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# PROLONGATIONS OF F-STRUCTURE TO THE TANGENT BUNDLE OF ORDER 2

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ABSTRACT. A study of prolongations of $F$-structure to the tangent bundle of order 2 has been presented.

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## 1. INTRODUCTION.

Let $F$ be a nonzero tensor field of type $(1,1)$ and of class $c^\infty$ on an $n$-dimensional manifold $V_n$ such that [1]

$$F^K + (-)^{K+1}F = 0 \text{ and } F^W + (-)^{W+1}F \neq 0 \qquad \text{for } 1 < W < K \tag{1.1}$$

where $K$ is a fixed positive integer greater than 2. Such a structure on $V_n$ is called an $F$-structure of rank '$r$' and degree $K$. If the rank of $F$ is constant and $r = r(F)$, then $V_n$ is called an $F$-structure manifold of degree $K(\geq 3)$. The case when $K$ is odd has been considered in this paper.

Let the operators on $V_n$ be defined as follows [1]:

$$l = (-)^K F^{K-1} \text{ and } m = I + (-)^{K+1}F^{K-1} \tag{1.2}$$

where $I$ denotes the identity operator on $V_n$.
From the operators defined by (1.2) we have

$$l + m = I \text{ and } l^2 = l; \text{ and } m^2 = m \tag{1.3}$$

For $F$ satisfying (1.1), there exist complementary distributions $L$ and $M$ corresponding to the projection operators $l$ and $m$ respectively.

If rank $(F)$ = constant on $V_n$ then $dim\ L = r$ and $dim\ M = (n-r)$. We have the following results [1]

$$Fl = lF = F \text{ and } Fm = mF = 0 \tag{1.4a}$$

$$F^{K-1}l = -l \text{ and } F^{K-1}m = 0 \tag{1.4b}$$

## 2. PROLONGATIONS OF F-STRUCTURE IN THE TANGENT BUNDLE OF ORDER 2.

Let $V_n$ be an $n$-dimensional differentiable manifold of class $c^\infty$ and $T_p(V_n) = \bigcup_{p \in V_n} T_p(V_n)$ is the tangent bundle over the manifold $V_n$.

Let us denote $T_s^r(V_n)$, the set of all tensor fields of class $c^\infty$ and of the type $(r,s)$ in $V_n$ and $T(V_n)$ be the tangent bundle over $V_n$.

Let us introduce an equivalence relation $\sim$ in the set of all differentiable mappings $F: R \to V_n$ where $R$ is the real line. Let $r \geq 1$ be a fixed integer. If two mappings $F: R \to V_n$ and $G: R \to V_n$ satisfy the conditions

$$F^h(0) = G^h(0), \ \frac{dF^h(0)}{dt} = \frac{dG^h(0)}{dt}, \ldots, \frac{dF^r(0)}{dt^r} = \frac{dG^r(0)}{dt^r},$$

the mapping $F$ and $G$ being represented respectively by $X^h = F^h(t)$ and $X^h = G^h(t)$, $(t \in R)$ with respect to local coordinates $X^h$ in a coordinate neighborhood $\{U, X^h\}$ containing the point $P = F(0) = G(0)$, then we say that the mapping $F$ is equivalent to $G$. Each equivalence class determined by the equivalence relation $\sim$ is called an $r$-jet of $V_n$ and denoted by $J_p^r(F)$. The set of all $r$-jets of $V_n$ is called the tangent bundle of order $r$ and denoted by $T_r(V_n)$. The tangent bundle $T_2(V_n)$ of order 2 has the natural bundle structure over $V_n$, its bundle projection $\pi_2: T_2(V_n) \to V_n$ being defined by $\pi_2(Jp^2(F)) = P$. If we introduce a mapping such that $P = F(0)$, then $T_2(V_n)$ has a bundle structure over $T(V_n)$ with projection $\pi_{12}$.

Let us denote $T_2(V_n)$, the second order tangent bundle over $V_n$ and let $F^{II}$ be the second lift of $F$ in $T_2(V_n)$. The second lift $F^{II}$ which belong to $T_s^r(T_2(V_n))$ has component of the form [3]

$$F^{II}: \begin{bmatrix} F_i^h & 0 & 0 \\ y^s \delta_s F_i^h & F_i^h & 0 \\ z^s \delta_s F_i^h + (1/2) y^t y^s \delta_t \delta_s F_i^h & y^s \delta_s F_i^h & F_i^h \end{bmatrix} \tag{2.1}$$

with respect to the induced coordinates in $T_2(V_n)$, $F_i^h$ being local components of $F$ in $V_n$.

Now we obtain the following results on the second lift of $F$ satisfying (1.1).

For any $F, G \in T_1^1(V_n)$, the following holds [3]:

$$\begin{aligned} (G^{II}F^{II})X^{II} &= G^{II}(FX^{II}), \\ &= G^{II}(FX)^{II} \\ &= (G(FX))^{II} \\ &= (GF)^{II}X^{II} \qquad \text{for every } X \in T_0^1(V_n), \end{aligned} \tag{2.2}$$

therefore we have

$$G^{II}F^{II} = (GF)^{II}$$

If $P(s)$ denote a polynomial of variable $s$, then we have

$$(P(F))^{II} = P(F^{II}), \text{ where } F \in T_1^1(V_n) \tag{2.3}$$

We have the following theorem:

THEOREM 2.1. The second lift $F^{II}$ defines a $F$-structure in $T_2(V_n)$ iff $F$ defines a $F$-structure in $V_n$.

PROOF. Let $F$ satisfy (1.1) then $F$ defines $F$-structure in $V_n$ satisfying

$$F^K + (-)^{K+1}F = 0.$$

which in view of equation (2.3) yields

$$(F^{II})^K + (-)^{K+1}F^{II} = 0. \tag{2.4}$$

Therefore $F^{II}$ defines a $F$-structure in $T_2(V_n)$. The converse can be proved in a similar manner.

THEOREM 2.2. The second lift $F^{II}$ is integrable in $T_2(V_n)$, iff $F$ is integrable in $V_n$.

PROOF. Let us denote $N_{II}$ and $N$, the Nijenhuis tensors of $F^{II}$ and $F$ respectively. Then we have [2]

$$N_{II}(X,Y) = (N(X,Y))^{II} \tag{2.5}$$

We know that $F$-structure is integrable in $V_n$, iff

$$N(X,Y) = 0,$$

which in view of (2.5) is equivalent to

$$N_{II}(X,Y) = 0. \tag{2.6}$$

Thus $F^{II}$ is integrable, iff $F$ is integrable in $V_n$.

THEOREM 2.3. The second lift $F^{II}$ of $F$ is partially integrable in $T_2(V_n)$, iff $F$ is integrable in $V_n$.

PROOF. We know that for $F$ to be partially integrable in $V_n$, the following holds [2]:

$$N(lX,lY) = 0$$

and

$$N(mX,mY) = 0,$$

which, in view of equation (2.5), takes the form

$$N_{II}(l^{II}X^{II}, l^{II}, Y^{II}) = 0$$

and

$$N_{II}(m^{II}X^{II}, m^{II}Y^{II}) = 0. \tag{2.7}$$

where $l^{II}, m^{II}$ are operators in $T_2(V_n)$ which define the distribution $L^{II}$ and $M^{II}$ respectively. Thus equation (2.7) gives the condition for $F^{II}$ to be partially integrable.

The converse follows in a similar manner.

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