinitely many non-trivial subgroups $\{G_\alpha : \alpha\epsilon I\}$, then G contains an R_0-set.

PROOF: To prove that G contains an R_0-set, we constuct an infinite subgroup H of G of infinite index with the above lemma. Indeed, write $I = I_1 \uplus I_2$ such that both I_1 and I_2 are infinite, then $H = \pi\left\{G_\alpha : \alpha\epsilon I_1\right\}$ is what we want.

LEMMA 2.7: If an infinite suchgroup H of an amenable group G contains an R_0-set A, then A is also an R_0-set in G.

PROOF: This is quite obvious since whenever $\overline{d}_H(A) = 0$ then $\overline{d}_G(A) = 0$ and $R^H \subset R^G$.

LEMMA 2.8: If H is an amenable subgroup of an amenable group G such that G/H has an R_0-set A', then $\theta^{-1}(A')$ is an R_0-set in G (θ is the natural homomorphism of G onto G/H).

PROOF: Let $A = \theta^{-1}(A')$ and $G' = G/H$. Then for each $\varphi\epsilon\mathrm{LIM}(G)$, $\varphi(\chi_{_A}) = \varphi(\chi_{_A} \cdot \theta) = \theta^*\varphi(\chi_{_{A'}}) = 0$ since $\theta^*\varphi\epsilon\mathrm{LIM}(G')$ and $\overline{d}_{G'}(A') = 0$. Then A satisfies $(R_0 \cdot 1)$.

The natural homomorphism θ of G onto G' can be extended to a continuous mapping of βG onto $\beta G'$. We will denote the extended mapping again by θ. It is not hard to check that $\theta^{-1}(R^{G'}) \subset R^G$, sine $A'^{-} \cap R^{G'} \neq \phi$, and $A^{-} \cap R^G \neq \phi$ either. Thus A satisfies $(R_0 \cdot 2)$, and A is an R_0-set as wanted.

Before providing our main result, we will need some definitions and some structure theorems for abelian groups. For the proofs see [6].

DEFINITION: Let $x\epsilon G$ and n be an integer; x is <u>divisible</u> by n in case there is an element $y\epsilon G$ with $ny = x$. A group G is <u>divisible</u> in case each $x\epsilon G$ is divisible by every $n > 0$.

A subgroup S of G is pure in G in case $nG \cap S = nS$ for every integer n.

If G is a periodic group, then a subgroup B of G is a basic subgroup of G in the following cases: (i) B is the direct sum of cyclic groups; (ii) B is pure in G; (iii) G/B is divisible.

THEOREM 2.9:

1. Every periodic group G contains a basic subgroup B.
2. Every periodic group G is a direct sum of p-groups.
3. Every periodic group is an extension of a direct sum of cyclic groups by a divisible group.
4. Every divisible subgroup D is a direct sum of copies of Q and of copies of $Z(p^{\infty})$.

We are now ready to state and prove the main results of this paper.

THEOREM 2.10: If G is an infinite abelian group, then G contains an R_0-set.

PROOF: We will consider two cases:

1. G: non-periodic. Then G contains an infinite cyclic subgroup which can be regarded as the additive group of integers Z. Thus by lemmas 2.3 and 2.7, there exists an R_0-set.
2. G: periodic. Then by 2.9, G contains a basic subgroup B, so that B is a direct sum of cyclic groups. Here we have two subcases:
   (a) If B is an infinite subgroup, by corollary 2.6 and lemma 2.7, there exists an R_0-set.
   (b) If B is a finite subgroup, the question group G/B is an infinite divisible that can be written as a direct sum of $\leq$ copies of $Z(P^{\infty})$ (see 2.9). Consider one of these copies and apply lemmas 2.4 and 2.7 to get an R_0-set in G. Then the proof of the theorem is completed.

THEOREM 2.11: If G is an infinite elementary amenable group, then G contains an R_0-set.

PROOF: If G is an abelian group, the theorem follows from the above one. If G is not abelian, then by lemma 1.2, G contains an infinite abelian subgroup H, and hence there exists an R_0-set A in H. By lemma 2.7, A is also an R_0-set in G.

THEOREM 2.12: If G is an infinite elementary amenable group, then

$$R^G \supsetneqq \overline{A^G} \quad \text{and} \quad R^G - K^G \neq \phi.$$

The proof follows from the above theorem and proposition 2.1.

COROLLARY 2.13: If G is an infinite elementary amenable group, then

$$R^G \subsetneqq WR^G.$$

This follows immediately from the above theorem and lemma 1.1.

CONJECTURE: Every infinite amenable group G contains an R_0-set, and therefore

$$A^G \subsetneqq R^G \quad \text{and} \quad R^G - K^G \neq \phi.$$

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