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CATEGORY BASES

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ABSTRACT. We study category bases by means of category decompositions. These enable us to obtain a better insight into the structure of category bases.

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1. INTRODUCTION.

John C. Morgan II introduced around 1975 and has developed since then a theory of category bases (cf. [1]). Its main feature is to present within a common frame measure and category as well as some other point set classifications. The aim of this paper is to study category bases by means of category decompositions. This concept is a compilation of some ideas used by Morgan throughout his development of category bases. It has been applied in [2] and [3] to settle some problems from category bases. An approach using category decomposition enables us to give elementary proofs of some known theorems about category bases as well as some new ones. In particular, we give a proof of the Morgan-Schilling theorem about invariance under operation (A) of the family of Baire sets with respect to a category base (Theorem 1 in section 1). In section 2 we characterize certain fields of sets which are generated by category bases (Theorem 3). Finally, we provide a characterization of category bases equivalent to topologies by means of lower densities (Theorem in section 3).

Let us recall some basic definitions and concepts of the theory of category bases.

A category base on a set X is a pair (X,C) such that X is a non-empty set and C is a family of non-empty subsets of X, called regions, satisfying the following axioms:

(1) $\cup C=X$;

(2) Let A be a region and D a non-empty family of disjoint regions which has power less than the power of C. Then

(i) if $A\cap(\cup D)$ contains a region, then there is a region $B\in D$ such that $A\cap B$ contains a region.

(ii) if $A\cap(\cup D)$ contains no region, then there is a region $B\subseteq A$ which is disjoint from $\cup D$.

Notice that parts (i) and (ii) of (2) can be rewritten in the following form:

(j) if $A\cap B$ contains no region for each $B\in D$, then there is a subregion of A which is disjoint from $\cup D$.

Also, axiom (1) can be skipped for without loss of generality, we may assume that X is a region. Standard examples of category bases include topologies (without the empty set) or sets of positive measure with respect to a σ-finite measure.

A set $E \subseteq X$ is singular if for each region A there exists a region $B \subseteq A$ such that $B \cap E = \phi$. A set $M \subseteq X$ is meager if M is a countable union of singular sets; non-meager sets are called abundant sets. A set Y is abundant everywhere in a region A if for every region $B \subseteq A, B \cap Y$ is abundant. The class of meager sets will be denoted $M(C)$. A set $G \subseteq X$ is Baire if for each region A there exists a region $B \subseteq A$ such that $B \cap G$ is meager or $B \cap (X - G)$ is meager. The class of all Baire sets will be denoted by $B(C)$. If C is a topology, then $M(C)$ is the family of sets of 1st category and $B(C)$ is the family of sets with the Baire property. Generally, $B(C)$ is a σ-complete field of subsets of X and $M(C)$ is a σ-additive complete ideal in $B(C)$.

If C is a category base on X and A is a region then $CIA = \{B \in C: B \subseteq A\}$ is a category base on A; let us call such a category base the restriction of C to A. It is easy to see that $M(CIA) = \{E \cap A: E \in M(C)\}$ and $B(CIA) = \{E \cap A: E \in B(C)\}$.

2. CATEGORY DECOMPOSITIONS.

Let C be a category base on X. We say that R is a category decomposition of X if R is a disjoint family of regions such that each region from C intersects a member of R on a set that contains a region.

It is known (cf. [4]) that the intersection of two regions either contains a region or is singular. This is also true for any finite number of regions. This is not true for an infinite number of regions as topological examples show.

PROPOSITION 1. Let C be a category base on X and let $R \subseteq C$ be a category decomposition of X. Then:

(i) $X - (\cup R)$ is a singular set;

(ii) if $E \subseteq X$ and $E \cap A$ is singular for each $A \in R$, then E is singular;

(iii) if $E \subseteq X$ and $E \cap A$ is meager for each $A \in R$, then E is meager;

(iv) if $E \subseteq X$ and $E \cap A$ is Baire for each $A \in R$, then E is Baire.

PROOF. (i) Let D be a region. There exist $A \in R$ and $B \in C$ such that $B \subseteq D \cap A$. Hence $B \subseteq D$ and $B \cap (X - (\cup R)) = \phi$.

(ii) Let E be a subset of X such that $E \cap A$ is singular for each $A \in R$. Let D be a region. There exists $A \in R$ and $B \in C$ such that $B \subseteq D \cap A$. Since $E \cap A$ is singular, there exists a subregion B' of B such that $B' \cap (E \cap A) = \phi$. Hence $B' \cap E = \phi$.

