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HANKEL TRANSFORMS IN GENERALIZED FOCK SPACES

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ABSTRACT. A classical Fock space consists of functions of the form,

$$\phi \leftrightarrow (\phi_0, \phi_1, \ldots, \phi_q),$$

where $\phi_0 \in \mathbf{C}$ and $\phi_q \in L^p(\mathbf{R}^q)$, $q \geq 1$. We will replace the ϕ_q, $q \geq 1$ with test functions having Hankel transforms. This space is a natural generalization of a classical Fock space as seen by expanding functionals having abstract Taylor Series. The particular coefficients of such series are multilinear functionals having distributions as their domain. Convergence requirements set forth are somewhat in the spirit of ultra differentiable functions and ultra distribution theory. The Hankel transform oftentimes implemented in Cauchy problems will be introduced into this setting. A theorem will be proven relating the convergence of the transform to the inductive limit parameter, s, which sweeps out a scale of generalized Fock spaces.

KEY WORDS AND PHRASES. Generalized Fock Spaces, ultra distributions, Hankel transforms, Abelian theorems.
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1. INTRODUCTION.

The test space, $\mathcal{H}_\mu$, $\mu \in (-\infty,\infty)$ consisting of continuous complex-valued function defined on the q-dimensional orthant, $E_q = \{t \in \mathbf{R}^q: 0 < t_\gamma < \infty, (1 \leq \gamma \leq q)\}$ and its dual space, $\mathcal{H}'_\mu$, are excellent candidates for examining the Hankel transform (Brychkov and Prudnikov [1], Koh [2], Pathak and Singh [3] and Zemanian [4]). The Hankel transform in this setting investigates spaces having test functions, $\phi \in \mathcal{H}_\mu$, defined on a finite number of independent variables. By this we mean the independent variables of a test function, $\phi(t_1,\ldots,t_q) \in \mathcal{H}_\mu$ has finitely many independent variables, $(t_1,\ldots,t_q)$, belonging to E_q. Our present development will indicate a process whereby the independent variables, t_γ, $1 \leq \gamma \leq q$, can become infinite in the sense that the dimension, $q \to \infty$.

The Hankel transform in classical analysis is oftentimes implemented to study abstract Cauchy problems involving the Bessel differential operator (Pathak [5]). Our main effort, however, will be to extend the transform to our spaces defined as generalized Fock spaces.

The need for this is essential in modern physics. A system whereby the number of particles are theoretically described to become infinite can be modeled by a state vector belonging to a direct sum of Hilbert spaces. The basic Hilbert space, $\mathcal{H}_1^{\otimes n}$, usually selected is the space of Lebesgue p-integrable functions, $L^p(\mathbf{R}^q)$ and the state vector, Φ, belongs to a direct sum of these Hilbert spaces, $\mathcal{H} \triangleq \bigoplus_{n=0}^{\infty} \mathcal{H}_1^{\otimes n}$. This direct sum is generally called a Fock space. A complete development in this setting can be found in reference (Bogolubov et al [6]). A state vector, Φ, belonging to this Fock space is described by an arbitrary sequence, $\Phi \triangleq \{\Phi_q\}_{q=0}^{\infty}$, satisfying the condition, $\|\Phi\|^2 \triangleq \sum_{q=0}^{\infty} \|\Phi_q\|^2 < \infty$. The Fock space is equipped with the natural scalar product given by the formula,

$$(\Phi,\Psi) \triangleq \sum_{q=0}^{\infty} (\Phi_q,\Psi_q),$$

where each (Φ_q,Ψ_q), $q \geq 0$ is the inner product given with the Hilbert space, $\mathcal{H}_1^{\otimes q}$. A principal problem with this development together with the test space, $\mathcal{H}_\mu$, is that the kernel of the Hankel transform is not a member of the test space, $\mathcal{H}_\mu$, and the Dirac delta is not a member of the space, $L^p(\mathbb{R}^q)$, (Zemanian [4]). These problems are overcome when one defines the distributional Hankel transform, H_μ, $\mu \geq -\frac{1}{2}$. These will be briefly reviewed in section 3. However the number of independent variables belonging to the q-dimensional orthant still remains to be finite.

Our present development will implement the procedures developed in Schmeelk [7] together with a general setting developed in Schmeelk and Takači [8]. With these settings in place, we will then extend the Hankel transform into inductive and projective limit spaces (Zarinov [9]). These spaces will enjoy all of the classical Hankel transform results together with an approach to solve the infinite number of independent variables problem.

We will conclude our paper with a generalization of the Hankel transform for the Dirac delta functional, $\delta^{(k)}(m^2+P)$, into our setting. The transform for $\delta^{(k)}(m^2+P)$ is developed in Aguirre and Trione [10] and is based on the notion of distributions applied to surfaces (Gelfand and Shilov [11]). The extension of a particular case of $\delta^{(k)}(m^2+P)$ will then enjoy the infinite number of independent variable setting.

2. SOME NOTIONS AND NOTATIONS.

We begin with recalling some fundamental conditions placed on our sequences of positive constants and sequences of functions. The prerequisites on the sequences lead us in a natural way into the approach in [12] and then into our generalized Fock spaces.

Throughout the paper we suppose that a monotonically increasing sequence of positive real numbers, $r = (m_q)_{q\in\mathbb{N}_0}$, is given. We assume that conditions (M_1)–(M_3) from [12] are satisfied:

(M_1) $m_q^2 \leq m_{q-1}m_{q+1}$, $q = 1,\ldots$;

(M_2) $\exists\, A, H \ni m_{q+1} \leq A \cdot H^q \cdot m_q$, $q = 0,\ldots,$;

(M_3) $\sum_{q=1}^{\infty} \frac{m_{q-1}}{m_q} \neq \infty$.

It is convenient to take $m_0 = 1$. One easily checks that, for instance, the sequence $m_q = q!^a$, $a > 1$, satisfy the three conditions, (M_1)–(M_3).

