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ON COMPLETE CONVERGENCE IN A BANACH SPACE

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ABSTRACT: Sufficient conditions are given under which a sequence of independent random elements taking values in a Banach space satisfy the Hsu and Robbins law of large numbers. The complete convergence of random indexed sums of random elements is also considered.

KEY WORDS AND PHRASES : complete convergence, strong law of large numbers, random elements, Banach space, random indexed sums.
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1. INTRODUCTION

Let $\{X_n,\ n \geq 1\}$ be a sequence of independent random elements taking values in a separable Banach space $(B, \|\ \|)$. Put $S_n = \sum_{i=1}^{n} X_i$. A sequence $\{X_n,\ n \geq 1\}$ of random elements is said to satisfy the law of large numbers of Hsu-Robbins type if for any given $\varepsilon > 0$

$$\sum_{n=1}^{\infty} P[\|S_n\| \geq \varepsilon n] < \infty. \tag{1.1}$$

Hsu and Robbins [1] proved that the existence of the second moment of independent, identically distributed random variables for which $EX_1 = 0$, implies the Hsu-Robbins type law of large numbers. Erdös [2] showed that the existence of the second moment of independent, identically distributed random variables and the condition $EX_1 = 0$ is also the necessary one for the Hsu-Robbins type law of large numbers. Considerations concerning (1.1) for sequences and subsequences of independent, identically distributed random variables can be found in Katz [3], Baum, Katz [4], Asmussen, Kurtz [5] and Gut [6]. The results in those cases are given under the assumption when there exists a finite moment of order r $(1 < r \leq 2)$.

Some conditions, which guarantee the convergence of (1.1) for sequences and subsequences in the case nonidentically distributed random variables can be found in Duncan, Szynal [7], Bartoszyński, Puri [8] and Kuczmaszewska, Szynal [9], [10]. For instance, it has been shown in Duncan, Szynal [7] that if a sequence $\{X_n,\ n \geq 1\}$ of independent random variables with $EX_n = 0$ and

$EX_n^2 < \infty$, $n \geq 1$ satisfy the conditions

$$(i) \qquad \sum_{n=1}^{\infty} \sum_{i=1}^{n} P[|X_i| \geq n\varepsilon] < \infty,$$

$$(ii) \qquad \sum_{n=1}^{\infty} n^{-4} \sum_{i=1}^{n} E(X_i I[|X_i| < n\varepsilon] - EX_i I[|X_i| < n\varepsilon])^4 < \infty,$$

$$(iii) \qquad \sum_{n=1}^{\infty} n^{-4} \sum_{m=2}^{n} \sigma^2(X_m I[|X_m| < n\varepsilon]) \sum_{i=1}^{m-1} \sigma^2(X_i I[|X_i| < n\varepsilon]) < \infty,$$

$$(iv) \qquad \sum_{n=1}^{\infty} n^{-4} (\sum_{i=1}^{n} E(X_i I[|X_i| < n\varepsilon]))^4 < \infty$$

then

$$\sum_{n=1}^{\infty} P[|S_n| \geq n\varepsilon] < \infty.$$

The following example shows that the assumptions (i)-(iv) which are sufficient conditions for (1.1) in the case of independent random variables are not sufficient if we consider sequences of independent random elements taking values in Banach space B.

EXAMPLE. Let l^1 denote the separable Banach space

$$l^1 = \{x \in R^\infty, \ \|x\| = \sum_{n=1}^{\infty} |x_n| < \infty\}$$

and e^n denote the element having 1 for its n-th coordinate and 0 in the other coordinates. Let $\{\xi_n, \ n \geq 1\}$ be a sequence of independent random variables defined as follows $P[\xi_n = 1] = P[\xi_n = -1] = 1/2$, $n \geq 1$, and define $X_n = \xi_n e^n$, $n \geq 1$. Thus $\{X_n, \ n \geq 1\}$ is a sequence of independent l^1 -valued random elements with symmetric distributions, such that $EX_n = 0$, $E\|X_n\|^2 = 1$, $E\|X_n\|^4 = 1$, $n \geq 1$, and $\{X_n, \ n \geq 1\}$ satisfies the assumptions (i)-(iv) but $\|n^{-1} \sum_{i=1}^{n} X_i\| = n^{-1} \sum_{i=1}^{n} 1 = 1$, which shows that the condition $\sum_{n=1}^{\infty} P[\|S_n\| \geq n\varepsilon] < \infty$ does not hold for all $\varepsilon > 0$.

The aim of this note is to give sufficient conditions, which guarantee the Hsu-Robbins type of large numbers for independent random elements taking values in Banach space B.

2. PRELIMINARIES

We need now an extension of Hoffman-Jörgensen inequality (cf. Hoffmann- Jörgensen [11], and Gut [6]).

LEMMA 1. Let $\{X_n, \ n \geq 1\}$ be a sequence of independent random elements taking values in a real separable Banach space $(B, \|\ \|)$ with a symmetric distribution. Then for every $j = 1, \ 2, \ldots, n$ and $t > 0$

$$P[\|S_n\| \geq 3^j t] \leq C_j \sum_{i=1}^{n} P[\|X_i\| \geq t] + D_j (P[\|S_n\| \geq t])^{2^j}, \tag{2.1}$$

where C_j and D_j are positive constants depending only on j.

