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A SELECTION AND A FIXED POINT THEOREM AND AN EQUILIBRIUM POINT OF AN ABSTREACT ECONOMY

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ABSTRACT. A selection theorem and a fixed point theorem are proved. The fixed point theorem is then applied to prove the existence of an equilibrium point of an abstract economy.

KEY WORDS AND PHRASES. Selection and Fixed Point Theorems; Equilibrium Point of an abstract economy.
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1. INTRODUCTION.

Bewley [1] proved the existence of an equilibrium point of an abstract economy with infinite dimensional commodity space.

In recent years, a number of authors [e.g., Yannelis and Prabhakar [9], Toussaint [8], Tarafdar [7] and Ding, Kim, and Tan [3]] have established the existence of an equilibrium point of an abstract economy with infinite dimensional commodity space and infinite agents.

The object of this paper is to prove a selection theorem from which we derive a fixed point theorem that is different from the one due to Tarafdar [7] in that the compactness condition is relaxed to some extent at the expense of assuming locally convex topological vector spaces in place of topological vector spaces.

According to Debreu [2] and Shafer and Sonnenschein [5], an abstract economy or a generalized qualitative game is $\mathcal{E} = \{X_\alpha, A_\alpha, U_\alpha : \alpha \in I\}$ in which I is finite or infinite (countable or uncountable) set of agents of players and for each $\alpha \in I, X_\alpha$ is the choice set or strategy set; $A_\alpha: X = \prod_{\alpha \in I} X_\alpha \to 2^{X_\alpha}$ is the constraint correspondence (set valued mapping) and $U_\alpha: X \to \mathbf{R}$ is the utility or pay-off function. X_α is a subset of a topological vector space for each $\alpha \in I$. The product $\prod_{\beta \in I, \beta \neq \alpha} X_\beta$ is denoted by $X_{-\alpha}$ and a generic element of $X_{-\alpha}$ by $x_{-\alpha}$.

We note that an abstract economy $\mathcal{E} = \{X_\alpha, A_\alpha, U_\alpha : \alpha \in I\}$ may also be given by $\{X_\alpha, P_\alpha, U_\alpha : \alpha \in I\}$ in which for each $\alpha \in I, P_\alpha: X \to 2^{X_\alpha}$ is the preference correspondence. The relationship between the utility function U_α and the preference correspondence P_α can be exhibited by the definition

$$P_\alpha(x) = \{y_\alpha \in X_\alpha : U_\alpha([y_\alpha, x_{-\alpha}]) > U_\alpha(x)\},$$

where $x_{-\alpha}$ is the projection of x onto $X_{-\alpha}$ for each α and $[y_\alpha, x_{-\alpha}]$ is that point of X which has y_α as its α^{th} coordinate.

A point $\bar{x} \in X$ of an economy $\mathcal{E} = \{X_\alpha, A_\alpha, U_\alpha : \alpha \in I\}$ is called an equilibrium point or a generalized Nash equilibrium point of $\mathcal{E}$ if

$$U_\alpha(\bar{x}) = U_\alpha[\bar{x}_\alpha, \bar{x}_{-\alpha}] = \sup_{z_\alpha \in A_\alpha(\bar{x})} U[z_\alpha, \bar{x}_{-\alpha}]$$

for each α in which $\bar{x}_\alpha$ and $\bar{x}_{-\alpha}$ are respectively projective of $\bar{x}$ onto X_α and $X_{-\alpha}$. In this case an equilibrium point is the natural extension of the equilibrium point introduced by Nash (1950). If $\mathcal{E} = \{X_\alpha, A_\alpha, U_\alpha : \alpha \in I\}$ is an abstract economy and for each $\alpha \in I, P_\alpha$ is defined as above, then it is easy to see that a point $\bar{x} \in X$ is an equilibrium point of $\mathcal{E}$ if and only if for each $\alpha \in I, P_\alpha(\bar{x}) \cap A_\alpha(\bar{x}) = \emptyset$ and $\bar{x}_\alpha \in A_\alpha(\bar{x})$. Thus if an abstract economy is given by $\{X_\alpha, P_\alpha, A_\alpha : \alpha \in I\}$, then its equilibrium point can be defined as follows: A point $\bar{x} \in X$ is an equilibrium point of the abstract economy $\{X_\alpha, P_\alpha, A_\alpha : \alpha \in I\}$ if for each $\alpha \in I, P_\alpha(\bar{x}) \cap A_\alpha(\bar{x}) = \emptyset$ and $\bar{x}_\alpha \in A_\alpha(\bar{x})$, where $\bar{x}_\alpha$ is the projection of $\bar{x}$ onto X_α.

