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INVOLUTIONS WITH FIXED POINTS IN 2-BANACH SPACES

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ABSTRACT. Some results on fixed points of involution maps in 2-Banach spaces have been obtained. These are extensions of those proved earlier by Goebel-Zlotkiewicz, Sharma-Sharma, Assad-Sessa and Iśeki.

KEY WORDS AND PHRASES. Involutions, 2-Banach spaces, coincidence points, fixed points.
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1. INTRODUCTION.

Gähler ([1]-[3]) initiated the concepts of 2-metric and 2-Banach spaces in a series of papers. These new spaces have subsequently been studied by several mathematicians in recent years. Like other spaces, the fixed point theory has also been developed in the frame work of these spaces. It was Iśeki ([4], [5]) who for the first time, obtained basic results on fixed points in 2-metric and 2-Banach spaces. Since then quite a number of authors have extended and generalized fixed point theorems of Iśeki and various other results involving contraction type mappings. For an extensive bibliography one is referred to Iśeki ([6]).

In this note, some fixed point theorems for certain involutions in 2-Banach spaces have been obtained which can be viewed as a 2-Banach space extension of a result due to Assad and Sessa [7], which in turn generalizes a fixed point theorem of Goebel and Zlotkiewiez [8] concerning an involution of a closed convex subset of a Banach space. The work of Assad and Sessa [7] was inspired by the contractive condition introduced by Delbosco [9]. Our result also generalizes a theorem of Sharma and Sharma [10]. It is important to note that in our proof continuity of the map under consideration is not essentially needed, and hence the same is unnecessarily stringent in Sharma and Sharma [10] and Iśeki [5].

2. PRELIMINARIES.

We assume the familiarity with the basic theory of 2-Banach spaces as given in White [11]. But for the sake of completeness we present here some pertinent definitions.

The following notions are essentially due to Gähler [1].

DEFINITION 2.1. Let X be a linear space, and $\|.,.\|$ be a real-valued function defined on X. Then the pair $(X\,\|.,.\|)$ is called a **linear 2-normed space** if, for $a,b,c \in X$,

(i) $\|a,b\| = 0$ if and only if a and b are linearly dependent,

(ii) $\|a,b\| = \|b,a\|$,

(iii) $\|a,\beta b\| = |\beta|\,\|a,b\|$, $\quad \beta$ being real,

(iv) $\|a,b+c\| \le \|a,b\| + \|a,c\|$.

Here $\|.,.\|$ is called a 2-norm and is a non-negative function.

DEFINITION 2.2. A sequence $\{x_n\}$ in a linear 2-normed space X is called a **convergent sequence** if there is an element $x \in X$ such that the $\lim_{n\to\infty} \|x_n - x, y\| = 0$ for all $y \in X$. If $\{x_n\}$ converges to x, we write $x_n \to x$ and call x the limit of $\{x_n\}$. Of course, here $dim X \ge 2$ otherwise every sequence of points in such a space would converge to every point of the space.

DEFINITION 2.3. A sequence $\{x_n\}$ in a linear 2-normed space X is called a **Cauchy sequence** if $\lim_{m,n\to\infty} \|x_m - x_n, y\| = 0$ for every $y \in X$.

DEFINITION 2.4. A linear 2-normed space in which every Cauchy sequence is a convergent sequence is called a **2-Banach space**.

We also need the following notion from Assad and Sessa [7].

Let Φ be the family of continuous functions $\phi: \Re_+^3 \to \Re_+$, (where $\Re_+$ stands for the set of non-negative reals) satisfying the following conditions:

(i) $\phi(1,1,1) = k < 2$,

(ii) for $s \ge 0, t \ge 0$, the inequality $s \le \phi(t, 2t, s)$ implies that $s \le kt$.

3. RESULTS.

Throughout this section, X stands for a 2-Banach space with $dim X \ge 2$, and I denotes the identity map on X.

THEOREM 3.1. Let T be a self-mapping of X and $\phi \in \Phi$ such that

(A) $T^2 = I$,

(B) $\|Tx - Ty, a\| \le \phi(\|x-y,a\|, \|x - Tx, a\|, \|y - Ty, a\|)$, for all x,y,a in X. Then T has at least one fixed point.