PROOF. Let $T = inf\{n \geq 1,\ \|S_n\| \geq t\}$. Then

$$P[\|S_n\| \geq 3t] = \sum_{i=1}^{n} P[\|S_n\| \geq 3t,\ T = i]$$

$$= \sum_{i=1}^{n} P[\|S_n\| \geq 3t,\ \|S_1\| < t, \ldots, \|S_{i-1}\| < t,\ \|S_i\| \geq t]$$

$$= \sum_{i=1}^{n} P[\|S_n - S_i + S_{i-1} + X_i\| \geq 3t,\ \|S_1\| < t, \ldots,\ \|S_{i-1}\| < t,\ \|S_i\| \geq t]$$

$$\leq \sum_{i=1}^{n} P[\|S_n - S_i\| \geq 3t - \|S_{i-1}\| - \|X_i\|,\ \|S_1\| < t, \ldots,\ \|S_{i-1}\| < t,\ \|S_i\| \geq t]$$

$$\leq \sum_{i=1}^{n} P[\|S_n - S_i\| \geq 2t - \|X_i\|,\ T = i] \leq \sum_{i=1}^{n} P[\|X_i\| \geq t,\ T = i]$$

$$+ \sum_{i=1}^{n} P[\|S_n - S_i\| \geq t,\ T = i] \leq \sum_{i=1}^{n} P[\|X_i\| \geq t]$$

$$+ \sum_{i=1}^{n} P[\|S_n - S_i\| \geq t] \cdot P[T = i].$$

Moreover,

$$P[\|S_n - S_i\| \geq t] \leq P[max(\|S_n - S_i\|,\ \|S_n - S_i + S_i\|) \geq t]$$

$$\leq 2P[\|S_n\| \geq t],$$

as $S_n - S_i$ and S_i are independent, symmetrically distributed random elements.

Hence

$$P[\|S_n\| \geq 3t] \leq \sum_{i=1}^{n} P[\|X_i\| \geq t] + 2P[\|S_n\| \geq t] \cdot \sum_{i=1}^{n} P[T = i]$$

$$\leq \sum_{i=1}^{n} P[\|X_i\| \geq t] + 2P[\|S_n\| \geq t] \cdot P[\max_{1 \leq j \leq n} \|S_j\| \geq t]$$

$$\leq \sum_{i=1}^{n} P[\|X_i\| \geq t] + 4(P[\|S_n\| \geq t])^2.$$

By the induction principle, we get

$$P[\|S_n\| \geq 3^{j+1}t] = P[\|S_n\| \geq 3 \cdot 3^j t]$$

$$\leq \sum_{i=1}^{n} P[\|X_i\| \geq 3^j t] + 4(P[\|S_n\| \geq 3^j t])^2$$

$$\leq \sum_{i=1}^{n} P[\|X_i\| \geq t] + 4(C_j \sum_{i=1}^{n} P[\|X_i\| \geq t] + D_j \cdot P^{2^j}[\|S_n\| \geq t])^2$$

$$\leq C_{j+1} \sum_{i=1}^{n} P[\|X_i\| \geq t] + D_{j+1}(P[\|S_n\| \geq t])^{2^{j+1}}.$$

Moreover, we shall use the following lemmas.

LEMMA 2. (Yurinski [12]) Let $X_1, \ldots, X_n$ be independent B-valued random elements with $E\|X_i\| < \infty$ ($i = 1, \ldots, n$). Let $\mathcal{F}_k$ be the σ-field generated by $(X_1, \ldots, X_k)$, $(k = 1, \ldots, n)$ and let $\mathcal{F}_0 = \{\emptyset, \Omega\}$. Then for $1 \le k \le n$

$$|E(\|S_n\||\mathcal{F}_k) - E(\|S_n\||\mathcal{F}_{k-1})| \le \|X_k\| + E\|X_k\|. \tag{2.2}$$

LEMMA 3. (Loève [13]) For every $\varepsilon > 0$

$$P[\|X - med\, X\| \ge \varepsilon] \le 2 \cdot P[\|X^s\| \ge \varepsilon], \tag{2.3}$$

$$P[\sup_{j \le n} \|X_j - med\, X_j\| \ge \varepsilon] \le 2 \cdot P[\sup_{j \le n} \|X_j^s\| \ge \varepsilon], \tag{2.4}$$

where X^s is a symmetrized version of X.

In what follows we shall use the strong law of large numbers for a sequence of independent, identically distributed random elements $\{X_n, n \ge 1\}$ in a separable Banach space given in Taylor [14].

THEOREM. Let $\{X_n, n \ge 1\}$ be a sequence of independent identically distributed B-valued random elements such that $E\|X_1\| < \infty$.

Then $\|n^{-1} \sum_{i=1}^{n} X_i - EX_1\| \to 0$ a.s. as $n \to \infty$.

3. RESULTS

THEOREM 1. Let $\{X_n, n \ge 1\}$ be a sequence of independent, symmetrically distributed, B-valued random elements. Suppose that $\{n_k, k \ge 1\}$ is a strictly increasing sequence of positive integers. If for some positive integer j and any given $\varepsilon > 0$

$$(i) \qquad \sum_{k=1}^{\infty} \sum_{i=1}^{n_k} P[\|X_i\| \ge n_k \varepsilon / 3^j] < \infty,$$

$$(ii) \qquad \sum_{k=1}^{\infty} (n_k^{-4} \sum_{i=1}^{n_k} E\|X_i\|^4 I[\|X_i\| < n_k \varepsilon])^{2^j} < \infty,$$

$$(iii) \qquad \sum_{k=1}^{\infty} (n_k^{-4} \sum_{m=2}^{n_k} E\|X_m\|^2 I[\|X_m\| < n_k \varepsilon] \sum_{i=1}^{m-1} E\|X_i\|^2 I[\|X_i\| < n_k \varepsilon])^{2^j} < \infty,$$

then

$$\sum_{k=1}^{\infty} P[\|S_{n_k}\| \ge n_k \varepsilon] < \infty$$

iff

$$\|S_{n_k}/n_k\| \to 0 \text{ in probability as } k \to \infty. \tag{3.1}$$

PROOF. It is enough to show that under the conditions (i)-(iv) $\|S_{n_k}/n_k\| \to 0$ in probability as $k \to \infty$ implies that $\sum_{k=1}^{\infty} P[\|S_{n_k}\| \ge n_k \varepsilon] < \infty$.