We next suppose that a sequence, $\mathcal{M}_0 = \left(M_p(\cdot)\right)_{p\in\mathbb{N}_0}$, of continuous functions on $\mathbb{R}^q$ is given. We require the usual conditions, (P),(M) and (N) hold as in reference [13] as well as the inequalities,

$$M_0(t) \leq M_1(t) \leq \ldots, \quad t \in \mathbb{R}^q.$$

Then, an infinitely differentiable function, $\phi(t)$, on $\mathbb{R}^n$ is in the space, $\mathcal{K}(M_p)$, if for every $p \in \mathbb{N}_0$ the following norms are finite,

$$\|\phi\|_p = \sup\{M_p(t)|D^j\phi| : t \in \mathbb{R}^q, j_\gamma \leq p, 1 \leq \gamma \leq q\}, \tag{2.1}$$

where

$$D^j\Phi \triangleq \frac{\partial^{j_1+\ldots+j_q}}{\partial t_1^{j_1}\ldots\partial t_q^{j_q}}\phi(t_1,\ldots,t_q).$$

The family of norms, $(\|\cdot\|_p)_{p\in\mathbb{N}_0}$, defines a locally convex topology on $\mathcal{K}(M_p)$which in view of condition (P) turns this space into a Frèchet space. It also has several other mathematical properties. For a detailed account of spaces of type $\mathcal{K}(M_p)$see references [11, 14].

Let us denote by $\|\cdot\|_{-p}$ following norm on $\mathcal{K}'(M_p)$,

$$\|x\|_{-p} = \sup\{|\langle x,\phi\rangle| : \|\phi\|_p \le 1|\}. \tag{2.2}$$

Observe that the sequence of norms,$\{\|\cdot\|_{-p}\}_{p\in\mathbb{N}_0}$, satisfies $\|x\|_0 \ge \|x\|_{-1} \ge \dots$, for any $x \in \mathcal{K}'(M_p)$. The sequence of positive numbers, $r = (m_q)_{q\in\mathbb{N}_0}$, and the sequence of continuous functions, $\mathcal{M}_0 = (M_p(\cdot))_{p\in\mathbb{N}_0}$, will play an essential role in the definition of the generalized Fock space, $\Gamma^{r,\mathcal{M}_0}$, in Section 4. Throughout the paper the notation, N, will indicate the natural numbers and N_0 indicates the natural numbers and zero.

3. THE SPACES,$\mathcal{H}_\mu$ AND $\mathcal{H}'_\mu$.

We briefly recall the definition of the space, $\mathcal{H}_\mu$. For brevity let I_q denote the set of q-tuples, $i = (i_1, \dots, i_q)$ of nonnegative integers, i_γ, $1 \le \gamma \le q$. A continuous complex valued function, $\phi(t)$, defined on E_q will belong to the space, $\mathcal{H}_\mu$, if for each pair of q-tuples, $p = (p,\dots,p)$ and $k \in I_q$, then the condition,

$$\|\phi(t)\|_p^\mu \triangleq \sup\left[[t]^p \left|S_\mu^k(\phi(t))\right| : t \in E_q, k_\gamma \le p,\ 1 \le \gamma \le q\right] < \infty,$$

is satisfied. Herein the notation denotes,

$$[t]^p = t_1^p \cdot \dots \cdot t_q^p, \qquad p \in N_0,$$

and

$$S_\mu^k(\phi(t)) = \prod_{\gamma=1}^q \left(t_\gamma^{-1}\frac{\partial}{\partial t_\gamma}\right)^{k_\gamma}(t_\gamma)^{-\mu-\frac{1}{2}}\phi(t_1,\dots,t_q).$$

The Hankel transform, H_μ, $\mu \ge -\frac{1}{2}$ is then defined on the space, $\mathcal{H}_\mu$, as

$$(H_\mu\phi)(y_1,\dots,y_q) \triangleq \int_0^\infty \cdots \int_0^\infty \phi(t_1,\dots,t_q)\prod_{\gamma=1}^q (t_\gamma y_\gamma)^{\frac{1}{2}} J_\mu(t_\gamma y_\gamma)dt_1,\dots,dt_q.$$

The function, $J_\mu(t_\gamma y_\gamma)$, $(1 \le \gamma \le q)$, $\mu \ge -\frac{1}{2}$, is the Bessel function of the first kind given by the formula,

$$J_\mu(w) = \sum_{n=\pm}^{\infty} \frac{(-1)^n}{n!\Gamma(n+\mu+1)}\left(\frac{w}{2}\right)^{2n+\mu} \tag{3.1}$$

Several properties regarding this definition of the Hankel transform on functions defined on $\mathbf{R}'$ can be found in reference [4] and the $\mathbf{R}^q$, $q \ge 2$, case in reference [2].

For $\mu \ge -\frac{1}{2}$ the generalized Hankel transform, H'_μ defined on distributions, $F \in \mathcal{H}'_\mu$, is taken to be the adjoint of the Hankel transform, H_μ given by the equation,

$$\langle H'_\mu F,\phi\rangle \triangleq \langle F,H_\mu\phi\rangle, \tag{3.2}$$

for every $\phi \in \mathcal{H}_\mu$ and $F \in \mathcal{H}'_\mu$. A survey of the many properties for this definition of the generalized Hankel transform can be found in references [4, 5, 14, 1].

4. GENERALIZED FOCK SPACES, $\Gamma^{r,\mathcal{M}_0}$.

Let the sequence $r = (m_q)_{q\in\mathbb{N}_0}$ and $\mathcal{M}_0 = (M_p(\cdot))_{p\in\mathbb{N}_0}$ be given with the properties given in Section 2. We then define

$$a_q: \mathcal{K}'(M_p) \underbrace{\times \ldots \times}_{\text{q-copies}} \mathcal{K}'(M_p) \mapsto \mathbb{C} \tag{4.1}$$

to be a multilinear continuous functional, $q \in \mathbb{N}$, and by definition select $a_0 \in \mathbb{C}$. Then the formal sum

$$\Phi = \sum_{q=0}^{\infty} a_q[\underbrace{\cdot,\ldots,\cdot}_{\text{q-spaces}}], \tag{4.2}$$

is in the space, $\Gamma^{s,r,\mathcal{M}_0}$, $s > 1$, if the norm,

$$|||\Phi|||^{(p)}_{s,r,\mathcal{M}_0} = \sup\left\{\frac{\| a_q \|_p \cdot m_q}{s^q} : q \in \mathbb{N}_0\right\}, \tag{4.3}$$

is finite for every $p \in \mathbb{N}_0$. Here

$$\| a_q \|_p = \sup\left\{|a_q[x,\ldots,x]| : \| x \|_{-p} \le 1,\ x \in \mathcal{K}'(M_p)\right\}. \tag{4.4}$$

Recall the definition of $\| x \|_{-p}$ as given in expression (2.2).