(iii) If $E \subseteq X$ is a set such that $E \cap A$ is meager for each $A \in R$, then let $E \cap A = \cup \{F_n(A): n \in \omega\}$, where $F_n(A)$ is singular for each $n \in \omega$ and $A \in R$. By virtue of (ii), the set $F_n = \cup \{F_n(A): A \in R\}$ is singular for each $n \in \omega$. Hence the set E is meager being the union of the following singular sets: F_n, $n \in \omega$ and $E \cap (X - (\cup R))$.

(iv) Let E be a subset of X such that $E \cap A$ is Baire for each $A \in R$. Let D be a region. There exist $A \in R$ and $B \in C$ such that $B \subset D \cap A$. There exists a subregion B' of B such that $B' \cap (E \cap A)$ is meager or $B' \cap (X - (E \cap A))$ is meager. Since $B' \cap E = B' \cap (E \cap A)$ and $B' \cap (X - E) = B' \cap (X - (E \cap A))$, property (iv) follows.

PROPOSITION 2. Let C be a category base on X such that each region contains a subregion with property P. Then there exists a category decomposition of X consisting exclusively of regions with property P.

PROOF. Let $C = \{A_\alpha : \alpha < \kappa\}$, where $|C| = \kappa$. For each $\beta < \kappa$ we are going to define a region B_β so that the following are satisfied:

(1) $\{B_\alpha : \alpha \leq \beta\}$ is a disjoint family for each $\beta < \kappa$;

(2) B_β has property P for each $\beta < \kappa$;

(3) A_α intersects some B_β with $\beta \leq \alpha$ on a set containing a region.

Let B_O be a subregion of A_O with property P. Let $\beta < \kappa$ and suppose B_α has been defined for each $\alpha < \beta$.

If $A_\beta \cap B_\alpha$ contains a region for some $\alpha < \beta$, then we set $B_\beta = B_O$. In the opposite case, since $\alpha < \beta$ and C is a category base, there exists a subregion B of A_β disjoint with every $B_\alpha, \alpha < \beta$. We take B_β to be a subregion of B with property P. The induction is finished.

From conditions (1), (2), and (3) it follows immediately that $R = \{B_\beta : \beta < \kappa\}$ is a required category decomposition of X.

COROLLARY 1. Let C be a category base on X. A set $E \subseteq X$ is singular if and only if there exists a category decomposition R of X such that $E \subseteq X - (\cup R)$.

PROOF. The "if" part follows immediately from the definition of category decomposition. To show the "only if" part, apply Proposition 2 for the property P being "disjoint with E".

COROLLARY 2. Let C be a category base on X and let R_1, R_2 be two category decompositions of X. Then there exists a category decomposition R of X which is a common refinement of R_1 and R_2, i.e., for each $A \in R$ there exist $B_1 \in R_1$ and $B_2 \in R_2$ such that $A \subseteq B_1 \cap B_2$.

PROOF. Let us take an arbitrary region A. There exists a region $A_1 \subseteq A \cap B_1$ for some $B_1 \in R_1$. There exists a region $A_2 \subseteq A_1 \cap B_2$ for some $B_2 \in R_2$. Hence the region A contains a subregion A_2 contained in $B_1 \cap B_2$ for some $B_1 \in R_1$ and $B_2 \in R_2$. Hence the corollary follows from Proposition 2.

COROLLARY 3. (= Morgan's Fundamental Theorem [1]). Let C be a category base on X. Then for each abundant subset Y of X there exists a region A such that Y is abundant everywhere in A.

PROOF. Suppose otherwise. Then every region contains a subregion intersecting Y on a meager set. By virtue of Proposition 2, there exists a category decomposition R of X consisting only of regions that intersect Y on a meager set. By virtue of Proposition 1 (iii), Y would be a meager set; a contradiction.

COROLLARY 4. For each category base there exists a category decomposition such that each member is either meager or abundant everywhere in itself.

PROOF. Each region is either abundant everywhere in itself or contains a meager subregion. Hence by an application of Proposition 2 for the above property P we get the required category decomposition.

COROLLARY 5. Let C be a category base on X. Then X can be decomposed into two sets G and H such that H is meager and G is abundant everywhere in any region A such that $A - G$ is singular.