Put $X_j' = X_j I[\|X_j\| < n_k \varepsilon]$, $S_n' = \sum_{i=1}^{n} X_i'$ and $Y_{n_k,i} = E(\|S_{n_k}'\||\mathcal{F}_i) - E(\|S_{n-k}'\||\mathcal{F}_{i-1})$ where $\mathcal{F}_i = \sigma(X_1', X_2', \ldots, X_i')$ and $\mathcal{F}_0 = \{\emptyset, \Omega\}$. Then we have

$$P[\|S_{n_k}\| \geq n_k\varepsilon] \leq C_j \sum_{i=1}^{n_k} P[\|X_i\| \geq n_k\varepsilon/3^j] + D_j(P[\|S_{n_k}\| \geq n_k\varepsilon/3^j])^{2^j}.$$

Moreover,

$$\sum_{k=1}^{\infty}(P[\|S_{n_k}\| \geq n_k\varepsilon/3^j])^{2^j}$$

$$\leq 2^{2^j-1}\{\sum_{k=1}^{\infty}(\sum_{i=1}^{n_k} P[\|X_i\| \geq n_k\varepsilon/3^j])^{2^j} + \sum_{k=1}^{\infty}(P[\|S'_{n_k}\| \geq n_k\varepsilon/3^j])^{2^j}\}.$$

Note that $\|S'_{n_k}\| - E\|S'_{n_k}\| = \sum_{i=1}^{n_k} Y_{n_k,i}$ and

$$\sum_{k=1}^{\infty}(P[|\|S'_{n_k}\| - E\|S'_{n_k}\|| \geq n_k\varepsilon/3^j])^{2^j} = \sum_{k=1}^{\infty}(P[(\sum_{i=1}^{n_k} Y_{n_k,i})^2 \geq n_k^2(\varepsilon/3^j)^2])^{2^j}$$

$$= \sum_{k=1}^{\infty}\{P[\sum_{i=1}^{n_k} Y_{n_k,i}^2 + 2\sum_{m=2}^{n_k} Y_{n_k,m}\sum_{i=1}^{m-1} Y_{n_k,i} \geq n_k^2(\varepsilon/3^j)^2]\}^{2^j}$$

$$\leq \sum_{k=1}^{\infty}(P[\sum_{i=1}^{n_k} Y_{n_k,i}^2 \geq n_k^2(\varepsilon/3^j)^2/2] + P[\sum_{m=2}^{n_k} Y_{n_k,m}\sum_{i=1}^{m-1} Y_{n_k,i} \geq n_k^2(\varepsilon/3^j)^2/4])^{2^j}.$$

Now putting $Z_{n_k,i} = Y_{n_k,i}^2 - EY_{n_k,i}^2$ and using the inequality (2.2) we get for $\varepsilon' = (\varepsilon/3^j)^2/2$

$$\sum_{k=1}^{\infty}(P[|\sum_{i=1}^{n_k} Z_{n_k,i}| \geq n_k^2\varepsilon'])^{2^j} \leq (\varepsilon')^{-2^{j+1}}\sum_{k=1}^{\infty}(n_k^{-4}E|\sum_{i=1}^{n_k} Z_{n_k,i}|^2)^{2^j}$$

$$= (\varepsilon')^{-2^{j+1}}\sum_{k=1}^{\infty}(n_k^{-4}\sum_{i=1}^{n_k} EZ_{n_k,i}^2)^{2^j} \leq (\varepsilon')^{-2^{j+1}}\sum_{k=1}^{\infty}(n_k^{-4}\sum_{k=1}^{n_k} EY_{n_k,i}^4)^{2^j}$$

$$\leq (\varepsilon')^{-2^{j+1}} \cdot 2^{2^{j+2}}\sum_{k=1}^{\infty}(n_k^{-4}\sum_{i=1}^{n_k} E\|X'_i\|^4)^{2^j} < \infty.$$

Moreover, we see that (ii) and (iii) imply

$$n_k^{-2}\sum_{i=1}^{n_k} EY_{n_k,i}^2 \leq 8n_k^{-2}\sum_{i=1}^{n_k} E\|X'_i\|^2 = o(1)$$

as

$$(n_k^{-2}\sum_{i=1}^{n_k} E\|X'_i\|^2)^2 = n_k^{-4}\sum_{i=1}^{n_k} E\|X'_i\|^4 + n_k^{-4}2\sum_{m=2}^{n_k} E\|X'_m\|^2\sum_{i=1}^{m-1} E\|X'_i\|^2$$

implies

$$(n_k^{-2}\sum_{i=1}^{n_k} E\|X'_i\|^2)^2 \to 0 \text{ as } k \to \infty.$$

Now we see that $\{Y_{n_k,i}\sum_{j=1}^{i-1} Y_{n_k,j},\ 2 \leq i \leq n\}$ and $\{Y_{n_k,i},\ 1 \leq i \leq n\}$ are martingale differences for fixed n. Therefore

$$\sum_{k=1}^{\infty}(P[\sum_{m=2}^{n_k} Y_{n_k,m}\sum_{i=1}^{m-1} Y_{n_k,i} \geq n_k^2\varepsilon'/2])^{2^j}$$

$$\leq 2^{2^j+1}(\varepsilon')^{-2^{j+1}}\sum_{k=1}^{\infty}\{n_k^{-4}\sum_{m=2}^{n_k} E(Y_{n_k,m}\sum_{i=1}^{m-1} Y_{n_k,i})^2\}^{2^j}$$

$$\leq 2^{2^j+1}(\varepsilon')^{-2^{j+1}}\sum_{k=1}^{\infty}\{n_k^{-4}\sum_{m=2}^{n_k} E[(\|X'_m\| + E\|X'_m\|)^2(\sum_{i=1}^{m-1} Y_{n_k,i})^2]\}^{2^j}$$

$$\leq 2^{2^j+1}(\varepsilon')^{-2^{j+1}}\sum_{k=1}^{\infty}\{n_k^{-4}\sum_{m=2}^{n_k} E(\|X'_m\| + E\|X'_m\|)^2\sum_{i=1}^{m-1} E(\|X'_i\| + E\|X'_i\|)^2\}^{2^j}$$