REMARK. Physicists prefer to represent the elements from our generalized Fock space, $\Gamma^{s,r,\mathcal{M}_0}$, as column vectors, for instance,

$$\Phi \Leftrightarrow \begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_q \\ \vdots \end{bmatrix}$$

Since this is convenient also when working with the Hankel transform, we shall do likewise.

Let us first observe that in view of (4.3) and (4.4), the canonical inclusion

$$\Gamma^{s,r,\mathcal{M}_0} \mapsto \Gamma^{s',r,\mathcal{M}_0}, \tag{4.5}$$

is continuous provided that $s' > s > 1$. So in view of reference [9], we can now give the following definition.

DEFINITION 4.1. A generalized Fock space, $\Gamma^{r,\mathcal{M}_0}$, is the inductive limit of the spaces, $\Gamma^{s,r,\mathcal{M}_0}$, i.e.,

$$\Gamma^{r,\mathcal{M}_0} = \operatorname*{ind}_{s\to\infty} \Gamma^{s,r,\mathcal{M}_0}.$$

For the development of the inductive limit one can consult reference [9]. In quantum theory the state vectors, as already indicated, satisfy constraints of the form,

$$(\Phi,\Phi) \triangleq |k_0| + \sum_{q=1}^{\infty} \int_{\mathbb{R}^q} |k_q(t_1,\ldots,t_q)|^2\, dt_1,\ldots,dt_q < \infty.$$

In keeping with the spirit of such a constraint, we shall indicate that the elements from the inductive limit, $\Gamma^{r,\mathcal{M}_0}$, are L^r summable for any $r \in (1,\infty)$. For this result we cite a well known lemma.

LEMMA 4.1. Conditions (M_1) and (M_3) on the sequence, $r = \{m_q\}_{q \in N_0}$, imply that for any real number, t, we have

$$\sum_{q=0}^{\infty} \frac{|t|^q}{m_q} < \infty. \tag{4.6}$$

PROOF. See reference [8].

The state vectors, $\Phi \in \Gamma^{r,\mathcal{M}_0}$, can also enjoy an alternate representation called its kernel representation. However for this to be true we must require that each member, $M_p(\cdot) \in \mathcal{M}_0$, decrease sufficiently fast as infinity so that our test space, $\mathcal{K}(M_p)$, for example contain the rapid descent test functions [7]. Assuming this to be true, we briefly review the kernel construction. Since each a_q, $q \geq 1$ is a multilinear functional on $\underbrace{\mathcal{K}'(M_p) \times \ldots \times \mathcal{K}'(M_p)}_{q\text{-copies}}$ we can define

$$\phi_q(t_1^0,\ldots,t_q^0) \triangleq a_q\,[\delta_{t_1^0},\ldots,\delta_{t_q^0}] \tag{4.7}$$

where each $\delta_{t_\gamma^0}$ $(1 \leq \gamma \leq q)$ is the translate of the Dirac delta distribution. Recall this translate satisfies

$$\langle \delta_{t_\gamma^0}, \psi(t) \rangle = \psi(t_\gamma^0) \tag{4.8}$$

for every test function, $\psi(t) \in \mathcal{K}(M_p)$. As was shown in reference [8], each $\phi_q(t_1^0,\ldots,t_q^0)$ defined in expression (4.7) is a rapid descent test function. Thus for each $\Phi \in \left(\Gamma^{r,\mathcal{M}_0}\right)$, we have an alternate representation,

$$\Phi \Leftrightarrow \begin{bmatrix} \phi_0 \\ \phi_1 \\ \vdots \\ \phi_q \\ \vdots \end{bmatrix}, \tag{4.9}$$

where $a_0 \triangleq \phi_0$ is a scalar and ϕ_q, $q \geq 1$ are each defined in expression (4.7). We use this alternate representation given in expression (4.9) when addressing the dual space of $\Gamma^{r,\mathcal{M}_0}$.

5. GENERALIZED DUAL FOCK SPACE $\left(\Gamma^{s,r,\mathcal{M}_0}\right)'$.

We now examine the dual of the inductive limit space, $\left(\Gamma^{r,\mathcal{M}_0}\right)'$, by first analyzing the dual to each space, $\Gamma^{s,r,\mathcal{M}_0}$. The dual is presented in the spirit of reference [7]. In our present environment a member, F, belonging to the dual, $\left(\Gamma^{s,r,\mathcal{M}_0}\right)'$, will also enjoy a sequence representation,

$$F \leftrightarrow (F_0, F_1, \ldots, F_q, \ldots)$$

where F_0 is a scalar and F_q, $q \geq 1$ are tempered distributions of order $\leq$ m. Moreover, F satisfies the constraint,

$$|||F|||^{(p)}_{-[s,r,\mathcal{M}_0]} = \sup\left\{ \frac{|\langle\langle F,\Phi\rangle\rangle|}{|||\Phi|||^{(p)}_{s,r,\mathcal{M}_0}} \;\middle|\; |||\Phi|||^{(p)}_{s,r,\mathcal{M}_0} \neq 0 \right\}$$

$$= \sum_{q=0}^{\infty} (s)^q (m_q^{-1})\, \|F_q\|_{L_p} < \infty. \tag{5.1}$$

The value, $\langle\langle F,\Phi\rangle\rangle$, is computed as

$$\langle\langle F,\Phi\rangle\rangle = \sum_{q=0}^{\infty} \langle F_q, \phi_q\rangle \tag{5.2}$$

where F_0 and ϕ_0 are scalars and F_q, ϕ_q $q \geq 1$ are the already respectively defined tempered distributions and rapid descent test functions. We can now equip, $\left(\Gamma^{r,\mathcal{M}_0}\right)'$ with a projective limit. Again projective limits are extensively developed in reference [9].

