Internat. J. Math. & Math. Sci.
VOL. 17 NO. 3 (1994) 511-514

CR-SUBMANIFOLDS OF A LOCALLY CONFORMAL KAEHLER SPACE FORM

M. HASAN SHAHID

Department of Mathematics
Faculty of Natural Sciences
Jamia Millia Islamia, New Delhi - 110 025

(Received November 14, 1991 and in revised form May 20, 1992)

ABSTRACT. (Bejancu [1,2]) The purpose of this paper is to continue the study of CR-submanifolds, and in particular of those of a locally conformal Kaehler space form (Matsumoto [3]). Some results on the holomorphic sectional curvature, D-totally geodesic, D^1-totally geodesic and D^1-minimal CR-submanifolds of locally conformal Kaehler (l.c.k.)-space from $\bar{M}(c)$ are obtained. We have also discussed Ricci curvature as well as scalar curvature of CR-submanifolds of $\bar{M}(c)$.

KEY WORDS AND PHRASES. CR-submanifolds, D-totally goedesic, D^1-totally geodesic and minimal CR-submanifolds.
1991 AMS SUBJECT CLASSIFICATION CODE. 53B25.

1. PRELIMINARIES.

Let $\bar{M}$ be a Hermitian manifold with complex structure J. Let Ω denote the fundamental 2-form of a Hermitian manifold $\bar{M}$ defined by $g(JX,Y)=\Omega(X,Y)$, where g is the Hermitian metric and X,Y are arbitrary vector fields on $\bar{M}$. $\bar{M}$ is called a locally conformal Kaehler (l.c.k.) manifold [4] if there is a closed 1-form called the Lee form on $\bar{M}$ such that $d\Omega=\Omega\wedge\omega$ where d and $\wedge$ denoting exterior derivative operator and wedge product. In a l.c.k. manifold $\bar{M}$, we define a symmetric tensor field $P(X,Y)$ as

$$P(Y,X) = -(\bar{\nabla}_Y\,\alpha)(X) - \alpha(X)\alpha(Y) + \tfrac{1}{2}\|\alpha\|^2 g(X,Y), \tag{1.1}$$

where $\|\alpha\|$ denotes the length of the Lee form with respect to g. Moreover, we assume that the tensor field P is hybrid, that is,

$$P(Y,JX) = -P(JY,X) \tag{1.2}$$

A l.c.k. manifold $\bar{M}$ is called a l.c.k.-space form if it has a constant holomorphic sectional curvature c, and will be denoted by $\bar{M}(c)$. Let $\bar{M}(c)$ be a l.c.k. - space form, and M be a Riemannian manifold isometrically immersed in $\bar{M}$. We denote by g the metric tensor field of $\bar{M}(c)$ as well as that induced on M. Let $\bar{\nabla}$ (resp. ∇) be the covariant differentiation with respect to the Levi-Civita connection in $\bar{M}$ (resp. M). Then the Gauss and Weingarten formulas for M are respectively given by

$$\bar{\nabla}_X Y = \nabla_X Y + h(X,Y), \qquad \bar{\nabla}_X N = -A_X N + \nabla_X^{\perp} N, \tag{1.3}$$

for any $X,Y\in TM$, where h (resp. A) is the second fundamental form (resp. tensor) of M and $\nabla^{\perp}$ denotes the operator of the normal connection. Moreover

$$g(h(X,Y),N) = g(A_N X,Y). \tag{1.4}$$

The curvature tensor $\bar{R}$ of a l.c.k. space form $\bar{M}(c)$ is given by Matsumoto [3]