$$\leq A_j\sum_{k=1}^{\infty}\{n_k^{-4}\sum_{m=2}^{n_k} E\|X'_m\|^2\sum_{i=1}^{m-1} E\|X'_i\|^2\}^{2^j} < \infty,$$

where A_j is a positive constant depending only on j and ε.
Thus we have proved that

$$\sum_{k=1}^{\infty}(P[|\|S'_{n_k}\| - E\|S'_{n_k}\|| \geq n_k\varepsilon/3^j])^{2^j} < \infty, \tag{3.2}$$

which implies that

$$P[|\|S'_{n_k}\| - E\|S'_{n_k}\|| \geq n_k\varepsilon] \rightarrow 0 \text{ as } k \rightarrow \infty. \tag{3.3}$$

Moreover, we state that (3.1) and (i) imply

$$P[\|S'_{n_k}\| \geq n_k\varepsilon] = P[\|S'_{n_k}\| \geq n_k\varepsilon,\ S_{n_k} = S'_{n_k}] + P[\|S'_{n_k}\| \geq n_k\varepsilon,\ S_{n_k} \neq S'_{n_k}]$$

$$\leq P[\|S_{n_k}\| \geq n_k\varepsilon] + \sum_{i=1}^{n_k} P[\|X_i\| \geq n_k\varepsilon] \rightarrow 0 \text{ as } k \rightarrow \infty$$

or

$$P[\|S'_{n_k}\| \geq n_k\varepsilon] \rightarrow 0 \text{ as } k \rightarrow \infty. \tag{3.4}$$

Hence by (3.3) and (3.4) we get

$$E\|S'_{n_k}\|/n_k \rightarrow 0 \text{ as } k \rightarrow \infty,$$

which together with (3.2) gives

$$\sum_{k=1}^{\infty}(P[\|S'_{n_k}\| \geq n_k\varepsilon/3^j])^{2^j} < \infty.$$

Taking into account that

$$\sum_{k=1}^{\infty}(P[\|S_{n_k}\| \geq n_k\varepsilon/3^j])^{2^j}$$

$$\leq 2^{2^j-1}\{\sum_{k=1}^{\infty}(\sum_{i=1}^{n_k} P[\|X_i\| \geq n_k\varepsilon/3^j])^{2^j} + \sum_{k=1}^{\infty}(P[\|S'_{n_k}\| \geq n_k\varepsilon/3^j])^{2^j}\}$$

and using (i) we complete the proof of Theorem 1.

COROLLARY 1. Let $\{X_n,\ n \geq 1\}$ be a sequence of independent, symmetrically distributed, B-valued random elements. Suppose that $\{n_k,\ k \geq 1\}$ is a strictly increasing sequence of positive integers. If for some positive integer j and any given $\varepsilon > 0$

$$(i) \qquad \sum_{k=1}^{\infty}\sum_{i=1}^{n_k} P[\|X_i\| \geq n_k\varepsilon/3^j] < \infty,$$

$$(ii) \qquad \sum_{k=1}^{\infty}(n_k^{-2}\sum_{i=1}^{n_k} E\|X_i\|^2 I[\|X_i\| < n_k\varepsilon])^{2^j} < \infty,$$

then

$$\sum_{k=1}^{\infty} P[\|S_{n_k}\| \geq n_k\varepsilon] < \infty$$

iff

$$\|S_{n_k}/n_k\| \rightarrow 0 \text{ in probability as } k \rightarrow \infty.$$

Now we consider the Hsu and Robbins law of large numbers for subsequences of independent, nonsymmetrically distributed random elements taking values in a real separable Banach space.

THEOREM 2. Let $\{X_n,\ n \geq 1\}$ be a sequence of independent, B-valued random elements. Suppose that $\{n_k,\ k \geq 1\}$ is a strictly increasing sequence of positive integers. If for some positive integer j and any given $\varepsilon > 0$

$$(I) \qquad \sum_{k=1}^{\infty}\sum_{i=1}^{n_k} P[\|X_i\| \geq n_k\varepsilon/(2\cdot 3^j)] < \infty,$$

$$(II) \qquad \sum_{k=1}^{\infty}(n_k^{-4}\sum_{i=1}^{n_k} E\|X_i\|^4 I[\|X_i\| < 2n_k\varepsilon])^{2^j} < \infty,$$

$$(III) \qquad \sum_{k=1}^{\infty}(n_k^{-4}\sum_{m=2}^{n_k} E\|X_m\|^2 I[\|X_m\| < 2n_k\varepsilon]\sum_{i=1}^{m-1} E\|X_i\|^2 I[\|X_i\| < 2n_k\varepsilon])^{2^j} < \infty,$$

then

$$\sum_{k=1}^{\infty} P[\|S_{n_k}\| \geq n_k\varepsilon] < \infty$$

iff

$$\|S_{n_k}/n_k\| \rightarrow 0 \text{ in probability as } k \rightarrow \infty.$$

PROOF. Assume that $\{X_n,\ n \geq 1\}$ is a sequence of symmetrically distributed random elements. Then by Theorem 1 we conclude that conditions (I) - (III) are sufficient for the Hsu and Robbins law of large numbers, i.e.

