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REMARKS ON THE EXISTENCE AND DECAY OF THE NONLINEAR BEAM EQUATION

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Abstract

We will consider a class of nonlinear beam equation and we will prove the existence and decay weak solution

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1 Introduction

In this paper we will consider the abstract problem associated with the nonlinear beam equation.

$$Ku_{tt} + A^2u + M(||A^{\frac{1}{2}}u||^2)Au + u_t = 0 \tag{1.1}$$

$$u(0) = u_0, Ku_t(0) = Ku_1$$

Where A is a selfadjoint positive definide operator in a Hilbert space $\mathcal{H}$ with domain $D(A)$ dense in $\mathcal{H}$ and the embedding of $D(A^r)$ into $D(A^s)$ is compact for $r > s \geq 0$. We will denote by $(\cdot,\cdot)$ and $||\cdot||$ the inner product and the norm of $\mathcal{H}$ respectively. In Pereira [3] the author proves that there exists a weak solution u for equation (1.1) satisfying

$$u \in L^\infty([0,T], D(A)) \quad Ku_t \in L^\infty([0,T],\mathcal{H}) \cap L^2([0,T],\mathcal{H})$$

in the sence

$$\frac{d}{dt}(Ku_t, w) + (Au, Aw) + M(||A^{\frac{1}{2}}u||^2) + (u_t, w) = 0 \text{ in } D'(I\!R^+)$$

when the following hypothese holds

- (i) $M(\xi) \geq -\beta$, $\forall \xi \geq 0$, and $0 < \beta < \lambda_1$; λ_1 is the first eigenvalue of A
- (ii) K is a simetric bounded operator in $\mathcal{H}$ such that $(Kw, w) \geq 0 \ \forall w \in \mathcal{H}$,

and in order to obtain the exponential decay, the author considers the aditional assumption on K

- (iii) $(Kw, w) \geq c||w||^2 \ \forall w \in \mathcal{H}$,

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where $c > 0$. In this paper we will prove that the lower bound $-\beta$ in item (i) does not depend on the expectral properties of the operator A. That is the exist a solution u for equation (1.1) when

$$M(\xi) \geq -\beta, \quad for\ any\ fixed\ \beta \in I\!R \tag{1.2}$$

and hypotheses (ii) holds, moreover, this solution satisfies

$$u \in C([0,+\infty[, D(A)); \ K u_t \in C([0,+\infty[,\mathcal{H}); \ u_t \in L^2_{loc}([0,+\infty[,\mathcal{H}) \tag{1.3}$$

Finally we prove that the energy associated to system (1.1) has exponential decay when (i) and (ii) holds, that is, hypotheses (iii) is not necessary.

2 Existence results

In order to obtain the existence result we will use the Galerkin method and to show the exponentially decay of the energy, we will reasoning as in Zuazua [4]. Let us denote by w_ν and λ_ν the sequence of eigenvector and eigenvalues of A and by V_m the space generated by the first m eigenvector of A. Then by Carateodory's theorem there exist a maximal local solution $w_{m,\epsilon}$ defined in $[0,T_{m,\epsilon}]$ satisfying for any w in V_m the following $m\epsilon$-system:

$$((K+\epsilon I)u_{tt}^{m\epsilon}, w) + \left(A^2 u^{m\epsilon}, w\right) + M(||A^{\frac{1}{2}}u^{m\epsilon}||^2)(Au^{m\epsilon}, w) + (u_t^{m\epsilon}, w) = 0$$

$$u_{m\epsilon}(0) = u_{0m} = P_m u_0; \quad u_t^{m\epsilon}(0) = u_{1m} = P_m u_1$$

where

$$P_m v = \sum_{i=1}^{m}(v, w_i)w_i \quad and \quad u_{m\epsilon} = \sum_{i=1}^{m} g_i^{m\epsilon}(t) w_i$$

Since u_0 and u_1 are functions of $D(A)$ and $\mathcal{H}$, the corresponding sequences u_{0m}, u_{1m} converge strongly in $D(A)$ and $\mathcal{H}$ respectively.