$$\bar{R}(X,Y,Z,W) = \frac{c}{4}[g(X,W)g(Y,Z) - g(X,Z)g(Y,W) + g(JX,W)g(JY,Z) - g(JX,Z)g(JY,W) - 2g(JX,Y)g(JZ,X)] + \frac{3}{4}[P(X,W)g(Y,Z) - P(X,Z)g(Y,W) + g(X,W)P(Y,Z) - g(X,Z)P(Y,W)] + \frac{1}{4}[P(X,JW)g(JY,Z) - P(X,JZ)g(JY,W) + g(JX,W)P(Y,JZ) - g(JX,Z)P(Y,JW) - 2P(X,JY)g(JZ,W) - 2P(Z,JW)g(JX,Y)], \tag{1.5}$$

where $\bar{R}(X,Y,Z,W) = g(\bar{R}(X,Y)Z,W)$ and

$$P(X,Y) = P(Y,X),\ P(X,JY) = -P(JX,Y),\ P(JX,JY) = P(X,Y).$$

The Gauss equation is given by

$$R(X,Y,Z,W) = \bar{R}(X,Y,Z,W) + g(h(X,W),h(Y,Z)) - g(h(X,Z),h(Y,W)), \tag{1.6}$$

where R (resp. $\bar{R}$) is the curvature of M and (resp. $\bar{M}(c)$).

DEFINITION 1.1. A submanifold M of a l.c.k. space form $\bar{M}(c)$ is called a CR-submanifold if there exists a differentiable distribution $D: x \to D_x \subset T_xM$ on M satisfying the following condition:

(i) D is holomorphic i.e. $JD_x = D_x$ for each $x \in M$ and

(ii) the complementary orthogonal distribution $D^{\perp}: x \to D_x^{\perp} \subset T_xM$ is totally real, i.e. $JD_x^{\perp} \subset T_x^{\perp}M$ for each $x \in M$.

For any vector field X tangent to M, we put

$$X = TX + FX, \tag{1.7}$$

where TX and FX belong to the distribution D and $D^{\perp}$ respectively.

2. SECTIONAL CURVATURE OF CR-SUBMANIFOLDS.

Let M be a CR-submanifold of a l.c.k. space form $\bar{M}(c)$. Then using Gauss equation (1.6), the curvature tensor of M is given by

$$R(X,Y,Z,W) = \frac{c}{4}[g(X,W)g(Y,Z) - g(X,Z)g(Y,W) + g(JTX,W)g(JTY,Z) - g(JTX,Z)g(JTY,W) - 2g(JTX,Y)g(JTZ,W)] + \frac{3}{4}[P(X,W)g(Y,Z) - P(X,Z)g(Y,W) + g(X,W)P(Y,Z) - g(X,Z)P(Y,W)] + \frac{1}{4}[P(X,JTW)g(JTY,Z) - P(X,JTZ)\, g(JTY,W) + g(JTX,W)\, P(Y,JTZ) - g(JTX,Z)P(Y,JTW) - 2g(JTZ,W)P(X,JTY) - 2P(Z,JTW)g(JTX,Y)] + g(h(X,W),h(Y,Z)) - g(h(X,Z),h(Y,W)) \tag{2.1}$$

for $X,Y,Z,W \in TM$.

Let $\bar{H}(X)$ be the holomorphic sectional curvature of M determined by a unit vector X and JX. Then from (1.5) we have

$$\bar{H}(X) = \bar{R}(X,JX,JX,X) = -\frac{c}{2} + \frac{7}{4}P(X,X). \tag{2.2}$$

Now suppose that $\bar{K}(X \wedge Y)$ be the sectional curvature of $\bar{M}$ determined by a unit vector X and JX. Then from (1.5) we have

$$\bar{K}(X \wedge Y) = \bar{R}(X,Y,Y,X) = \frac{c}{4}[1 + g(JX,Y)^2 + 2g(JX,Y)] + \frac{3}{4}[P(X,X) + P(Y,Y)] + P(X,JY)g(JX,Y). \tag{2.3}$$