$$\sum_{k=1}^{\infty} P[\|S_{n_k}\| \geq n_k\varepsilon] < \infty$$

To remove the symmetry assumption we argue as follows. Let $\{X_n^s,\ n \geq 1\}$ be a sequence of the symmetrized version of X, i.e. $X_k^s = X_k - X_k^*$, $k \geq 1$, where X_k and X_k^* are independent and have the same distribution. Then by (I) we get for $\varepsilon' = \varepsilon/3^j$

$$\sum_{k=1}^{\infty}\sum_{i=1}^{n_k} P[\|X_i^s\| \geq n_k\varepsilon'] = \sum_{k=1}^{\infty}\sum_{i=1}^{n_k} P[\|X_i - X_i^*\| \geq n_k\varepsilon']$$

$$\leq 2\sum_{k=1}^{\infty}\sum_{i=1}^{n_k} P[\|X_i\| \geq n_k\varepsilon'/2] < \infty$$

and by (I) and (II) we have

$$\sum_{k=1}^{\infty}(n_k^{-4}\sum_{i=1}^{n_k} E\|X_i^{s'}\|^4)^{2^j} = \sum_{k=1}^{\infty}(n_k^{-4}\sum_{i=1}^{n_k} E\|X_i - X_i^*\|^4 I[\|X_i - X_i^*\| < n_k\varepsilon])^{2^j}$$

$$= \sum_{k=1}^{\infty}(n_k^{-4}\sum_{i=1}^{n_k} E\|X_i - X_i^*\|^4 I[\|X_i - X_i^*\| < n_k\varepsilon,\ \|X_i^*\| < n_k\varepsilon]$$

$$+ n_k^{-4}\sum_{i=1}^{n_k} E\|X_i - X_i^*\|^4 I[\|X_i - X_i^*\| < n_k\varepsilon,\ \|X_i^*\| \geq n_k\varepsilon])^{2^j}$$

$$\leq 2^{2^j+1-1}\sum_{k=1}^{\infty}(n_k^{-4}\sum_{i=1}^{n_k} E\|X_i\|^4 I[\|X_i\| < 2n_k\varepsilon])^{2^j} + 2^{2^j-1}\varepsilon^{2^j+2}\sum_{k=1}^{\infty}\sum_{i=1}^{n_k}(P[\|X_i\| \geq n_k\varepsilon])^{2^j} < \infty.$$

Now we see (II) and (III) imply

$$n_k^{-2}\sum_{i=1}^{n_k} E\|X_i\|^2 I[\|X_i\| < 2n_k\varepsilon] \to 0 \text{ as } k \to \infty \tag{3.5}$$

since

$$(n_k^{-2}\sum_{i=1}^{n_k} E\|X_i\|^2 I[\|X_i\| < 2n_k\varepsilon])^2 = n_k^{-4}\sum_{i=1}^{n_k} E\|X_i\|^4 I[\|X_i\| < 2n_k\varepsilon]$$

$$+ 2n_k^{-4}\sum_{m=2}^{n_k} E\|X_m\|^2 I[\|X_m\| < 2n_k\varepsilon]\sum_{i=1}^{m-1} E\|X_i\|^2 I[\|X_i\| < 2n_k\varepsilon].$$

Therefore by (I), (III) and (3.5) we obtain

$$\sum_{k=1}^{\infty}(n_k^{-4}\sum_{m=2}^{n_k} E\|X_m^{s\prime}\|^2\sum_{i=1}^{m-1} E\|X_i^{s\prime}\|^2)^{2^j}$$

$$\leq C\{\sum_{k=1}^{\infty}(n_k^{-4}\sum_{m=2}^{n_k} E\|X_m\|^2 I[\|X_m\| < 2n_k\varepsilon]\cdot\sum_{i=1}^{m-1} E\|X_i\|^2 I[\|X_i\| < 2n_k\varepsilon])^{2^j}$$

$$+\sum_{k=1}^{\infty}(n_k^{-2}\sum_{m=2}^{n_k} E\|X_m\|^2 I[\|X_m\| < 2n_k\varepsilon]\cdot\sum_{i=1}^{m-1} P[\|X_i\| \geq n_k\varepsilon])^{2^j}$$

$$+\sum_{k=1}^{\infty}(n_k^{-2}\sum_{m=2}^{n_k} P[\|X_m\| \geq n_k\varepsilon]\cdot\sum_{i=1}^{m-1} E\|X_i\|^2 I[\|X_i\| < 2n_k\varepsilon])^{2^j}$$

$$+\sum_{k=1}^{\infty}\sum_{i=1}^{n_k} P[\|X_i\| \geq n_k\varepsilon]\} < \infty,$$

where C is a positive constant depending only on j and ε.
Hence by Theorem 1 we obtain

$$\sum_{k=1}^{\infty} P[\|S_{n_k}^s\| \geq n_k\varepsilon] < \infty.$$

Taking into account the symmetrization inequality (2.3)

$$P[\|S_{n_k}/n_k - med(S_{n_k}/n_k)\| \geq \varepsilon] \leq 2P[\|S_{n_k}^s\| \geq n_k\varepsilon]$$

we have

$$\sum_{k=1}^{\infty} P[\|S_{n_k}/n_k - med(S_{n_k}/n_k)\| \geq \varepsilon] < \infty.$$

But the assumption $P[\|S_{n_k}\| \geq n_k\varepsilon] \to 0$ as $k \to \infty$

$$\|med(S_{n_k}/n_k)\| \to 0 \text{ as } k \to \infty,$$

which together with

$$\sum_{k=1}^{\infty} P[\|S_{n_k}/n_k - med(S_{n_k}/n_k)\| \geq \varepsilon] < \infty$$

gives

$$\sum_{k=1}^{\infty} P[\|S_{n_k}\| \geq n_k\varepsilon] < \infty.$$

COROLLARY 2. Let $\{X_n, n \geq 1\}$ be a sequence of independent, B-valued random elements. Suppose that $\{n_k, k \geq 1\}$ is a strictly increasing sequence of positive integers. If for some positive integer j and any given $\varepsilon > 0$

$$(I') \qquad \sum_{k=1}^{\infty}\sum_{i=1}^{n_k} P[\|X_i\| \geq n_k\varepsilon/(2\cdot 3^j)] < \infty,$$

$$(II') \qquad \sum_{k=1}^{\infty}(n_k^{-2}\sum_{i=1}^{n_k} E\|X_i\|^2 I[\|X_i\| < 2n_k\varepsilon])^{2^j} < \infty,$$

