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GENERALIZED COMMON FIXED POINT THEOREMS FOR A SEQUENCE OF FUZZY MAPPINGS

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ABSTRACT. We obtain generalized common fixed point theorems for a sequence of fuzzy mappings, which is a generalization of the result of Lee and Cho [6].

KEY WORDS AND PHRASES. Fuzzy set, fuzzy mapping, upper semi-continuous, common fixed point.

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1. INTRODUCTION.

Heilpern [3] first introduced the concept of fuzzy mappings and proved a fixed point theorem for fuzzy contraction mappings, which is a fuzzy analogue of the fixed point theorems for multi-valued mappings ([2], [4], [9]) and the well-known Banach fixed point theorem. Bose and Sahani [1], in their first theorem, extended Heilpern's result for a pair of generalized fuzzy contraction mappings. They also, in their second theorem, proved a fixed point theorem for non-expansive fuzzy mappings on a compact star-shaped subset of a Banach space. Lee and Cho [5] proved a fixed point theorem for a contractive-type fuzzy mapping which is an extension of the result of Heilpern [3]. Also, they [6] obtained common fixed point theorems for a sequence of fuzzy mappings which are generalizations of their result in [5]. Lee et al. [7] obtained a common fixed point theorem for a sequence of fuzzy mappings satisfying certain conditions, which is a generalization of the second theorem of Bose and Sahani. They also showed common fixed point theorems for a pair of fuzzy mappings in [8], which is an extension of the first theorem of Bose and Sahani [1].

In this paper, we prove generalized common fixed point theorems for a sequence of fuzzy mappings satisfying certain conditions which are generalizations of the result of Lee and Cho [6].

2. PRELIMINARIES.

Let (X,d) be a linear metric linear space. A fuzzy set A in X is a function from X into $[0,1]$. If $x \in X$, the function value $A(x)$ is called the *grade of membership* of X in A. The α-level set of A, denote by A_α, is defined by

$$A_\alpha = \{x: A(x) \geq \alpha\} \text{ if } \alpha \in (0,1], \quad A_0 = \overline{\{x: A(x) > 0\}},$$

where $\overline{B}$ denotes the closure of the nonfuzzy set of B.

Let $W(X)$ be the collection of all the fuzzy sets A in X such that A_α is compact and convex for each $\alpha \in [0,1]$, and $\sup_{x \in X} A(x) = 1$. For $A, B \in W(X), A \subset B$ means $A(x) \leq B(x)$ for each $x \in X$.

DEFINITION 2.1. Let $A, B \in W(X)$ and $\alpha \in [0,1]$. Then we define

$$P_\alpha(A,B) = \inf_{x \in A_\alpha, y \in B_\alpha} d(x,y), \quad P(A,B) = \sup_\alpha P_\alpha(A,B)$$

and

$$D(A,B) = \sup_\alpha d_H(A_\alpha, B_\alpha),$$

where d_H is the Hausdorff metric induced by the metric d. We note that P_α is a nondecreasing function of α and D is a metric on $W(X)$.

DEFINITION 2.2. Let X be an arbitrary set and Y be any linear metric space. F is called a *fuzzy mapping* if and only if F is a mapping from the set X into $W(Y)$.

In the following section, we will use the following lemmas.

LEMMA 2.1 [5]. Let (X,d) be a complete linear metric space, F a fuzzy mapping from X into $W(X)$ and $x_0 \in X$, then there exists $x_1 \in X$ such that $\{x_1\} \subset F(x_0)$.

LEMMA 2.2 [8]. Let $A, B \in W(X)$. Then for each $\{x\} \subset A$ there exists $\{y\} \subset B$ such that $D(\{x\},\{y\}) \leq D(A,B)$.

We can easily prove the following lemma.

LEMMA 2.3. Let $x \in X$ and $B \in W(X)$. If $\{y\} \subset B$, then $P(\{x\}, B) \leq d(x,y)$.

3. COMMON FIXED POINTS THEOREMS FOR A SEQUENCE OF FUZZY MAPPINGS.

THEOREM 3.1. Let g be a non-expansive mapping from a complete linear metric space (X,d) into itself. If $(F)_{i=1}^{\infty}$ is a sequence of fuzzy mappings from X into $W(X)$ satisfying the following condition: For each pair of fuzzy mappings, F_i, F_j and for any $x \in X, \{u_x\} \subset F_i(x)$, there exists $\{v_y\} \subset F_j(y)$ for all $y \in X$ such that

$$D(\{u_x\},\{v_y\}) \leq a_1 d(g(x), g(u_x)) + a_2 d(g(y), g(v_y)) + a_3 d(g(y), g(u_x)) + a_4 d(g(x), g(v_y)) + a_5 d(g(x), g(y)),$$

where a_1, a_2, a_3, a_4, a_5 are nonnegative real numbers, $a_1 + a_2 + a_3 + a_4 + a_5 < 1$ and $a_3 \geq a_4$. Then there exists $p \in X$ such that $\{p\} \subset \bigcap_{i=1}^{\infty} F_i(p)$.