PROOF. Let R be a category decomposition of X guaranteed by Corollary 4. Then we set $H = \cup\{B \in R : B \text{ is meager}\} \cup (X - (\cup R))$ and G is the rest of X, i.e., $G = \cup\{B \in R : B \text{ is abundant everywhere in itself}\}$. By virtue of Proposition 1 (i) and (iii), H is meager. Suppose that A is a region and $A - G$ is singular. Let A_1 be any subregion of A. Then A_1 must intersect one of the members of R, which is abundant everywhere in itself, on a non-singular set. Hence that intersection contains a region and therefore abundant.

COROLLARY 6. Let A be a region and let S be a Baire set such that $S \cap A$ is abundant. Then there exists a subregion B of A such that S is abundant everywhere in B yet $B - S$ is meager.

PROOF. According to Corollary 3, there exists a region B_1 such that $S \cap A$ is abundant everywhere in B_1. Hence $B_1 \cap A$ is abundant and therefore there exists a region $B_2 \subseteq B_1 \cap A$. Since S is Baire, there exists a subregion B of B_2 such that either $B \cap S$ is meager or $B \cap (X - S)$ is meager. Since $S \cap A$ is abundant everywhere in B_1 and B is a subregion of $B_1, B \cap S$ cannot be meager. Hence $B - S$ is meager and S is abundant everywhere in B.

A set E is an essential hull for a set S if E is a Baire set, $S - E$ is meager, and if F is a Baire set such that $S - F$ is meager, then $E - F$ is meager. J. Morgan [1] has shown that in the case of *ccc* category bases, each set has an essential hull. We can show that this is true in general.

COROLLARY 7. Let C be a category base on X and Y be a subset of X. Then there exists an essential hull, $b(Y)$, for Y. Moreover, $b(Y)$ is the union of a disjoint family of regions which are abundant everywhere in themselves and a meager set.

PROOF. By virtue of Proposition 2, there exists a category decomposition R of X such that for each $A \in R$ either $A \cap Y$ is meager or Y is abundant everywhere in A. Let $R' = \{A \in R: Y$ is abundant everywhere in $A\}$ and $b'(Y) = \cup R'$. By virtue of Proposition 1 (iv) and (iii), $b'(Y)$ is Baire and the set $Y - b'(Y)$ is meager. We set $b(Y) = b'(Y) \cup (Y - b'(Y))$. Let B' be a Baire set containing Y. We shall show that the Baire set $F = b'(Y) - B'$ is meager.

Suppose otherwise. Then there exists A in R' such that $A \cap F$ is abundant. By virtue of Corollary 4, there exists a subregion D of A such that, in particular, $D - F$ is meager. Since $D \cap B' \subseteq D - F, D \cap B'$ is meager. Since $Y \subseteq B', D \cap Y$ is meager, too. But this contradicts the fact that $A \in R'$.

In 1933 E. Szpilrajn-Marczewski proved a general theorem on families to be closed under operation (A). This theorem enables us to give a simple proof of the Morgan-Schilling theorem.

THEOREM. (E. Szpilrajn-Marczewski [5]). Let K be a σ-field of subsets of a set X and $N(K)$ be the subclass of K consisting of sets all of whose subsets are also in K. Suppose that K and $N(K)$ satisfy the following condition:

(#) if $E \subseteq X$, then there exists a set $A \in K$ such that $E \subseteq A$ and if $B \in K$ contains E then $A - B \in N(K)$.

Then K is closed under operation (A).

THEOREM 1. (J. Morgan and K. Schilling [6]). Let C be a category base on X. Then $B(C)$ is closed under operation (A).

PROOF. The family $B(C)$ together with $M(C)$ satisfies the assumptions of Marczewski's theorem above. Indeed, $B(C)$ is a σ-field and by virtue of Corollary 7, (#) holds since each subset of a meager set is Baire.

REMARK. (a) The referee has pointed out that our Corollary 7 can be derived from Theorem 15 or Theorem 16 [1; pp 35,37].

(b) Corollary 7 has recently been obtained, independently, by A. Kucia of Katowice.

3. FIELDS AND CATEGORY BASES.

A κ-complete field on a set X is a non-empty family F of subsets of X such that:

(a) if $R \subseteq F$ and $|R| < \kappa$ then $\cup R \in F$;

(b) if $A \in F$, then $X - A \in F$.

ω-complete fields are called fields and ω_1-complete fields are called σ-fields.

Let F be a field on X. A κ-ideal of F is a proper subfamily I of F such that:

(i) if $R \subseteq I$ and $|R| < \kappa$ then $\cup R$ is contained in a member of I;

(ii) if $A \in I$ and $B \in F$ and $B \subseteq A$ then $B \in I$.

ω-ideals are called ideals and ω_1-ideals are called σ-ideals.