Theorem 2.1 *Let us suppose that (ii) and (1.2) holds, then for any u_0 in $D(A)$ and u_1 in $\mathcal{H}$, there exists a weak solution for system (1.1) satisfying (1.3).*

Proof.- Taking $w = u_t^{m\epsilon}$ in the aproximates $m\epsilon$-system we have

$$\frac{d}{dt}\left\{||(K+\epsilon I)^{\frac{1}{2}}u_t^{m\epsilon}||^2 + \hat{M}(||A^{\frac{1}{2}}u^{m\epsilon}||^2) + ||Au^{m\epsilon}||^2\right\} + 2||u_t^{m\epsilon}||^2 = 0 \tag{2.1}$$

Integration from 0 to $t < T_{m\epsilon}$ using (1.2) and identity $\frac{d}{dt}||A^{\frac{1}{2}}u^{m\epsilon}||^2 = 2(Au^{m\epsilon}, u_t^{m\epsilon})$ we have

$$||(K+\epsilon I)^{\frac{1}{2}}u_t^{m\epsilon}||^2 + ||Au_t^{m\epsilon}||^2 + 2\int_0^t ||u_t^{m\epsilon}||^2\,d\tau \leq c_m + \beta||A^{\frac{1}{2}}u_{0m}||^2 + \beta^2\int_0^t ||Au^{m\epsilon}(\tau)||^2\,d\tau$$

By Gronwall's inequality we conclude that

$$(u^{m\epsilon}, u_t^{m\epsilon}) \ \ are\ \ bounded\ \ in\ \ L^\infty([0,T], D(A)) \times L^2_{loc}([0,+\infty[,\mathcal{H}) \ \ \forall\epsilon > 0\ and\ \forall T > 0.$$

Then so is

$$||P_m\left\{|(K+\epsilon I)^{\frac{1}{2}}u_{tt}^{m\epsilon}\right\}||^2 + ||Au_t^{m\epsilon}||^2$$

in $L^2(0,T;V_m)$, (by a constant which we will denote in the same way) and a function u^m satisfying

$$A^{\frac{1}{2}}u^{m\epsilon} \rightarrow A^{\frac{1}{2}}u^m \ \ strongly\ in\ C(0,T;V_m)$$

$$P_m\{(K+\epsilon I)u_t^{m\epsilon}\} \rightarrow P_m\{Ku_t^m\} \ \ strongly\ in\ C(0,T;V_m)$$

$$u_t^{m\epsilon} \rightarrow u_t^m \ weak\ in\ L^2(0,T;V_m)$$

Moreover u^m satisfies the following m-approximated system

$$(Ku_{tt}^m, w) + \left(A^2u^m(t), w\right) + M\left(||A^{\frac{1}{2}}u^m(t)||^2\right)(Au^m(t), w) + (u_t^m(t), w) = 0$$

$$u^m(0) = u_{0m}; \quad P_m\{Ku_t^m(0)\} = P_m\{Ku_{0m}\}$$

Taking $w = u_t^m$ in the above equation we have

$$\frac{d}{dt}\left\{||K^{\frac{1}{2}}u_t^m||^2 + ||Au^m||^2 + \hat{M}||A^{\frac{1}{2}}u^m||^2\right\} + ||u_t^m||^2 = 0 \tag{2.2}$$

and using the same above reasoning we conclude that

$$u_t^m \;\; is\ bounded\ in \;\; L^\infty([0,T], \mathcal{H}) \tag{2.3}$$

$$u^m \;\; is\ bounded\ in \;\; L^\infty([0,T], D(A)) \tag{2.4}$$

By Lions-Aubin theorem, there exists a subsequence (which we still denoting on the same way) and a function u satisfying

$$u^m \to u \;\; strongly\ in \;\; L^\infty([0,T], D(A^{\frac{1}{2}}))$$

moreover we can obtain other subsequence for which we have

$$||A^{\frac{1}{2}}u^m(t)||^2 \to ||A^{\frac{1}{2}}u(t)||^2 \;\; a.\ e.\ in \;\; [0,T]$$

From Lebesgues's dominated convergence Theorem follows that $M(||A^{\frac{1}{2}}u^m(t)||^2)$ defines a Cauchy's sequence in $L^2(0,T)$, then for any $\epsilon > 0$ there exists a positive number N such that for m, $\mu \geq N$ we have

$$\int_0^T |M(||A^{\frac{1}{2}}u^m(\sigma)||^2) - M(||A^{\frac{1}{2}}u_\mu(\sigma)||^2)|\, d\sigma \leq \epsilon \tag{2.5}$$

Putting $U = u^m - u_\mu$, with $m > \mu$ and $g_i^\mu = 0$ for $\mu < i \leq m$, follows that

$$(KU_{tt}(t), w) + \left(A^2U(t), w\right) + (U_t(t), w) =$$

$$\left\{M(||A^{\frac{1}{2}}u^m||^2) - M(||A^{\frac{1}{2}}u_\mu||^2)\right\}(Au^m(t), w) + M(||A^{\frac{1}{2}}u_\mu||^2)(AU(t), w)$$