PROOF. Let x be an arbitrary point in X. Put $y = \frac{1}{2}(Tx + x)$, $z = Ty$ and $u = 2y - z$. It is easy to observe that

$$2\|Tx - y, a\| = \|x - Tx, a\| = 2\|x - y, a\|.$$

Now we have

$$\begin{aligned}\|x - z, a\| &= \|T^2x - Ty, a\| \\ &\le \phi(\|Tx - y, a\|, \|Tx - T^2x, a\|, \|y - Ty, a\|) \\ &= \phi(\|x - y, a\|, 2\|x - y, a\|, \|y - Ty, a\|\end{aligned}$$

and also

$$\begin{aligned}\|u - x, a\| &= \|2y - Ty - x, a\| = \|Tx - Ty, a\|, \\ &\le \phi(\|x - y, a\|, \|x - Tx, a\|, \|y - Ty, a\|) \\ &= \phi(\|x - y, a\|, 2\|x - y, a\|, \|y - Ty, a\|\end{aligned}$$

On the other hand, we have

$$\| u - z, a \| = 2 \| y - Ty, a \|$$

Hence

$$\| y - Ty, a \| \leq \phi(\| x - y, a \|, 2 \| x - y, a \|, \| y - Ty, a \|).$$

By the hypothesis (ii), we obtain

$$\| y - Ty, a \| \leq k \| x - y, a \| = \frac{k}{2} \| x - Tx, a \|.$$

Let us put $Gx = \frac{1}{2}(Tx + x)$, for any x in X. Then by the foregoing inequality, we get

$$\begin{aligned}
\| G^2 x - Gx, a \| &= \| Gy - y, a \| \\
&= \frac{1}{2} \| y - Ty, a \| \\
&\leq \frac{k}{4} \| x - Tx, a \| \\
&\leq \frac{k}{4} \| x - (2Gx - x), a \| \\
&= \frac{k}{2} \| Gx - x, a \|,
\end{aligned}$$

for all x, a in X.

Now, for an arbitrary point x_o in X, let $x_n = G^n x_o = Gx_{n-1}, n = 1, 2, \cdots$ *If* $m \geq n \geq 1$, then

$$\begin{aligned}
\| x_m - x_n, a \| &= \| G^m x_o - G^n x_o, a \| \\
&\leq \| G^m x_o - G^{m-1} x_o, a \| + \cdots + \| G^{n+1} x_o - G^n x_o, a \| \\
&= \sum_{r=n}^{m-1} \left(\frac{k}{2}\right)^r \| Gx_o - x_o, a \| \\
&\leq \left(\frac{k}{2}\right)^n \left(\frac{1}{1 - k/2}\right) \| Gx_o - x_o, a \|.
\end{aligned}$$

From this, it follows that $\{x_n\}$ is a Cauchy sequence which converges in X. Put $x^* = \lim_{n \to \infty} x_n$. Now consider

$$\begin{aligned}
\| x^* - Gx^*, a \| &\leq \| x^* - x_{n+1}, a \| + \| Gx_n - Gx^*, a \| \\
&\leq \| x^* - x_{n+1}, a \| + \frac{1}{2} \| x_n - x^*, a \| + \frac{1}{2} \| Tx_n - Tx^*, a \| \\
&\leq \| x^* - x_{n+1}, a \| + \frac{1}{2} \| x_n - x^*, a \| + \frac{1}{2} \phi(\| x_n - x^*, a \|, \| x_n - Tx_n, a \|, \| x^* - Tx^*, a \|)
\end{aligned}$$

$$\leq \| x^* - x_{n+1}, a \| + \frac{1}{2} \| x_n - x^*, a \| + \frac{1}{2} \phi(\| x_n - x^*, a \|, 2 \| (Gx_n - x_n), a \|, 2 \| (x^* - Gx^*), a \|).$$

Letting $n \to \infty$, we get

$$2 \| x^* - Gx^*, a \| \leq \phi(0, 0, 2 \| x^* - Gx^*, a \|).$$

So again by condition (ii), we get

$$\| x - Gx^*, a \| = 0, \text{ for all } a \text{ in } X.$$

Hence, $(x^* - Gx^*)$ and a are linearly dependent for all a in X. Since $dim X \geq 2$, the only way $(x^* - Gx^*)$ can be linearly dependent with all a in X, is that $x^* - Gx^* = 0$. Hence $x^* = Tx^*$ as required. This completes the proof.