A family R of sets is said to be κ-saturated if it does not contain κ pairwise disjoint elements. We set $sat(R) = inf\{\kappa : R \text{ is } \kappa\text{-saturated}\}$.

THEOREM 1. Let F be a field on X. If I is a κ-ideal of F such that $sat(F-1) \leq \kappa$, then $C = F - I$ is a category base on X.

PROOF. Since $F \neq I$, $X \in C$ and therefore $\cup C = X$. Let R be a disjoint subfamily of C and let A be a fixed element of C. Suppose that $A \cap B \notin C$ for each $B \in R$. Since F is a field, $A \cap B \in I$ for each $B \in R$. Since $|R| \leq sat(C) \leq \kappa$, there exists a member D of I such that $A \cap B \subseteq D$ for each $B \in R$. Hence $G = A - D \in C, G \subseteq A$, and $G \cap (\cup R) = \phi$.

Let F be a field on X. A κ-complete ideal of F is a κ-ideal I of F which is complete in the sense that if $E \in I$ and $Y \subseteq E$, then $Y \in I$.

THEOREM 2. Let F be a κ-complete field on X, where $\kappa > \omega$. If I is a κ-complete ideal of F such that $sat(F - I) \leq \kappa$, then $C = F - I$ is a category base on X with $M(C) = I$ and $B(C) = F$.

PROOF. By virtue of Theorem 1, C is a category base on X with $F \subseteq B(C)$ and $I \subseteq M(C)$. To prove that $M(C) \subseteq I$ it is enough if we show that the ideal I contains all singular sets, since I is κ-complete and $\kappa > \omega$. So let E be singular. Take a maximal pairwise disjoint family $R \subseteq C$ such that $E \cap (\cup R) = \phi$. Since $sat(C) \leq \kappa$ and F is κ-complete, $G = X - (\cup R)$ is in F. Since R is maximal and $G \in F, G$ is singular. Since no singular set belongs to the category base, $G \in I$. Hence $E \in I$ being a subset of G.

To prove that $B(C) \subseteq F$ let us take arbitrary non-meager Baire set B. Let R be a maximal pairwise disjoint subfamily of C such that $A \cap (X - B) \in I$ for all $A \in R$. By virtue of κ-completeness of F and I it follows that the set $D = \cup R \in C$ and that $D \cap (X - B) \in I$. The set $B - D$ is Baire, too. By virtue of maximality of $R, B - D$ must be meager. Since $M(C) = I, B = (D - E_1) \cup E_2$, where $D \in C$ and $E_1, E_2 \in I$. Hence $B \in F$.

A field F has the κ-*cc* subset property with respect to an ideal I of F if for each $A \in F - I$ there is $B \in F - I$ such that $B \subseteq A$ and $sat(\{D \in F - I : D \subseteq B\}) \leq \kappa$.

A family R is a decomposition of a family $F \subseteq P(X)$ if R is a disjoint collection of elements of F such that:

(1) $\cup R = X$;

(2) for each $A \in F$ there exist $B \in R$ and $C \in F$ such that $C \subseteq A \cap B$;

(3) if $Y \subseteq X$ and $Y \cap B \in F$ for all $B \in R$, then $Y \in F$.

It is easy to see that if (X, M, μ) is a decomposable measure space (cf. E. Hewitt and K. Stromberg [7; p. 317]) and N is the ideal of μ-zero sets, then there exists a decomposition of $M - N$.

THEOREM 3. Let F be a κ-complete field on X and let I be a κ-complete ideal of F such that F has the κ-*cc* subset property with respect to I, where $\kappa > \omega$. Then there exists a category base $C \subseteq F$ with $M(C) = I$ and $B(C) = F$ if and only if there exists a decomposition R of $F - I$ with $sat(\{D \in F - I : D \subseteq B\}) \leq \kappa$ for all $B \in R$.

PROOF. Let $C \subseteq F$ be a category base such that $B(C) = F$ and $M(C) = I$. We shall show that each region contains a subregion B such that $sat(\{D \in F - I : D \subseteq B\}) \leq \kappa$. So let $A \in C$. Since $M(C) = I$ and I is identical with the family of all singular sets, $A \in F - I$. There exists $B_1 \in F - I$ such that $B_1 \subseteq A$ and $sat(\{D \in F - I : D \subseteq B_1\}) \leq \kappa$. Hence B_1 is an abundant Baire set. There exists a subregion B of A such that $B - B_1$ is meager and thus $B - B_1$ belongs to I. Hence B satisfies the required condition.