then

$$\sum_{k=1}^{\infty} P[\|S_{n_k}\| \geq n_k\varepsilon] < \infty$$

iff

$$\|S_{n_k}/n_k\| \to 0 \text{ in probability as } k \to \infty.$$

COROLLARY 3. Let $\{X_n, n \geq 1\}$ be a sequence of independent, B-valued random elements. Suppose that $\{n_k, k \geq 1\}$ is a strictly increasing sequence of positive integers. If for some positive integer j and any given $\varepsilon > 0$

$$(I") \qquad \sum_{k=1}^{\infty}\sum_{i=1}^{n_k} P[\|X_i\| \geq n_k\varepsilon/(2\cdot 3^j)] < \infty,$$

$$(II") \qquad \sum_{k=1}^{\infty}(n_k^{-4}\sum_{i=1}^{n_k} E\|X_i\|^4 I[\|X_i\| < 2n_k\varepsilon])^{2^j} < \infty,$$

$$(III") \qquad \sum_{k=1}^{\infty}(n_k^{-2}\sum_{i=1}^{n_k} E\|X_i\|^2 I[\|X_i\| < 2n_k\varepsilon])^{2^{j+1}} < \infty,$$

then

$$\sum_{k=1}^{\infty} P[\|S_{n_k}\| \geq n_k\varepsilon] < \infty$$

iff

$$\|S_{n_k}/n_k\| \rightarrow 0 \text{ in probability as } k \rightarrow \infty.$$

Some results concerning the independent identically distributed random elements can be obtained as corollaries of Theorem 2.

COROLLARY 4. Let $\{X_n, n \geq 1\}$ be a sequence of independent, identically distributed B-valued random elements. Suppose that $\{n_k, k \geq 1\}$ is a strictly increasing sequence of positive integers. If for some positive integer j and any given $\varepsilon > 0$

$$(I*) \qquad \sum_{k=1}^{\infty} n_k P[\|X_1\| \geq n_k\varepsilon/(2 \cdot 3^j)] < \infty,$$

$$(II*) \qquad \sum_{k=1}^{\infty} (n_k^{-3} E\|X_1\|^4 I[\|X_1\| < 2n_k\varepsilon])^{2^j} < \infty,$$

$$(III*) \qquad \sum_{k=1}^{\infty} (n_k^{-1} E\|X_1\|^2 I[\|X_1\| < 2n_k\varepsilon])^{2^{j+1}} < \infty,$$

then

$$\sum_{k=1}^{\infty} P[\|S_{n_k}\| \geq n_k\varepsilon] < \infty$$

iff

$$\|S_{n_k}/n_k\| \rightarrow 0 \text{ in probability as } k \rightarrow \infty.$$

COROLLARY 5. (Theorem of Hsu and Robbins for random elements taking values in Banach space) If $\{X_n, n \geq 1\}$ is a sequence of independent, identically distributed B-valued random elements with $EX_1 = 0$ and $E\|X_1\|^2 < \infty$, then

$$\sum_{k=1}^{\infty} P[\|S_{n_k}\| \geq n_k\varepsilon] < \infty.$$

PROOF. It is easy to see the that conditions (I*) - (III*) from Corollary 4 are satisfied by the assumptions $EX_1 = 0$ and $E\|X_1\|^2 < \infty$. Moreover, by the strong law of large numbers for a sequence $\{X_n, n \geq 1\}$ of independent, identically distributed random elements we conclude that

$$\|S_n/n\| \rightarrow 0 \text{ in probability as } n \rightarrow \infty.$$

COROLLARY 6. Let $\{X_n, n \geq 1\}$ be a sequence of independent, identically distributed B-valued random elements with $EX_1 = 0$ and let $\{n_k, k \geq 1\}$ be a strictly increasing sequence of positive integers. Suppose that for some r, $1 < r \leq 2$,

$$x^{-r} M(\psi(x)) \rightarrow \infty \; as \; x \rightarrow \infty, \qquad (3.6)$$

where $\psi(x) = card\{k : n_k \le x\}$, $x > 0$, $\psi(0) = 0$, $M(x) = \sum_{k=1}^{[x]} n_k$, $x > 0$.

If

$$\sum_{k=1}^{\infty} n_k P[\|X_1\| \ge n_k\varepsilon] < \infty \tag{3.7}$$

then

$$\sum_{k=1}^{\infty} P[\|S_{n_k}\| \ge n_k\varepsilon] < \infty.$$

PROOF. The assumption (3.7) implies that $EM(\psi(\|X_1\|)) < \infty$ which with (3.6) gives $E\|X_1\|^r < \infty$ for some r, $1 < r \le 2$.

Now it is easy to show that there exists some positive integer j, for which

$$\sum_{k=1}^{\infty}(n_k^{-3}E\|X_1\|^4 I[\|X_1\| < 2n_k\varepsilon])^{2^j} \le \sum_{k=1}^{\infty}(n_k^{-3}E\|X_1\|^r(2n_k\varepsilon)^{4-r})^{2^j}$$

$$\le C \cdot \sum_{k=1}^{\infty} n_k^{(1-r)2^j}(E\|X_1\|^r)^{2^j} < \infty.$$

and

$$\sum_{k=1}^{\infty}(n_k^{-1}E\|X_1\|^2 I[\|X_1\| < 2n_k\varepsilon])^{2^{j+1}} \le \sum_{k=1}^{\infty}(n_k^{-1}E\|X_1\|^r(2n_k\varepsilon)^{2-r})^{2^{j+1}}$$

$$\le C' \sum_{k=1}^{\infty} n_k^{(1-r)2^{j+1}}(E\|X_1\|^r)^{2^{j+1}} < \infty.$$