Taking $w = U_t$ and applying (2.3) and (2.4) we have

$$\frac{d}{dt}\left\{||K^{\frac{1}{2}}U_t||^2 + ||AU||^2\right\} + ||U_t||^2 \leq$$

$$C\left\{||A^{\frac{1}{2}}u^m(t)||^2) - M(||A^{\frac{1}{2}}u_\mu(t)||^2)\right\}^2 + C||AU(t)||^2$$

Integrating the last expression from 0, to t by (2.5) and Gronwall's inequality we conclude that u^m, u_t^m and $K^{\frac{1}{2}}u_t^m$ are Cauchy's sequences. Then we have that

$$u^m \to u \;\; strongly\ in \;\; C(0,T; D(A))$$

$$K^{\frac{1}{2}}u_t^m \to K^{\frac{1}{2}}u_t \;\; strongly\ in \;\; C(0,T; \mathcal{H})$$

$$u_t^m \to u_t \;\; strongly\ in \;\; L^2(0,T; \mathcal{H})$$

For any $T > 0$. By standard methods we can proof that u is a weak solution of system (1.1) (see [1]) and the proof is now complete.

Q.E.D.

Theorem 2.2 *With the hypothesis (i) and (ii) we have*

$$||K^{\frac{1}{2}}u_t(t)||^2 + ||Au(t)||^2 \leq Ce^{-\gamma t}$$

Proof.- Denoting by

$$E_m(t) = ||K^{\frac{1}{2}}u_t^m||^2 + \hat{M}(||A^{\frac{1}{2}}u^m||^2) + ||Au^m||^2.$$

From (2.2) we have

$$\frac{d}{dt}E_m(t) = -2||u_t^m||^2 \tag{2.6}$$

Taking $w = u^m$ in the aproximated $m-$system we conclude that

$$\frac{d}{dt}(Ku_t^m, u^m) = -||Au^m||^2 - M(||A^{\frac{1}{2}}u^m||^2)||A^{\frac{1}{2}}||^2 - (u_t^m, u^m) + (Ku_t^m u^m)$$

By hypothesis (i) there exists $\delta > 0$ ($\delta < 1$) such that $\frac{\beta}{\lambda_1} - 1 < -\delta$. Using $(u, u_t) \leq \delta||Au||^{\frac{1}{2}} + c(\delta)||u_t||^2$ and taking $c_0 = (1 - \delta - \frac{\beta}{\lambda_1})$ we obtain from (2.6) and the last identity that

$$\frac{d}{dt}\{E_m(t) + \epsilon(Ku_t^m, u^m)\} \leq -\epsilon c_0\left\{||u_t^m||^2 + ||Au^m||^2\right\} \tag{2.7}$$

for ϵ satisfying $2 - \{||K|| + C(\delta)\}\epsilon \geq \epsilon c_0$, where by $||K||$ we are denoting the norm of the operator K. On the other hand we have that

$$E_m(t) + \epsilon(Ku_t^m, u^m) \leq C\left\{||u_t^m||^2 + ||Au^m||^2\right\}. \tag{2.8}$$

Multiplying inequalities (2.7) and (2.8) by C and $c_0\epsilon$ respectively and adding the inequalities result we have

$$c_0\epsilon\frac{d}{dt}\{E_m(t) + \epsilon(Ku_t^m, u^m)\} + C\{E_m(t) + \epsilon(Ku_t^m, u^m)\} \leq 0.$$

Multiplying by $e^{\gamma t}$ for $\gamma = \frac{C}{c_0\epsilon}$ and integrating from 0 to t we get

$$E_m(t) + \epsilon(Ku_t^m, u^m) \leq E_m(0) + \epsilon(Ku_{0m}, u_{0m}).$$

Since

$$\{E_m(t) + \epsilon(Ku_t^m, u^m)\} \geq \delta\left\{||K^{\frac{1}{2}}u_t^m||^2 + ||Au_m||^2\right\}$$

for ϵ such that $1 - c(\delta)\epsilon > \delta$, then we have

$$||K^{\frac{1}{2}}u_t^m(t)||^2 + ||Au_m(t)||^2 \leq \frac{1}{\delta}\{E_m(0) + \epsilon(Ku_{1m}, u_{0m})\}e^{-\gamma t}$$

Finally from the uniformly convergence of u^m the result follows.

Q.E.D.

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