6. THE HANKEL TRANSFORM IN $\left(\Gamma^{r,\mathcal{M}_0}\right)'$.

We first examine the Hankel transform in each space, $\Gamma^{s,r,\mathcal{M}_0}$. The space $\mathcal{H}_\mu$, already defined in section 3 consists of functions defined on the q-dimensional orthant, E_q, and each member satisfies

$$\|\phi(t)\|_p^\mu = \sup\left\{\left|[t]^p \prod_{\gamma=1}^{q}\left(t_\gamma^{-1}\frac{\partial}{\partial t_\gamma}\right)^{k_\gamma}(t_\gamma)^{-\mu-\frac{1}{2}}\phi(t_1,\ldots,t_q)\right| : t\epsilon E_q, k_\gamma \leq p,\ 1 \leq \gamma \leq q\}\right\} < \infty$$

for every $p \in N_0$ In this section we identify a $\Phi \in \Gamma^{s,r,\mathcal{M}_0}$ if and only if it has the representation,

$$\Phi \Leftrightarrow \begin{bmatrix} \phi_0 \\ \phi_1 \\ \vdots \\ \phi_q \\ \vdots \end{bmatrix}, \tag{6.1}$$

where ϕ_0 is a scalar and $\phi_q \in \mathcal{H}_\mu\ (\mathbb{R}^q)$, $q \geq 1$, $\mu \geq -\frac{1}{2}$. Moreover each Φ must satisfy the condition,

$$|||\Phi|||^{(p)}_{s,r,\mathcal{M}_0} = \sup\left\{\frac{\|\phi_q\|_q^\mu \cdot m_q}{s^q} : q \in \mathbb{N}_0\right\} < \infty, \tag{6.2}$$

for every $p \in \mathbb{N}_0$.

LEMMA 6.1. If $\Phi \in \Gamma^{s,r,\mathcal{M}_0}$, then Φ enjoys the sum integrable property

$$|||\Phi||| = |\phi_0| + \sum_{q=1}^{\infty} \int_{E_q} |\phi_q(t_1,\ldots,t_q)|\ dt_1,\ldots,dt_q < \infty. \tag{6.3}$$

PROOF. We decompose the q-dimensional orthant, E_q, into its q-dimensional unit sphere, $S_q = \{0< \sqrt{t_1^2+\ldots+t_q^2} \leq 1\}$ and $CS_q = \{1 \leq \sqrt{t_1^2+\ldots+t_q^2}<\infty\}$. Thus we have $E_q = S_q \cup CS_q$. First since $\phi_q(t_1,\ldots,t_q) \in \mathcal{H}_\mu(\mathbb{R}^q)$ we have that

$$\begin{aligned}
&\int_{S_q}\cdots\int |\phi_q(t_1,\ldots,t_q)|\, dt_1,\ldots,dt_q \\
&= \int_{S_q}\cdots\int \left[t_1^{\mu+1/2}\cdot\ldots\cdot t_q^{\mu+1/2}\right]\left[t_1^{-\mu-1/2}\cdot\ldots\cdot t_q^{-\mu-1/2}\right]\phi(t_1,\ldots,t_q)dt_1,\ldots,dt_q \\
&\leq \|\phi\|_0^\mu \int_{S_q}\cdots\int t_1^{\mu+1/2}\cdot\ldots\cdot t_q^{\mu+1/2}dt_1,\ldots,dt_q < \infty.
\end{aligned} \tag{6.4}$$

Secondly $\phi(t_1,\ldots,t_q)$ is of rapid descent as $t\to\infty$ [4] so by a generalization of the proof in reference [17, pg. 434] the desired result easily follows.

DEFINITION 6.1. The Hankel transform $\mathcal{H}_\mu$, $\mu \geq -\frac{1}{2}$ is defined on each $\Gamma^{s,r,\mathcal{M}_0}$ as follows;

$$\mathcal{H}_\mu : \begin{bmatrix} \phi_0 \\ \phi_1 \\ \vdots \\ \phi_q \\ \vdots \end{bmatrix} \to \begin{bmatrix} \phi_0 \\ H_\mu(\phi_1) \\ \vdots \\ H_\mu(\phi_q) \\ \vdots \end{bmatrix} \tag{6.5}$$

where

$$H_\mu(\phi_q) \triangleq \int\limits_{E_q} \phi_q(t_1,\ldots,t_q) \prod_{\gamma=1}^{q} \left(t_\gamma y_\gamma\right)^{1/2} J_\mu\left(t_\gamma y_\gamma\right) dt_1 \ldots dt_q \tag{6.6}$$

for $q \geq 1$.

LEMMA 6.2. $\mathcal{H}_\mu$, $\mu \geq -\frac{1}{2}$ is well defined for every $\Gamma^{s,r,\mathcal{M}_0}$.

PROOF. We decompose the q-dimensional orthant, Eq, into the portion contained in the q-dimensional unit sphere, $S_q = \{0 < \sqrt{t_1^2+\ldots+t_q^2} \leq 1\}$ and its complement, $CS_q = \{1 < \sqrt{t_1^2+\ldots+t_q^2} < \infty\}$. We then have $E_q = S_q \cup CS_q$. We will estimate the integrals over S_q and CS_q. The estimate over S_q will use the formula (18, pg 75] for the volume of a unit sphere $\mathbb{R}^q$. It is given by the formula

$$S_q = \frac{2}{q}\frac{\pi^{q/2}}{\Gamma(\frac{q}{2})}, \tag{6.7}$$

where $\Gamma(\frac{q}{2})$ is the classical Gamma function. We select a $\Phi\epsilon\Gamma^{s,r,\mathcal{M}_0}$ and our norm requirement given in expression (6.2) for $p = 0$ implies

$$\sup_q \frac{\left|\prod_{\gamma=1}^{q}\left(t_\gamma\right)^{-\mu-1/2}\phi(t_1,\ldots,t_q)\right| \cdot m_q}{s^q} \leq C$$

thus

$$\sup_q \left|\prod_{\gamma=1}^{q}\left(t_\gamma\right)^{-\mu-1/2}\phi(t_1,\ldots,t_q)\right| \leq \frac{Cs^q}{m_q} \tag{6.8}$$

and likewise for $p = 2$ we have

$$\sup_q \left|\prod_{\gamma=1}^{q}(t_\gamma)^2\left(t_\gamma\right)^{-\mu-1/2}\phi(t_1,\ldots,t_q)\right| \leq \frac{Cs^q}{m_q}$$