Next, suppose that $K(X \wedge Y)$ be the sectional curvature of M determined by orthonormal tangent vectors $\{X,Y\}$ of M. Then using (1.6) and (2.3), we have

$$K(X \wedge Y) = \frac{c}{4}[1 + g(JTX,Y)^2 + 2g(JTX,Y)] + \frac{3}{4}[P(X,X) + P(Y,Y)] + P(X,JTY)\, g(JTX,Y) + g(h(X,X),h(Y,Y)) - \| h(X,Y) \|^2, \tag{2.4}$$

for all X,Y tangent to M. From this, we have

PROPOSITION 2.1. Let M be a CR-submanifold of a l.c.k. space form $\bar{M}(c)$. If M is totally geodesic in $\bar{M}(c)$, then the sectional curvature of M is given by

$$K(X \wedge Y) = \frac{c}{4}[1 + g(JTX,Y)^2 + 2g(JTX,Y)] + \frac{3}{4}[P(X,X) + P(Y,Y)] + P(X,JTY)\, g(JTX,Y) \tag{2.5}$$

for all X,Y tangent to M.

DEFINITION 2.1. A CR-submanifold M of a l.c.k. space form $\bar{M}(c)$ is said to be D-totally (resp. $D^{\perp}$-totally geodesic) if $h(X,Y)=0$ (resp. $h(Z,W)=0$) for all $X,Y \in D,(Z,W \in D^{\perp})$.

Thus as an immediate consequence of (2.5) we have

COROLLARY 2.2. Let M be a CR-submanifold of a l.c.k. space form $\bar{M}(c)$. If M is $D^{\perp}$-totally geodesic in $\bar{M}(c)$, then the sectional curvature of M is given by

$$K(X \wedge Y) = \frac{c}{4} + \frac{3}{4}[P(X,X) + P(Y,Y)] \qquad \text{for all } X,Y \in D. \tag{2.6}$$

The holomorphic sectional curvature H of M determined by a unit vector $X \in D$ is the sectional curvature determined by $\{X, JX\}$. Hence from (2.2), we have

$$H(X) = -\frac{c}{2} + \frac{7}{4}P(X,X) + g(h(X,X),h(JX,JX)) - \| h(X,JX) \|^2. \tag{2.7}$$

LEMMA [1]. Let M be a CR-submanifold of a Kaehler manifold $\bar{M}$. Then the holomorphic distribution D is involutive if and only if

$$h(JX,Y) = h(X,JY), \qquad \forall X,Y \in D. \tag{2.8}$$

Making use of (2.8) in (2.7), we have

PROPOSITION 2.3. Let M be a CR-submanifold of a l.c.k.-space form $\bar{M}(c)$ with involutive distribution D, then

$$H(X) \le \frac{7}{4}P(X,X), \qquad \forall X \in D.$$

Moreover from (2.7), we have

PROPOSITION 2.4. A CR-submanifold M of a l.c.k. space form $\bar{M}(c)$ is D-totally geodesic if and only if the following conditions are satisfied:

(a) the holomorphic distribution D is involutive, and (b) $H(X) = \frac{7}{4}P(X,X) - \frac{c}{2}$, $\forall X \in D$.

Let $\{E_1, E_2, \cdots, E_m\}$ be a local field of orthogonal frames of M such that $\{E_1, E_2, \cdots, E_P, E_{P+1} = JE_1, \cdots, E_{2P} = JE_P\}$ (resp. $\{E_{2P+1} \cdots E_{2P+q}\}$) is a local field of frames in D (resp. $D^{\perp}$).