Similary, as in the proof of Corollary 5, by the strong law of large numbers for a sequence $\{X_n,\ n \ge 1\}$ of independent, identically distributed random elements we conclude that

$$\|S_{n_k}/n_k\| \to 0 \text{ in probability as } k \to \infty.$$

REMARK. Note that the WLLN is implied by the additional conditions: $EX_n = 0$ and B is of the type 2 since

$$P[\|S_{n_k}\| \ge n_k\varepsilon] \le P[\|S_{n_k} - ES_{n_k}\| \ge n_k\varepsilon]$$

$$\le P[\|S'_{n_k} - ES'_{n_k}\| \ge n_k\varepsilon] + \sum_{i=1}^{n_k} P[\|X_i\| \ge n_k\varepsilon/(2\cdot 3^j)]$$

$$\le \varepsilon^{-2}n_k^{-2}\sum_{i=1}^{n_k} E\|X_i'\|^2 + \sum_{i=1}^{n_k} P[\|X_i\| \ge n_k\varepsilon/(2\cdot 3^j)] = o(1).$$

Now we are going to present some results on complete convergence for randomly indexed partial sums of independent, non-identically distributed random elements.

THEOREM 3. Let $\{X_n,\ n \ge 1\}$ be a sequence of independent, B-valued random elements and $\{T_n,\ n \ge 1\}$ be positive integer valued random variables. Let $\{a_n,\ n \ge 1\}$ be strictly increasing positive integers and $\{\beta_n,\ n \ge 1\}$ be positive constants such that $a_n \to \infty$ as $n \to \infty$, $lim\ sup_{n\to\infty}\beta_n = \beta < 1$ and

$$\sum_{n=1}^{\infty} P[|T_n/a_n - N| \ge \beta_n] < \infty, \tag{3.8}$$

where N is a positive random variables such that for some A, B, where $\beta < A < B < \infty$, $P[A < N < B] = 1$.

If for some positive integer j and for any given $\varepsilon > 0$

$$(a) \qquad \sum_{k=1}^{\infty} \sum_{i=1}^{[a_k(B+\beta_k)]} P[\|X_i\| \geq a_k\varepsilon(A-\beta)\beta/(2\cdot 3^j)] < \infty,$$

$$(b) \qquad \sum_{k=1}^{\infty}(a_k^{-4} \sum_{i=1}^{[a_k(B+\beta_k)]} E\|X_i\|^4 I[\|X_i\| < 2a_k\varepsilon(A-\beta)])^{2^j} < \infty,$$

$$(c) \qquad \sum_{k=1}^{\infty}(a_k^{-4} \sum_{m=2}^{[a_k(B+\beta_k)]} E\|X_m\|^2 I[\|X_m\| < 2a_k\varepsilon(A-\beta)] \sum_{i=1}^{m-1} E\|X_i\|^2 I[\|X_i\| < 2a_k\varepsilon(A-\beta)])^{2^j} < \infty,$$

then

$$\sum_{k=1}^{\infty} P[\|S_{T_k}\| \geq T_k\varepsilon] < \infty \tag{3.9}$$

if

$$\|S_{[a_k(B+\beta_k)]}/[a_k(B+\beta_k)]\| \to 0 \text{ in probability as } k \to \infty.$$

PROOF. Note that

$$P[\|\sum_{i=1}^{T_n} X_i\| \geq T_n\varepsilon]$$

$$\leq P[\|\sum_{i=1}^{T_n} X_i\| \geq T_n\varepsilon,\ |T_n/a_n - N| < \beta_n] + P[|T_n/a_n - N| \geq \beta_n]$$

$$\leq P[\max_{a_n(A-\beta_n)<j<a_n(B+\beta_n)} \|S_j\| \geq a_n\varepsilon(A-\beta)] + P[|T_n/a_n - N| \geq \beta_n] \tag{3.10}$$

Now assuming that X_n, $n \geq 1$, are symmetrically distributed random elements we get by the Lévy's inequality

$$P[\max_{a_n(A-\beta_n)<j<a_n(B+\beta_n)} \|S_j\| \geq a_n\varepsilon(A-\beta)]$$

$$\leq 2P[\|\sum_{i=1}^{[a_n(B+\beta_n)]} X_i\| \geq a_n\varepsilon(A-\beta)].$$

But under the assumptions of Theorem 3 one can verify after using Theorem 1 with $n_k = [a_k(B+\beta_k)]$ that

$$\sum_{k=1}^{\infty} P[\|\sum_{i=1}^{[a_k(B+\beta_k)]} X_i\| \geq a_k\varepsilon(A-\beta)] < \infty.$$

This bound and the assumption (3.8) together with (3.10) imply (3.9) for symmetrically distributed random elements.

To remove the symmetry assumption we proceed similar as it has been done in the proof of Theorem 2.

$$\sum_{n=1}^{\infty} P[\|S_{T_n}/T_n - med(S_{T_n}/T_n)\| \geq \varepsilon] < \infty. \tag{3.11}$$

Now we note that

$$P[\|\sum_{i=1}^{T_n} X_i\| \geq T_n\varepsilon]$$

$$\le P[\max_{a_n(A-\beta_n)<j<a_n(B+\beta_n)} \|\sum_{i=1}^{j} X_i I[\|X_i\| < a_n(A-\beta)\varepsilon]\| \ge a_n\varepsilon(A-\beta)]$$

$$+ \sum_{i=1}^{[a_k(B+\beta_k)]} P[\|X_i\| \ge a_k\varepsilon(A-\beta)] + P[|T_n/a_n - N| \ge \beta_n]$$

But by the Kolmogorov's inequality

$$P[\max_{a_n(A-\beta_n)<j<a_n(B+\beta_n)} \|\sum_{i=1}^{j} X_i I[\|X_i\| < a_n(A-\beta)\varepsilon]\| \ge a_n\varepsilon(A-\beta)]$$

$$\le (\varepsilon(A-\beta))^{-2} a_n^{-2} \sum_{i=1}^{[a_n(B+\beta_n)]} E\|X_i\|^2 I[\|X_i\| < a_n\varepsilon(A-\beta)].$$

Taking into account that

$$a_n^{-2} \sum_{i=1}^{[a_n(B+\beta_n)]} E\|X_i\|^2 I[\|X_i\| < a_n\varepsilon(A-\beta)] \;\rightarrow\; 0 \text{ as } n \;\rightarrow\; \infty$$

(cf. the proof of Theorem 1), (3.8) and assumption (a) we have

$$P[\|\sum_{i=1}^{T_n} X_i\| \ge T_n\varepsilon] \;\rightarrow\; 0 \text{ as } n \;\rightarrow\; \infty. \qquad (3.12)$$

Therefore, (3.11) and (3.12) imply that

$$\|med(S_{T_n}/T_n)\| \;\rightarrow\; 0 \text{ as } n \rightarrow \infty,$$

and complete the proof of the Theorem 3.