Next we examine each component in $E_q = S_q \cup CS_q$. We see that

$$\begin{aligned} H_\mu(\phi_q) &= \int\limits_{E_q} \phi_q(t_1,\ldots,t_q)\left(\prod_{\gamma=1}^{q} t_\gamma y_\gamma\right)^{1/2} J_\mu\left(t_\gamma y_\gamma\right) dt_1\ldots dt_q \\ &= \int\limits_{S_q}\cdots\int \phi_q(t_1,\ldots,t_q)\left(\prod_{\gamma=1}^{q} t_\gamma y_\gamma\right)^{1/2} J_\mu\left(t_\gamma y_\gamma\right) dt_1\ldots dt_q \\ &+ \int\limits_{CS_q}\cdots\int \phi_q(t_1,\ldots,t_q)\left(\prod_{\gamma=1}^{q} t_\gamma y_\gamma\right)^{1/2} J_\mu\left(t_\gamma y_\gamma\right) dt_1\ldots dt_q. \end{aligned} \tag{6.9}$$

We also have for $\mu \geq -\frac{1}{2}$ that

$$\sqrt{t_\gamma y_\gamma}\, J_\mu(t_\gamma y_\gamma) = \mathcal{O}(t_\gamma^{\mu+1/2}) \tag{6.10}$$

as $t_\gamma \to 0^+$ and

$$\sqrt{t_\gamma y_\gamma}\, J_\mu(t_\gamma y_\gamma) = \mathcal{O}(1) \tag{6.11}$$

as $t_\gamma \to \infty$ for $1 \leq \gamma \leq q$. Employing these two observations into expression (6.9) gives us

$|H_\mu\,(\phi_q)|$

$$\leq \underset{S_q}{\int \cdots\cdot \int} K^q \prod_{\gamma=1}^{q} \left|t_\gamma^{\mu+1/2}\right| \left|\phi_q(t_1,\ldots,t_q)\right| dt_1\ldots dt_q \tag{6.12}$$

$$+ \underset{CS_q}{\int \cdots\cdot \int} (K')^q \left|\phi_q(t_1,\ldots,t_q)\right| dt_1\ldots dt_q$$

$$= K^q \underset{S_q}{\int \cdots\cdot \int} \prod_{\gamma=1}^{q} \left|t_\gamma^{\mu+1/2}\right| \frac{t_\gamma^{-\mu-1/2}}{2t_\gamma^{-\mu-1/2}} \left|\phi_q(t_1,\ldots,t_q)\right| dt_1\ldots dt_q \tag{6.13}$$

$$+ (K')^q \underset{CS_q}{\int \cdots\cdot \int} \prod_{\gamma=1}^{q} \frac{t_\gamma^2}{t_\gamma^2} |\phi_q(t_1,\ldots,t_q)|\, dt_1\ldots dt_q$$

$$= K^q \underset{S_q}{\int \cdots\cdot \int} \prod_{\gamma=1}^{q} \left|t_\gamma^{2\mu+1}\right| t_\gamma^{-\mu-1/2} \left|\left|\phi_q(t_1,\ldots,t_q)\right|\right| dt_1\ldots dt_q \tag{6.14}$$

$$+ (K')^q \underset{CS_q}{\int \cdots\cdot \int} \prod_{\gamma=1}^{q} \frac{1}{t_\gamma^2} |t_\gamma^2\, |\phi_q|(t_1,\ldots,t_q)|\, dt_1\ldots dt_q$$

$$\leq \frac{K^q C s^q}{m_q} \underset{S_q}{\int \cdots\cdot \int} \prod_{\gamma=1}^{q} \left|t_\gamma^{2\mu+1}\right| dt_1\ldots dt_q \tag{6.15}$$

$$+ \frac{(K')^q C' s^q}{m_q} \underset{CS_q}{\int \cdots\cdot \int} \prod_{\gamma=1}^{q} \frac{1}{t_\gamma^2}\, dt_1\ldots dt_q$$

$$\leq \frac{K^q C s^q}{m_q}\left[\frac{2}{q}\,\frac{\pi^{q/2}}{\Gamma(\frac{q}{2})}\right] + \frac{(K')^q C' s^q}{m_q} \tag{6.16}$$

$$= \frac{(s)^q}{m_q}\left[\frac{2K^q C}{q}\,\frac{\pi^{q/2}}{\Gamma(\frac{q}{2})} + (K')^q\, C'\right] \tag{6.17}$$

$$\leq \frac{(s)^q}{m_q}\left[2K^q C \pi^{q/2} + (K')^q\, C'\right] \tag{6.18}$$

$$\leq \frac{(K'')^q \cdot C'' s^q}{m_q}, \tag{6.19}$$

where $K'' \triangleq \max\{K\pi,\, K'\}$ and $C'' \triangleq \max\{C,\, C'\}$. We now consider the sum of the components for the vector,

$$\begin{bmatrix} \phi_0 \\ H_\mu(\phi_1) \\ \vdots \\ H_\mu(\phi_q) \\ \vdots \end{bmatrix}, \tag{6.20}$$

which gives us

$$|\phi_0| + \left| \sum_{q=1}^{\infty} \int_{E_q} \phi_q(t_1,\ldots,t_q) \left(\prod_{\gamma=1}^{q} |t_\gamma y_\gamma| \right)^{1/2} J_\mu\left(t_\gamma y_\alpha\right) dt_1 \ldots dt_\gamma \right| \tag{6.21}$$

$$\leq |\phi_0| + \sum_{q=1}^{\infty} \frac{(K'')^q C'' s^q}{m_q} = |\phi_0| + \sum_{q=1}^{\infty} C'' \frac{(K''s)^q}{m_q} < \infty \tag{6.22}$$

whenever we select $s' \triangleq K''s$ and the use of lemma 4.1.

THEOREM 6.1. The Hankel transform $\mathcal{H}_\mu$, $\mu \geq -\frac{1}{2}$, is a linear continuous transformation on $\Gamma^{r,\mathcal{M}_0}$.