PROOF. Let $x_0 \in X$. Then we can choose $x_1 \in X$ such that $\{x_1\} \subset F_1(x_0)$ by Lemma 2.1. By our assumptions, there exists $x_2 \in X$ such that $\{x_2\} \subset F_2(x_1)$ and

$$\begin{aligned} D(\{x_1\},\{x_2\}) &\leq a_1 d(g(x_0), g(x_1)) + a_2 d(g(x_1), g(x_2)) + a_3 d(g(x_1), g(x_1)) \\ &\quad + a_4 d(g(x_0), g(x_2)) + a_5 d(g(x_0), g(x_1)) \\ &\leq a_1 d(x_0,x_1) + a_2 d(x_1,x_2) + a_3 d(x_1,x_1) + a_4 d(x_0,x_2) + a_5 d(x_0,x_1). \end{aligned}$$

Again we can find $x_3 \in X$ such that $\{x_3\} \subset F_3(x_2)$ and

$$D(\{x_2\},\{x_3\}) \leq a_1 d(x_1,x_2) + a_2 d(x_2,x_3) + a_3 d(x_2,x_2) + a_4 d(x_1,x_3) + a_5 d(x_1,x_2).$$

Inductively, we obtain a sequence (x_n) in X such that $\{x_{n+1}\} \subset F_{n+1}(x_n)$ and

$$\begin{aligned} D(\{x_n\},\{x_{n+1}\}) &\leq a_1 d(x_{n-1},x_n) + a_2 d(x_n,x_{n+1}) + a_3 d(x_n,x_n) \\ &\quad + a_4 d(x_{n-1},x_{n+1}) + a_5 d(x_{n-1},x_n). \end{aligned} \tag{3.1}$$

Since $D(\{x_n\},\{x_{n+1}\}) = d(x_n, x_{n+1})$, by (3.1)

$$\begin{aligned} d(x_n,x_{n+1}) &\leq a_1 d(x_{n-1},x_n) + a_2 d(x_n,x_{n+1}) \\ &\quad + a_4 d(x_{n-1},x_n) + a_4 d(x_n,x_{n+1}) + a_5 d(x_{n-1},x_n). \end{aligned}$$

Hence

$$d(x_n, x_{n+1}) \le [(a_1 + a_4 + a_5)/(1 - a_2 - a_4)]d(x_{n-1}, x_n).$$

Let $r = (a_1 + a_4 + a_5)/(1 - a_2 - a_4)$. Since $a_3 \ge a_4$, $0 < r < 1$. Moreover, we have $d(x_n, x_{n+1}) \le r^n d(x_0, x_1)$. We can easily show that $(x_n)_{n=1}^{\infty}$ is a Cauchy sequence in X. Since X is complete, there exists $p \in X$ such that $\lim_{n\to\infty} x_n = p$. Let F_m be an arbitrary member of $(F_i)_{i=1}^{\infty}$. Since $\{x_n\} \subset F_n(x_{n-1})$ for all n, there exists $v_n \in X$ such that $\{v_n\} \subset F_m(p)$ for all n and

$$D(\{x_n\}, \{v_n\}) \le a_1 d(x_{n-1}, x_n) + a_2 d(p, v_n) + a_3 d(p, x_n) + a_4 d(x_{n-1}, v_n) + a_5 d(x_{n-1}, p). \quad (3.2)$$

From (3.2), we have

$$d(x_n, v_n) \le a_1 d(x_{n-1}, x_n) + a_2 d(p, x_n) + a_2 d(x_n, v_n) + a_3 d(p, x_n) + a_4 d(x_{n-1}, x_n) + a_4 d(x_n, v_n) + a_5 d(x_{n-1}, p)$$

Thus we have

$$(1 - a_2 - a_4) d(x_n, v_n) \le a_1 d(x_{n-1}, x_n) + a_2 d(p, x_n) + a_3 d(p, x_n) + a_4 d(x_{n-1}, x_n) + a_5 d(x_{n-1}, p).$$

Since $x_n \to p$ as $n \to \infty$, $(1 - a_2 - a_4) d(x_n, v_n) \to 0$ as $n \to \infty$. Hence $d(x_n, v_n) \to 0$ as $n \to \infty$. Since $d(p, v_n) \le d(p, x_n) + d(x_n v_n)$, $v_n \to p$ as $n \to \infty$. Since $F_m(p) \in W(X)$, $F_m(p)$ is upper semi-continuous and thus

$$\lim_{n\to\infty} sup[F_m(p)](v_n) \le [F_m(p)](p).$$

Since $\{v_n\} \subset F_m(p)$ for all n, $[F_m(p)](p) = 1$. Hence $\{p\} \subset F_m(p)$. Since F_m is arbitrary, $\{p\} \subset \bigcap_{i=1}^{\infty} F_i(p)$.

Putting $g(x) = x$, we get the following corollary from Theorem 3.1.