COROLLARY 3.1 (Sharma and Sharma [10]). Let $T:X \to X$ be such that $T^2 = I$ and

$$\| Tx - Ty, a | \leq \alpha \| x - y, a \| + \beta(\| x - Tx, a \| + \| y - Ty, a \|), \tag{3.1}$$

for all x, y, a in X, where $\alpha \geq 0, \beta \geq 0$ and $\alpha + 4\beta < 2$. Then T has at least one fixed point.

PROOF. The condition (3.1) implies that

$$\begin{aligned} \| Tx - Ty, a \| &\leq \left(\frac{\alpha}{2}\right) \cdot 2 \| x - y, a \| + 2\beta \cdot \tfrac{1}{2}(\| x - Tx, a \| + \| y - Ty, a \|) \\ &\leq \left(\frac{\alpha}{2} + 2\beta\right) max\{2 \| x - y, a \|, \tfrac{1}{2}(\| x - Tx, a \| + \| y - Ty, a \|)\} \\ &\leq \left(\frac{\alpha}{2} + 2\beta\right) max\{2 \| x - y, a \|, \| x - Tx, a \|, \| y - Ty, a \| \}. \end{aligned}$$

Now if we assume that

$$\phi(p, q, r) = \left(\frac{\alpha}{2} + 2\beta\right) max\{2p, q, r\},$$

then, by Theorem 3.1, T has at least one fixed point. This completes the proof.

REMARKS.

(a) A critical observation of the proof of the main theorem in Sharma and Sharma [10], reveals that they have used the continuity of the involution map but failed to mention the same. However, in our proof this additional condition is not required.

(b) When X is the usual Banach space, Corollary 3.1 reduced to a theorem of Iśeki [5]. In a private communication Professor Iśeki agreed that the continuity of the involution map is essentially needed for his proof to hold.

COROLLARY 3.2. Under the hypothesis of Theorem 3.1, suppose, in addition, that at least one of the following strict inequality holds:

$$\begin{aligned} &\| x^* - Tx, a \| < \| x^* - x, a \| + \| x - Tx, a \|, \\ &\| x^* - x, a \| < \| x^* - Tx, a \| + \| Tx - x, a \| \end{aligned} \tag{3.2}$$

for all a and $x(\neq x^*)$ in X. Then x^* is the unique fixed point of T.

PROOF. By Theorem 3.1, $Tx^* = x^*$. Suppose also that $Ty^* = y^*$ for some $y^* \in X$. Assume that $x^* \neq y^*$. Then using (3.2), we have

$$\| x^* - y^*, a \| = \| x^* - Ty^*, a \| < \| x^* - y^*, a \| + \| y^* - Ty^*, a \| = \| x^* - y^*, a \|,$$

which is impossible. Therefore, $x^* = y^*$. Similarly, other condition in (3.2) also implies that $x^* = y^*$.

Now, we apply Theorem 3.1 to obtain a coincidence theorem.

THEOREM 3.2. Let T and S be the self-mappings of X, and $\phi \in \Phi$ such that the following hold:

(i) $T^2 = I, S^2 = I, TS = ST$,

(ii) $\| Tx - Ty, a \| \leq \phi(\| Sx - Sy, a \|, \| Sx - Tx, a \|, \| Sy - Ty, a \|)$, for all x, y, a in X. Then there exists at least one point x_o in X such that $Tx_o = Sx_o$.

PROOF. It follows from Theorem 3.1 that TS has at least one fixed point x_o. Then clearly $Tx_o = Sx_o$. This completes the proof.

REMARK. In case, one assumes some additional conditions on TS, as in Corollary 3.2, then x_o in Theorem 3.2 becomes the unique fixed point of TS. Then, commutativity of T, S and the uniqueness of x_o can be used to show that x_o is actually a common fixed point of S and T. Further, if S and T satisfies conditions similar to one in Corollary 3.2, then their common fixed point x_o is also unique.

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