PROOF. We consider the Hankel transform, $\mathcal{H}_\mu$, $\mu \geq -\frac{1}{2}$, restricted to anyone of the spaces, $\Gamma^{s,r,\mathcal{M}_0}$, $s \geq 1$, comprising the components of the inductive limit space, $\Gamma^{r,\mathcal{M}_0}$. Then as in reference [2] we have

$$|||\mathcal{H}_\mu(\Phi)|||^{p,\mu}_{s',r,\mathcal{M}_0} = \sup \left\{ \frac{\| H_\mu(\phi_q) \|^{\mu}_{p} \cdot m_q}{(s')^q} : q \in \mathbb{N}_0 \right\} \tag{6.23}$$

where

$$\| H_\mu(\phi) \|^{\mu}_{p} = \sup_{\substack{k_\gamma \leq p \\ 1 \leq \gamma \leq q}} \left| \prod_{\gamma=1}^{q} y^p_\gamma \left(y^{-1}_\gamma \frac{\partial}{\partial y_\gamma} \right)^{k_\gamma} (y_\gamma)^{-\mu-1/2} H_\mu(\phi_q) \right| = \tag{6.24}$$

$$\sup_{\substack{k_\gamma \leq p \\ \leq \gamma \leq q}} \prod_{\gamma=1}^{q} y^p_\gamma \left(y^{-1}_\gamma \frac{\partial}{\partial y_\gamma} \right)^{k_\gamma} (y_\gamma)^{-\mu-1/2}$$

$$\int_{E_q} \phi_q\left(t_1,\ldots,t_q\right) \left(\prod_{\gamma=1}^{q} t_\gamma y_\gamma \right)^{1/2} J_\mu\left(t_\gamma y_\gamma\right) dt_1 \ldots dt_q =$$

$$\sup_{\substack{k_\gamma \leq p \\ 1 \leq \gamma \leq q}} \int_{E_q} \phi_q\left(t_1,\ldots,t_q\right) \prod_{\gamma=1}^{q} t_\gamma^{1/2} (-1)^k \cdot \tag{6.25}$$

$$\prod_{\gamma=1}^{q} t_\gamma^{k_\gamma} y_\gamma^{-\mu-k_\gamma+p} J_{\mu+k_\gamma}\left(t_\gamma y_\gamma\right) dt_1 \ldots dt_q =$$

$$\sup_{\substack{k_\gamma \leq p \\ 1 \leq \gamma \leq q}} \int_{E_q} \phi_q(t_1,\ldots,t_q)(-1)^k \left(\prod_{\gamma=1}^{q} t_\gamma^{-\mu-1/2} \cdot \right. \tag{6.26}$$

$$\prod_{\gamma=1}^{q}\left(t_\gamma^{-1}\frac{\partial}{\partial t_\gamma}\right)^{p}\Big) y_\gamma^{-\mu-k_\gamma} t_\gamma^{\mu+k_\gamma+p} J_{\mu+k_\gamma+p}(x_\gamma y_\gamma)\, dt_1 \ldots dt_q =$$

$$\sup_{\substack{k_\gamma \le p\\ 1\le\gamma\le q}} (-1)^{k+p}\int_{E_q}\left(\prod_{\gamma=1}^{q} t_\gamma^{2\mu+2k_\gamma+p+1}\right)\cdot$$

$$(t^{-1}D_t)^p[t]^{-\mu-1/2}\phi(t)\prod_{\gamma=1}^{q}\left(t_\gamma y_\gamma\right)^{-\mu-k_\gamma}\cdot$$

$$J_{\mu+k_\gamma+p}\,(t_\gamma y_\gamma)\, dt_1 \ldots dt_q\,. \tag{6.27}$$

Equation (6.27) was obtained by integration by parts as in reference [2, pg 430-431] and so the limit terms vanished since $\phi(t)$ is rapid descent as $t \to \infty$ while

$$x_\gamma^{1/2}\, J_{\mu+1}(t_\gamma y_\gamma) = \mathcal{O}(t_\gamma) \tag{6.28}$$

and

$$\phi(t) = \mathcal{O}(1) \tag{6.29}$$

as $t_\gamma \to 0_+$, $1 \le \gamma \le q$. Also if p_γ is an integer no less than $\mu + p + \frac{1}{2}(p+1)$ then

$$t_\gamma^{2\mu+2k_\gamma+p+1} < \left(1 + t_\gamma^2\right)^{p_\gamma}$$

for $t_\gamma > 0$ and $1 \le \gamma \le q$. Thus equation (6.27) gives us

$$\| H_\mu(\phi) \|_p^{\mu} \le$$

$$\sup_{\substack{k_\gamma \le p\\ 1\le\gamma\le q}} (-1)^{k+p}\int_{E_q}\left(\prod_{\gamma=1}^{q} t_\gamma^{2\mu+2k_\gamma+p+1}\right)\cdot \tag{6.30}$$

$$(t^{-1}D_t)^p[t]^{-\mu-1/2}\phi(t)\prod_{\gamma=1}^{q}\left(t_\gamma y_\gamma\right)^{-\mu-k_\gamma}\cdot$$

$$J_{\mu+k_\gamma+p}\,(t_\gamma y_\gamma)\, dt_1 \ldots dt_q.$$

$$\le \int_{E_q}\prod_{\gamma=1}^{q}(1 + t_\gamma)^{p_\gamma+1}\,(t^{-1}D_t)^p\,[t]^{-\mu-1/2}\phi(t).$$

$$\prod_{\gamma=1}^{q}\frac{B_\gamma}{1+t_\gamma^2}\, dt_1 \ldots dt_q. \tag{6.31}$$

We now expand $(1+t_\gamma)^{p_\gamma+1}$ as $t_\gamma \to \infty$, $1 \le \gamma \le q$, using the binomial theorem and obtain the estimates,

$$(1+t_\gamma)^{p_\gamma+1} = \binom{p_\gamma+1}{0} + \binom{p_\gamma+1}{1}t_\gamma^2 + \ldots + \binom{p_\gamma+1}{p_\gamma+1}\left(t_\gamma^2\right)^{p_\gamma+1} \le 2^{p_\gamma+1}t_\gamma^{2(p_\gamma+1)} \text{ for } 1 \le \gamma \le q.$$