Given an abstract economy $\mathcal{E} = \{X_\alpha, P_\alpha, A_\alpha : \alpha \in I\}$, for each $x \in X$, we define

$$I(x) = \{\alpha \in I : P_\alpha(x) \cap A_\alpha(x) \neq \emptyset\}.$$

We assume that for each $x \in X$, $\bar{x}_\alpha \notin coP_\alpha(x)$, the convex hull of $P_\alpha(x)$ for each $\alpha \in I$. For each $\alpha \in I$, we define the set valued mapping $T_\alpha : X \to 2^{X_\alpha}$ by

$$T_\alpha(x) = \begin{cases} coP_\alpha(x) \cap A_\alpha(x) & \text{if } \alpha \in I(x) \\ A_\alpha(x) & \text{if } \alpha \notin I(x). \end{cases}$$

It is easy to see that $\bar{x} \in X$ is an equilibrium point of the economy $\mathcal{E}$ if and only if $\bar{x}$ is a fixed point of the set valued mapping $T : X \to X$ defined by $T(x) = \prod_{\alpha \in I} T_\alpha(x)$.

2. SELECTION AND FIXED POINT THEOREMS.

Here first we prove a selection theorem from which we derive fixed point theorems. One of these results contains Theorem 1 due to the second author [7].

THEOREM 2.1. Let X be a nonempty paracompact Hausdorff topological space and Y a nonempty convex subset of a topological vector space. Let $F : X \to 2^Y$ be a set valued mapping such that

(i) for each $x \in X, F(x)$ is a nonempty convex subset of Y;

(ii) for each $y \in Y, F^{-1}(y) = \{x \in X : y \in F(x)\}$ contains an open set O_y;

(iii) $\bigcup_{y \in Y} O_y = X$.

Then there is a continuous selection f of F (i.e., there is a continuous mapping $f : X \to Y$) such that $f(x) \in F(x)$ for each $x \in X$.

PROOF. Since X is a paracompact space, by (iii) there exists an open locally finite refinement $\mathcal{U} = \{U_a : a \in A\}$ of the family $\{O_y : y \in Y\}$ (see Lemma 1 of Michael [10]) in which A is an indexing set and each U_a is an open subset of X. Hence by Proposition 2 of Michael [10], there is a family $\{f_a : a \in A\}$ of continuous functions $f_a : X \to [0,1]$ with $f_a(x) = 0$ for $x \notin U_a$ and $\Sigma_{a \in A} f_a(x) = 1$ for all $x \in X$. Since $\mathcal{U}$ is a refinement of $\{O_y : y \in Y\}$, for each $a \in A$ we can choose $y_a \in Y$ such that $U_a \subset O_{y_a}$. We define $f : X \to Y$ by

$$f(x) = \Sigma_{a \in A} f_a(x) y_a, \quad x \in X.$$

Since $\mathcal{U}$ is a locally finite refinement it follows that for each $x \in X, f_a(x)$ is nonzero for at most finitely many $a \in A$. So f is well defined and is evidently continuous. For each $x \in X$, $f(x) \neq 0$ implies $x \in U_a \subset O_{y_a} \subset F^{-1}(y_a)$ i.e., $y_a \in f(x)$. Since $F(x)$ is convex, it follows that $f(x) \in F(x)$.

COROLLARY 2.1. Let X and Y be as in Theorem 2.1. Let $S: X \to 2^Y$ be a set valued mapping such that

(i) for each $x \in X, S(x) \neq \emptyset$

(ii) for each $y \in Y, S^{-1}(y)$ is open.

Then there is a continuous selection of the set valued mapping $T: X \to 2^Y$ defined by $T(x) = co\ S(x), x \in X$.