By virtue of Proposition 2, there exists a category decomposition R' of X consisting of regions B such that $sat(\{D \in F - I: D \subseteq B\}) \leq \kappa$. If $E = X - (\cup R')$, then $E \in I$, by Proposition 1 (i). Let us attach the set E to an element of R'. We get a decomposition of the family $F - I$. Indeed, $R \subseteq F - I$ and R is a covering of X. Let $Y \in F - I$. Hence $Y \in B(C) - M(C)$. There exists a region A such that $A - Y$ is meager, i.e., $A - Y$ is in I. Since A intersects a member of R on a set containing a region, Y intersects a member of R' on a set from $F - I$. The last property for R to be a decomposition of $F - I$ follows immediately from Proposition 1 (iii) and (iv).

To prove the converse implication, assume that R is a decomposition of $F - I$ consisting of sets B such that $sat(\{D \in F - I: D \subseteq B\}) \leq \kappa$. Define $C = \{D \in F - I: D \subseteq B$ for some $B \in R\}$. It is easy to see that C is a category base on X. The properties of C that $B(C) = F$ and $M(C) = I$ follow immediately from Theorem 2 by a simple application of property (3) for R to be a decomposition of $F - I$.

Let us give one more example of a field and an ideal bearing a decomposition.

Following D. Fremlin [8], a measurable space with negligibles is a triple (X, S, I), where X is a set, S is a σ-field of subsets of X and I is a σ-complete ideal generated by $I \cap S$. It is proto-decomposable if there is $W \subseteq S$ such that whenever $R \subseteq W$ is disjoint, then

(i) $\cup R \in S$,

(ii) if $\{E_r: r \in R\} \subseteq I$, then $\cup\{E_r \cap r: r \in R\} \in I$,

(iii) if $E \subseteq X$ and $E - (\cup R) \in S - I$, then there is a non-empty $V \in W$ such that $V \cap (\cup R) = \phi$ and $V - E \in I$.

PROPOSITION 3. Let (X, S, I) be a complete proto-decomposable measurable space with negligibles. Suppose that $W \subseteq S$ witnesses proto-decomposability of (X, S, I). If R is a maximal pairwise disjoint subfamily of $W - I$, then R is a decomposition of $S - I$.

PROOF. It follows easily from the definition of proto-decomposable measurable spaces that $X - (\cup R) \in I$. From the maximality of R it follows that R satisfies condition (2) for R to be a decomposition of $S - I$; condition (3) is shown in Fremlin's 1Ha from [8]; to get condition (1) simply attach $X - (\cup R)$ to an arbitrary member of R.

REMARK. The last proposition is also related to a characterization for measure algebras given in [2].

4. EQUIVALENCE PROBLEM.

Two category bases (X, C_1) and (X, C_2) are equivalent if $B(C_1) = B(C_2)$ and $M(C_1) = M(C_2)$. The Equivalence Problem of Morgan [4] asks if each category base is equivalent to topology. Here we elaborate further on this problem.

In the case C is a category base on X such that $X \in M(C), C$ is equivalent to the topology $T = \{X - E$: E is singular$\}$. It is known that category bases not exceeding ω_1 are equivalent to topologies (K. Schilling [9], Z. Piotrowski and A. Szymanski [10]).

PROPOSITION. Let C be a category base on X such that each region A contains a subregion B for which the category base $C|B$ is equivalent to a topology on B. Then C is equivalent to topology.

PROOF. By virtue of Proposition 2, there exists a category decomposition R of X such that $C|B$ is equivalent to a topology T_B for each $B \in R$. Then the topology T generated by $\cup\{T_B: B \in R\} \cup \{X\}$ is a topology on X equivalent to C.

From our proposition it follows that category bases that are locally of cardinality $\leq \omega_1$ are also equivalent to topologies. This, in turn, generalizes Morgan's theorem proved in [11]. Let S be a field of subsets of a set X and let $I \subseteq S$ be a complete ideal such that $X \notin I$. Then $E \Delta F$ means that both $E - F$ and $F - E$ are in I. A function $q: S \to S$ is called a lower density with respect to I (cf. [12]) if for any pair A, B in S one has:

(1) $q(A) \Delta A$

(2) $A \Delta B$ implies $q(A) = q(B)$

(3) $q(\phi) = \phi$ and $q(X) = X$

(4) $q(A \cap B) = q(A) \cap q(B)$

THEOREM. Let (X,C) be a category base such that $X \notin M(C)$. Then C is equivalent to a topology on X if and only if there exists a lower density on $B(C)$ with respect to $M(C)$.