Implementing this estimate into equation (6.31) will give us

$$\int_{E_q}\prod_{\gamma=1}^{q}(1 + t_\gamma)^{p_\gamma+1}\,(t^{-1}D_t)^p\,[t]^{-\mu-1/2}\phi(t). \tag{6.32}$$

$$\prod_{\gamma=1}^{q} \frac{B_\gamma}{1+t_\gamma^2} d_1^t \ldots d_q^t \leq$$

$$\int_{E_q} \prod_{\gamma=1}^{q} t_\gamma^{2(p_\gamma+1)} (t^{-1}D_t)^p \, [t]^{-\mu-1/2} \phi(t). \qquad (6.33)$$

$$\prod_{\gamma=1}^{q} \frac{2^{p_\gamma+1} \cdot B_\gamma}{1+t_\gamma^2} d_1^t \ldots d_q^t$$

$$\leq \left(\frac{\pi}{2}\right)^q \left(2^{p_\gamma+1}\right)^q \left(B_\gamma\right)^p \| \phi_q \|^{\mu}_{2(p_\gamma+1)}.$$

Now we return to our initial endeavor and compute

$$|||\mathcal{H}_\mu(\Phi)|||^{p,\mu}_{s',r,\mathcal{M}_0} = \sup \left\{ \frac{\| H_\mu(\gamma_q) \|^{\mu}_{p} \cdot m_q}{(s')^q} : q \epsilon \mathbb{N}_0 \right\}$$

$$\leq \sup \left\{ \frac{\left(\frac{\pi}{2}\right)^q \left(2^{p_\gamma+1}\right)^q \left(B_\gamma\right)^p \| \phi_q \|^{\mu}_{2(p_\gamma+1)} \cdot m_q}{(s')^q} : q \epsilon \mathbb{N}_0 \right\}$$

$$= \sup \left\{ \frac{\| \phi_q \|^{\mu}_{2(p_\gamma+1)} \cdot m_q}{s^q} \cdot \frac{s^q \left(\frac{\pi}{2}\right)^q \left(2^{p_\gamma+1}\right)^q \left(B_\gamma\right)^p}{(s')^q} : q \epsilon \mathbb{N}_0 \right\}. \qquad (6.34)$$

Selecting $s' \geq \frac{\pi}{2} \cdot 2^{p_\gamma+1} \cdot B_\gamma$ will give us the desired result.

COROLLARY 6.2. The Hankel transform $\mathcal{H}_\mu$, $\mu \geq -\frac{1}{2}$ is an automorphism on the space, $\Gamma^{r,\mathcal{M}_0}$.

PROOF. Since we apply our Hankel transform, $\mathcal{H}_\mu$, $\mu \geq -\frac{1}{2}$ to each component, $\phi_q(t_1, \ldots, t_q)$, of the vector, $\Phi \epsilon \Gamma^{r,\mathcal{M}_0}$, we can apply the classical theorem [4, pg 141] to each component. As in that result the Hankel transform is its own inverse namely $\mathcal{H}_\mu = \mathcal{H}_\mu^{-1}$ for $\mu \geq -\frac{1}{2}$ on each of our components. Since we have equipped, $\Gamma^{r,\mathcal{M}_0}$, with an inductive limit topology in s we have for each s and s′ such that our Hankel transform is one to one and onto between $\Gamma^{s,r,\mathcal{M}_0}$ and $\Gamma^{s',r,\mathcal{M}_0}$. The theorem 6.1 in this paper proves the continuity in both directions making it an automorphism on $\Gamma^{r,\mathcal{M}_0}$.

7. THE HANKEL TRANSFORM OF THE GENERALIZED DIRAC DELTA FUNCTIONAL.

One of the principal distributions utilized by physicists is the celebrated Dirac delta functional. Clearly in a contemporary setting the Dirac functional must be admitted into a generalized Fock space. There are several applications where this is beneficial and we merely select the application of annihilation and creation requirements as put forth in reference [8].

We select for $t_i^0 > 0$ our generalized Fock functional,

$$\delta_{t_i^0} \Leftrightarrow \begin{bmatrix} 1 \\ \delta_{t_i^0} \\ \vdots \\ \delta_{t_i^0} \otimes \ldots \otimes \delta_{t_i^0} \\ \vdots \end{bmatrix}, \qquad (7.1)$$

where $\delta_{t_i^0} \otimes \ldots \otimes \delta_{t_i^0}$ is the tensor product of q-copies of the translated Dirac delta functional already defined in expression (4.8). We immediately verify that for $p \geq 1$,

$$|||\delta_{t_i^0}|||^{(p)}_{-[s,r,\mathcal{M}_0]} = \sum_{q=0}^{\infty} (s)^q (m_q)^{-1} \| \delta_{t_i^0} \|_{-p} \leq \sum_{q=0}^{\infty} \frac{s^q}{m_q} \cdot 1 < \infty$$

making the generalized Delta given in expression (7.1) a member of $\left(\Gamma^{r,\mathcal{M}_0}\right)'$.

DEFINITION 7.1. The Hankel transform on the space, $\left(\Gamma^{r,\mathcal{M}_0}\right)$, is defined by the formula, $\langle\langle H'_\mu F, \Phi\rangle\rangle \triangleq \langle\langle F, H_\mu \Phi\rangle\rangle$ for $\mu \geq -\frac{1}{2}$.

We see from reference [15] that this definition for the generalized Hankel transform applied to the generalized Dirac functional given in expression (7.1) results in the vector,

$$\begin{bmatrix} 1 \\ \sqrt{t^0 y_1} \\ \vdots \\ \sqrt{t^0 y_1} J_u(t^0 y_1) \cdot \ldots \sqrt{t^0 y_1} J_u(t^0 y_1) \\ \vdots \end{bmatrix}. \tag{7.2}$$

Again recalling $\sqrt{t^0 y_\gamma}\ J_\mu\left(t^0 y_\gamma\right) = \mathcal{O}\left(y_\gamma^{\mu+1/2}\right)$ as $y_\gamma \to 0_+$ and $\sqrt{t^0 y_\gamma}\ J_\mu\left(t^0 y_\gamma\right) = \mathcal{O}(1)$ as $y_\gamma \to \infty$, $1 \leq \gamma \leq q$, it follows that the vector in expression (7.2) is a member of $\left(\Gamma^{r,\mathcal{M}_0}\right)'$ as a regular generalized Fock functional. The term regular has the obvious definition extended from the notion of regular distribution.