COROLLARY 3.1. Let (X, d) be a complete linear metric space. If $(F_i)_{i=1}^{\infty}$ is a sequence of fuzzy mappings from X into $W(X)$ satisfying the following condition $(*)$: For each pair of fuzzy mapping F_i, F_j and for any $x \in X, \{u_x\} \subset F_i(x)$, there exists $\{v_y\} \subset F_j(y)$ for all $y \in X$ such that

$$D(\{u_x\}, \{v_y\}) \le a_1 d(x, u_x) + a_2 d(y, v_y) + a_3 d(y, u_x) + a_4 d(x, v_y) + a_5 d(x, y),$$

where a_1, a_2, a_3, a_4, a_5 are nonnegative real numbers, $a_1 + a_2 + a_3 + a_4 + a_5 < 1$ and $a_3 \ge a_4$. Then there exists $p \in X$ such that $\{p\} \subset \bigcap_{i=1}^{\infty} F_i(p)$.

By Lemmas 2.2 and 2.3, we can obtain the following corollary from Corollary 3.1.

COROLLARY 3.2. Let (X, d) be a complete linear metric space and let $(F_i)_{i=1}^{\infty}$ be a sequence of fuzzy mappings from X into $W(X)$ satisfying the following condition $(**)$: For each pair of fuzzy mappings F_i, F_j,

$$D(F_i(x), F_j(y)) \le a_1 P(x, F_i(x)) + a_2 P(y, F_j(y)) + a_3 P(y, F_i(x)) + a_4 P(x, F_j(y)) + a_5 d(x, y),$$

for all x, y in X, where a_1, a_2, a_3, a_4, a_5 are nonnegative real numbers, $a_1 + a_2 + a_3 + a_4 + a_5 < 1$ and $a_3 \ge a_4$. Then there exists $p \in X$ such that $\{p\} \subset \bigcap_{i=1}^{\infty} F_i(p)$.

The following example shows that the condition $(*)$ in Corollary 3.1 does not imply the condition $(**)$ in Corollary 3.2.

EXAMPLE 3.1. Let $(F_i)_{i=1}^{\infty}$ be a sequence of fuzzy mappings from $[0, \infty)$ into $W([0, \infty))$, where $F_i(x): [0, \infty) \to [0, 1]$ is defined as follows

$$\text{if } x = 0, \quad [F_i(x)](z) = \begin{cases} 1, & z = 0 \\ 0, & z \ne 0, \end{cases}$$

$$\text{otherwise,} \quad [F_i(x)](z) = \begin{cases} 1, & 0 \le z \le x/2 \\ 1/2, & x/2 < z \le ix \\ 0, & z > ix. \end{cases}$$

Then the sequence $(F_i)_{i=1}^{\infty}$ satisfies the condition $(*)$ when $a_1 = a_2 = a_3 = a_4 = 0$, but does not satisfy the condition $(**)$.

Putting $a_1 = a_2 = a_2 = a_4 = 0$, we get the following corollary from Theorem 3.1.

COROLLARY 3.3 [6]. Let g be a non-expansive mapping from a complete linear metric space (X, d) into itself and $(F_i)_{i=1}^{\infty}$ a sequence of fuzzy mappings from X into $W(X)$ satisfying the following condition: There exists a constant k with $0 < k < 1$ such that for each pair of fuzzy mappings F_i, F_j and for any $x \in X, \{u_x\} \subset F_i(x)$, there exists $\{v_y\} \subset F_j(y)$ for all $y \in X$ such that

$$D(\{u_x\}, \{v_y\}) \leq kd(g(x), g(y)).$$

Then there exists $p \in X$ such that $\{p\} \subset \bigcap_{i=1}^{\infty} F_i(p)$.

By Lemma 2.2, we get the following corollary from Corollary 3.3.

COROLLARY 3.4 [6]. Let g be a non-expansive mapping from a complete linear metric space (X, d) into itself and $(F_i)_{i=1}^{\infty}$ a sequence of fuzzy mappings from X into $W(X)$ satisfying the following condition: There exists a constant k with $0 < k < 1$ such that for each pair of fuzzy mappings F_i, F_j,

$$D(F_i(x), F_j(y)) \leq kd(g(x), g(y)) \qquad \text{for all } x, y \text{ in } X,$$

Then there exists $p \in X$ such that $\{p\} \subset \bigcap_{n=1}^{\infty} F_i(p)$.

Putting $g(x) = x$, we get the following corollary from Corollary 3.4.

COROLLARY 3.5 [6]. Let (X, d) be a complete linear metric space and $(F_i)_{i=1}^{\infty}$ be a sequence of fuzzy mappings from X into $W(X)$ satisfying the following condition. There exists a constant k with $0 < k < 1$ such that for each pair of fuzzy mappings, F_i, F_j,

$$D(F_i(x), F_j(y)) \leq kd(x, y) \qquad \text{for all } x, y \text{ in } X$$

Then there exists $p \in X$ such that $\{p\} \subset \bigcap_{n=1}^{\infty} F_i(p)$.

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