DEFINITION 2.2. A CR-submanifold M is called D-minimal (resp. $D^{\perp}$-minimal) if

$$\sum_{i=1}^{2P} h(E_i, E_i) = 0, \ (\text{resp.} \sum_{i=1}^{q} h(E_{2P+i}, E_{2P+i}) = 0).$$

Thus we have,

PROPOSITION 2.5. Let M be a $D^{\perp}$-minimal CR-submanifold of a l.c.k. space form $\bar{M}(c)$. Then M is D-totally geodesic if and only if

$$K(X \wedge Y) = \frac{1}{4}[c + 3(P(X,X) + P(Y,Y))], \qquad \forall, X, Y \in D.$$

3. RICCI TENSOR AND SCALAR CURVATURE OF CR-SUBMANIFOLDS.

Let S be the Ricci tensor and ρ the scalar curvature of M. Then

$$S(X,Y) = \sum_i R\,(E_i, X; Y, E_i), \qquad \rho = \sum_j S(E_j, E_j),$$

for any vector fields X,Y tangent to M. By the straight forward calculation from (2.1), we get

$$S(X,Y) = \frac{c}{4}(m+2)g(X,Y) + \frac{3}{4}\sum_{i=1}^{m}\{P(E_i,E_i)g(X,Y) - P(E_i,Y)g(X,E_i) + mP(X,Y) - P(X,E_i)g(Y,E_i)\}$$

$$-\frac{5}{4}\sum_{i=1}^{m}\{P(JY,E_i)g(JX,E_i) + P(JX,E_i)g(JY,E_i)\} + \sum_{i=1}^{m}\{g(h(X,Y),h(E_i,E_i)) - g(h(E_i,X),h(E_i,Y))\},$$

since $g(JTE_i, E_i) = 0$.

The scalar curvature is given by

$$\rho = \frac{c}{4}m(m+2) + \sum_{i,j=1}^{m}\{g(h(E_j,E_j),h(E_i,E_i)) - g(h(E_i,E_j),\ h(E_i,E_j))\}.$$

Thus we have

PROPOSITION 3.1. Let M be a minimal CR-submanifold of a l.c.k. space form, then we have

(a) $S(X,Y)-\frac{c}{4}(m+2)g(X,Y)-\frac{3}{4}\sum_{i=1}^{m}\{P(E_i,E_i)g(X,Y)-P(E_i,Y)g(X,E_i)+mP(X,Y)-P(X,E_i)g(Y,E_i)\}$

$$+\frac{5}{4}\sum_{i=1}^{m}\{P(JY,E_i)g(JX,E_i)+P(JX,E_i)g(JY,E_i)\}$$

is semi-definite for all $X,Y\in D$.

(b) $\rho\leq\frac{c}{4}m(m+2)$.

Similarly we have:

PROPOSITION 3.2. Let M be a minimal CR-submanifold of a l.c.k.-space form. Then M is totally geodesic if and only if

(a) $S(X,Y)=(m+2)g(X,Y)+\frac{3}{4}\sum_{i=1}^{m}\{P(E_i,E_i)g(X,Y)-P(E_i,Y)g(X,E_i)+mP(X,Y)$

$-P(X,E_i)g(Y,E_i)\}-\frac{5}{4}\sum_{i=1}^{m}\{P(JY,E_i)g(JX,E_i)+P(JX,E_i)g(JY,E_i)\}$

(b) $\rho=m(m+2)$.

ACKNOWLEDGEMENT. The author expresses his sincere thanks to the referee for his suggestions.

REFERENCES

1. BEJANCU, A., CR-submanifolds of a Kaehler manifold I, Amer. Math. Soc. 69 (1978), 135-142.
2. BEJANCU, A., CR-submanifolds of a Kaehler manifold II, Trans. Amer. Soc. 250 (1979), 333-345.
3. MATSUMOTO, K., On CR-submanifolds of locally conformal Kaehler manifold, J. Korean Math. Soc. 21 (1984).
4. VAISMAN, I., On locally and globally conformal Kaehler manifolds, Trans. Amer. Math. Soc. 262 (1980), 533-542.
5. CHEN, B.Y., CR-submanifolds of a Kaehlerian manifold I, J. Diff. Geom. 16 (1981), 305-322.
6. YANO, K. & KON, M., Differential geometry of CR-submanifolds, Geometrical Dedicata 10 (1981), 369-391.