An alternate method must be selected for $\delta_{t_i^0}$ when $t_1^0 = 0$. This is because our q-dimensional orthant, E_q, does not contain the origin. Thus the delta functional concentrated at the origin is not a member of H'_μ. If we include the origin and consider the closed q-dimensional orthant, $\overline{E}_q$, then the Hankel transform does not have unique inverses. For further investigations surrounding this difficulty we refer the reader to reference [14].

To circumvent this difficulty we take an alternate definition for the distributional Hankel transform of $\delta^{(k)}(m^2+p)$ given in reference [10]. It is based upon distributions on surfaces developed in reference [11].

It defines the Hankel transform of a test function, $\phi(t)$, to be

$$H_{\frac{n-2}{2}}\{\phi(t)\}(y) \triangleq \frac{1}{2}\int_0^\infty \phi(t)\, t^{\frac{n-2}{2}}\, R_{\frac{n-2}{2}}(\sqrt{yt})\, dt$$

where $R_m(w) \triangleq \frac{J_m(w)}{w^m}$ and $J_m(w)$ is the Bessel function of the first kind given in expression (3.1).

We generalize this definition to the q-dimensional space, $q \geq 2$ to be

$$\left(H_{\frac{n-2}{2}}\phi\right)(y_1,\ldots,y_q) = \left(\frac{1}{2}\right)^q \int_0^\infty \cdots \int_0^\infty \phi(t_1,\ldots,t_q) \prod_{\gamma=1}^q (t_\gamma)^{\frac{n-2}{2}}\, R_{\frac{n-2}{2}}(\sqrt{y_r t_r})\, dt_1,\ldots,dt_q.$$

The Hankel transform is then extended to the distributional setting using the same technique as indicated in equation (3.2). Then the Hankel transform of the tensor product of q-copies of the Dirac delta concentrated at the origin becomes

$$\frac{1}{(2^{n/2})^q[\Gamma(\frac{n}{2})]^q}(y_1 \cdot y_2 \cdot \ldots \cdot y_q)^{\frac{n-2}{2}}.$$

Clearly for $\frac{n-2}{2} > 0$ and in particular if $\frac{n-2}{2}$ is an integer, we have a polynomial in [y] so our Hankel transform functional becomes a regular distribution and once again in our setting becomes a member of $\left(\Gamma^{r,\mathcal{M}_0}\right)'$. However the functional

$$\delta \Leftrightarrow \begin{bmatrix} 1 \\ \delta \\ \vdots \\ \delta \otimes \ldots \otimes \delta \\ \vdots \end{bmatrix},$$

is still not a member of $\left(\Gamma^{r,\mathcal{M}_0}\right)'$ as given in section 6. Therefore we must use a domain space as in reference [17]. This technique would provide procedures leading to excellent computational results.

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REFERENCES

1. BRYCHKOV, Y.A. and PRUDNIKOV, A.P. *Integral Transforms of Generalized Functions, Gordon and Breach Science Publishers, New York, 1989.*

2. *KOH, E.L The n-Dimensional Distributional Hankel Transformation, Can. J. Math 27(2) (1975), 423-433.*

3. *PATHAK, R.S., and SINGH, O.P., Finite Hankel Transforms of Distributions,* Pacific Journal of Mathematics 99(2) (1982), 439-458.

4. ZEMANIAN, A.H. *Generalized Integral Transformations*, John Wiley & Sons, Inc., New York, 1968.

5. PATHAK, R.S. *On Hankel Transformable Spaces and a Cauchy Problem,* Can. J. Math 27(1) (1985), 84-106.

6. BOGOLUBOV, N.N. LOGUNOV, A.A., OKSAK, A.I., and TODOROV, L.T., *General Principals of Quantum Field Theory*, Kluiver Academic Publishers, Boston, 1990.

7. SCHMEELK, J. *Infinite Dimensional Parametric Distributions*, Applicable Analysis, Vol. 24, (1987), 291-319.

8. SCHMEELK, J. and TAKAĈI, A. *Ultra Creation and Annihilation Operators,* Portugaliae Mathematica,49(3), 263-279 (1992).

9. ZARINOV, V.V. *Compact Families of Locally Convex Topological Vector Spaces, Fréchet-Schwartz Spaces*, Russian Math. Surveys, Vol. 34, No. 4(1979), 105-143.

10. AGUIRRE, M.A. and TRIONE, S.E. *The Distributional Hankel Transform of $\delta^{(k)}(m^2 + P)$*, Studies in Applied Mathematics 83, (1990), 111-121.

11. GELFAND, I.M. and SHILOV, G.E. *Generalized Functions*, Vol. I, Academic Press, New York, 1968.

12. KOMATSU, H. *Ultradistributions, I, Structural Theorems and a Characterization*, Journal of the Faculty of Science, Tokyo, Section 1A Mathematics Vol. 20, (1973), 25-105.

13. LIVERMAN, T.P.G., *Generalized Functions and Direct Operational Methods*, Prentice Hall, N.J., 1964.

14. MISRA, O.P. and LAVOINE, J.L. *Transform Analysis of Generalized Functions*, North-Holland Mathematics Studies 119, 1986.

15. PILIPOVIC, S. and TAKAĈI, A. *Space $H'(m)$ and Convolutors*, Proc. of the Moscow Conference on Generalized Functions, Moscow 1981, 415-427.

16. PILIPOVIC, S., STANKOVIC, B. and TAKAĈI, A. *Asymptotic Behaviour and Stieltjes Transformation of Distributions*, B.G. Teuner, Leipzig, 1990.

17. SCHMEELK, J. *Fourier Transforms in Generalized Fock Spaces*, Internat. J. Math and Math. Sci. 13(3)(1990), 431-442.

18. SHILOV, GEORGI, E., Generalized Functions and Partial Differential Equations, Translated from Russian, Bernard Seckler, Gordon and Breuch, New York, 1968.

19. TAKAĈI, A. *A note on the Distributional Stieltjes Transform*, Math. Proc. Camb. Phil. Soc., Vol. 94, (1983), 523-527.

20. TRIONE, S.E., *Distributional Products*, Cursos De Mathematica 3, 1980.