Note that Theorem 3 generalizes the results presented by Adler [15].

The following corollary is an extension of Adler's result to independent non-identically distributed B-valued random elements.

COROLLARY 7. Let $\{X_n,\ n \ge 1\}$ be a sequence of independent, B-valued random elements and $\{T_n,\ n \ge 1\}$ be positive integer valued random variables. Suppose that $\{a_n,\ n \ge 1\}$ is a strictly increasing sequence of positive integers and $\{\beta_n,\ n \ge 1\}$ is a sequence of positive constants such that $a_n \;\rightarrow\; \infty$ as $n \;\rightarrow\; \infty$, $lim\ sup_{n\rightarrow\infty}\beta_n = \beta < 1$ and

$$\sum_{n=1}^{\infty} P[|T_n/a_n - 1| \ge \beta_n] < \infty.$$

If for some positive integer j and for any given $\varepsilon > 0$ the assumptions (a)-(c) are satisfied then

$$\sum_{k=1}^{\infty} P[\|S_{T_k}\| \ge T_k\varepsilon] < \infty$$

if

$$\|S_{[a_k(1+\beta_k)]}/[a_k(1+\beta_k)]\| \;\rightarrow\; 0 \text{ in probability as } k \;\rightarrow\; \infty.$$

The next corollary is an extension of one of the results given in Adler [15] to the case of i.i.d. B-valued random elements.

COROLLARY 8. Let $\{X_n, n \geq 1\}$ be a sequence of independent identically distributed B-valued random elements with $EX_1 = 0$ and $\{T_n, n \geq 1\}$ be a sequence of positive integer-valued random variables. Suppose that $\{a_n, n \geq 1\}$ is a strictly increasing sequence of positive integers and $\{\beta_n, n \geq 1\}$ is a sequence of positive constants such that $a_n \to \infty$ as $n \to \infty$, $\limsup_{n\to\infty} \beta_n = \beta < 1$ and

$$\sum_{n=1}^{\infty} P[|T_n/a_n - 1| \geq \beta_n] < \infty.$$

Suppose that for some r, $1 < r \leq 2$, $x^{-r}M(\psi(x)) \to \infty$ as $x \to \infty$, where $\psi(x) = card\{k : a_k \leq x\}$, $x > 0$, $\psi(0) = 0$, $M(x) = \sum_{k=1}^{[x]} a_k$, $x > 0$.

If $\sum_{k=1}^{\infty} a_k P[\|X_1\| \geq a_k\varepsilon] < \infty$ then

$$\sum_{k=1}^{\infty} P[\|S_{T_k}\| \geq T_k\varepsilon] < \infty.$$

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REFERENCES

1. HSU, P.L. and ROBBINS, H. Complete Convergence and the Law of Large Numbers. Proc. Nat. Acad. Sci. U.S.A. 33 (1947) 25- 31.
2. ERDÖS, P. On a Theorem of Hsu-Robbins.Ann. of Math. Statist. 20 (1949) 286-291.
3. KATZ, M. The Probability in the Tail of Distribution. Ann. Math. Statist. 34 (1963) 312-318.
4. BAUM, L.E. and KATZ, M. Convergence Rates in the Law of Large Numbers. Trans. Amer. Math. Soc., 120 (1965) 108-123.
5. ASMUSSEN, S. and KURTZ, T. Necessary and Sufficient Conditions for Complete Convergence in the Law of Large Numbers.Ann. Probab., 8 (1980) 176-182.
6. GUT, A. On Complete Convergence in the law of Large Numbers for Subsequences. Ann. Probab. 13 (1985) 1286-1291.
7. DUNCAN, R. and SZYNAL, D. A Note on the Weak and Hsu-Robbins Law of Large Numbers. Bull. Ac. Pol. Math., 32 (1984) 729-735.
8. BARTOSZYNSKI, R. and PURI, P.S. On the Rate of Convergence for the Weak Law of Large Numbers. Probab. Math. Statist., 5 (1985) 91-97.
9. KUCZMASZEWSKA, A. and SZYNAL, D. On the Law of Large Numbers of the Hsu-Robbins Type. Probab. Math. Statist. 9 (1988) 85-93.
10. KUCZMASZEWSKA, A. and SZYNAL, D. On the Hsu-Robbins Law of Large Numbers for Subsequences. Bull. Ac. Pol. Math. 36 (1988) 69-79.
11. HOFFMANN-JÖRGENSEN, J. Sums of Independent Banach Spaces Valued Random Variables. Studia Math. 50 (1974) 159-186.
12. YURINSKI, V.V. Exponental Bounds for Large Deviations. Theor. Prob. Appl. 19 (1974) 154-155.
13. LOÈVE, M. Probability Theory. 4rd ed. Vol 1 Van Nostrand Princeton, (1977).
14. TAYLOR, R. Stochastic Convergence of Weighted Sums of Random Elements in Linear Space. Lecture Notes in Mathematics. 672 Springer- Verlag, Berlin, (1978).
15. ADLER, A. On Complete Convergence of the Sum of a Random Number of Random Variables. Calcutta Statist. Assoc. Bull., 37 (1988) 161-169.