PROOF. Suppose that there exists a topology T on X such that (X,C) and (X,T) are equivalent category bases. Let $B \subseteq X$ be a set with the Baire property (in T). Then there exists a unique open set $q(B)$ satisfying the following properties:

(i) $q(B)$ is regularly open,

(ii) each non-empty open subset of $q(B)$ is of 2nd category,

(iii) $q(B) \Delta B$.

To define $q(B)$ let us apply Corollary 7 with B as the set Y. We get an open set $b(B)$ such that $b(B) \Delta B$ and each non-empty open subset of $b(B)$ is of 2nd category. We set $q(B) = \mathit{int}\ \mathit{cl}\ b(B)$. Since $B(C)$ coincides with the family of all subsets of X with the Baire property, the correspondence $B \to q(B)$ is a function from $B(C)$ into $B(C)$. We shall show that it is a lower density.

Since two distinct regular open sets must differ by a non-empty open set and since sets of the form $q(B)$ are of 2nd category everywhere in themselves we have properties (1), (2), and (3) satisfied trivially for such defined q. It remains to show (4). Let A and B be two subsets of X with the Baire property. Let R_1, R_2 be category (= topology) decompositions of X yielding $b(A)$ and $b(B)$, respectively. By virtue of Corollary 2, there exists a category decomposition R of X which is a common refinement of R_1 and R_2. Then R is a category decomposition of X yielding $b(A \cap B)$.
Since $b(A) \cap b(B)$ and $b(A \cap B)$ differ only by a nowhere dense set we get $q(A \cap B) = \mathit{int}\ \mathit{cl}\ b(A \cap B) = \mathit{int}\ \mathit{cl}\ [b(A) \cap b(B)] = \mathit{int}\ \mathit{cl}\ b(A) \cap \mathit{int}\ \mathit{cl}\ b(B) = q(A) \cap q(B)$. The proof that q is a lower density on $B(C)$ with respect to $M(C)$ is finished.

Assume now that there exists a lower density q on $B(C)$ with respect to $M(C)$. Let T be the topology on X generated by sets of the form $q(B) - N$, where $B \in B(C)$ and $N \in M(C)$. Let us notice some obvious properties of T;

(a) each meager set is nowhere dense;

(b) each Baire set has the Baire property;

(c) if B is a Baire abundant set, then $\mathit{int}\ B \neq \phi$.

We shall show that:

(d) if E is nowhere dense, then E is meager.

For suppose otherwise. Then there exists a region A such that E is abundant everywhere in A. Since $\mathit{int}\ q(A) \neq \phi$, there exists a Baire set B and a meager set N such that $\phi \neq q(B) - N \subseteq q(A)$ and $[q(B) - N] \cap E = \phi$. Hence $B \cap A$ is abundant. By virtue of Corollary 6, there exists a region D which is abundant everywhere in itself such that $D \subseteq A$ and $D - B$ is meager. Since $q(B) \Delta B, D \cap E$ is meager which contradicts that E is abundant everywhere in A.

Property (d) together with (a) show that $M(C)$ coincides with the collection of all sets of 1st category. We shall show that:

(e) if $B \subseteq X$ has the Baire property, then $B \in B(C)$.

Let $B = (U - E) \cup F$, where U is open and E, F are sets of 1st category. Let A be a region. We are going to construct a subregion of A intersecting B or its complement on a set of 1st category. Without loss of generality we may assume that A is abundant everywhere in itself and that $q(A) \cap U \neq \phi$. Hence there exists a Baire set B_1 and a meager set N such that $\phi \neq q(B_1) - N \subseteq q(A) \cap U$. Hence $B_1 \cap A$ is abundant. By invoking Corollary 4 again, there exists a region D which is abundant everywhere in itself such that $D \subseteq A$ and $D - B_1$ is meager. In consequence, $D \cap B - N \subseteq E \cup F$ with is a meager set because of (d).

Property (e) together with (b) show that $B(C)$ coincides with the collection of all sets with the Baire property. Hence C and T are equivalent.

If κ is an uncountable cardinal, then any example of a κ-complete field S on a set X together with a κ-complete ideal $I \subseteq S$ such that $sat(S - I) \leq \kappa$ and such that there is no lower density on S with respect to I would be a counterexample to the Equivalence Problem. This is an immediate consequence of the Theorem in section 3 and Theorem 2 in section 2. Unfortunately, at present we are unaware of the existence of such an example.

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