PROOF. Set $O_y = S^{-1}(y)$ for each $y \in Y$. Then for each $y \in Y, O_y = S^{-1}(y) \subset T^{-1}(y)$ as $S(x) \subset co\ S(x) = T(x)$ for each $x \in X$. Also $\bigcup_{y \in y} O_y = X$ because if $x \in X$ then $S(x) \neq \emptyset$ implies there is $y \in S(x)$ and so $x \in S^{-1}(y) = O_y$. Now the corollary follows from Theorem 2.1.

Note Corollary 2.1 contains Theorem 1 of [3] as a special case.

LEMMA 2.1. Let D be a nonempty compact subset of a topological vector space. Then coD is paracompact.

See [3] for a simple proof.

THEOREM 2.2. Let $\{X_\alpha : \alpha \in I\}$ be a family of nonempty convex sets, each in a Hausdorff locally convex space E_α, where I is an indexing set. For each $\alpha \in I$, let D_α be a nonempty compact subset of X_α and $T_\alpha : X = \prod_{\alpha \in I} X_\alpha \to 2^{D_\alpha}$ a set valued mapping such that

(i) for each $x \in X, T_\alpha(x)$ is a nonempty convex subset of D_α;

(ii) for each $y_\alpha \in D_\alpha$, $T_\alpha{}^{-1}(y_\alpha)$ contains a relatively open subset O_{y_α} of X;

(iii) $\bigcup_{y_\alpha \in D_\alpha} O_{y_\alpha} = coD$, where $D = \prod_{\alpha \in I} D_\alpha$.

Then there is a point $\bar{x} \in D$ such that $\bar{x} \in T(\bar{x}) = \prod_{\alpha \in I} T_\alpha(\bar{x})$, i.e., $\bar{x}_\alpha \in T_\alpha(\bar{x})$ for each $\alpha \in I$ where $\bar{x}_\alpha$ is the projection of $\bar{x}$ onto X_α for each $\alpha \in I$. In other words, $\bar{x}$ is a fixed point of T.

PROOF. By Lemma 2.1, coD is a paracompact subset of X because D is compact by the Tychonoff Theorem. For each $\alpha \in I$, let $\hat{T}_\alpha$ denote the restriction of T_α to coD. Then clearly for each $\alpha \in I$ and each $x \in coD$, $\hat{T}_\alpha(x) = T_\alpha(x)$ is a nonempty convex subset of D_α and for each $y_\alpha \in D_\alpha$,

$$\begin{aligned}\hat{T}_\alpha{}^{-1}(y_\alpha) &= \{x \in coD : y_\alpha \in \hat{T}_\alpha(x)\} \\ &= \{x \in coD : y_\alpha \in T_\alpha(x)\} \\ &= coD \cap \overset{-1}{T}{}_\alpha(y_\alpha) = \hat{O}_{y_\alpha}, \text{ say.}\end{aligned}$$

Clearly $\hat{O}_{y_\alpha}$ is a relatively open subset of coD. Hence by Theorem 2.1, for each $\alpha \in I$, there is a continuous selection $\hat{f}_\alpha : co\ D \to D_\alpha$ of $\hat{T}_\alpha$, i.e., $\hat{f}_\alpha(x) \in \hat{T}_\alpha(x) = T_\alpha(x)$ for each $x \in co\ D$. Now we define $\hat{f} : coD \to D$ and $T: co\ D \to 2^D$ respectively by $\hat{f}(x) = \prod_{\alpha \in I} \hat{f}_\alpha(x)$ and $T(x) = \prod_{\alpha \in I} \hat{T}_\alpha(x) = \prod_{\alpha \in I} T_\alpha(x)$, $x \in co\ D$. Clearly $\hat{f}$ is continuous and so by Theorem 4.5.1. of Smart (1974), there exists a point $\bar{x} \in D$ such that $\bar{x} = \hat{f}(\bar{x}) \in T(\bar{x})$.

COROLLARY 2.2. Let $\{X_\alpha : \alpha \in I\}$ be a family of nonempty convex sets, each in a Hausdorff locally convex space E_α, in which I is an indexing set. For each $\alpha \in I$, let D_α be a nonempty compact subset of X_α and $S_\alpha : X = \prod_{\alpha \in I} X_\alpha \to 2^{D_\alpha}$ be a set valued mapping such that

(a) for each $x \in X, S_\alpha(x) \neq \emptyset$

(b) for each $y_\alpha \in D_\alpha, S_\alpha^{-1}(y_\alpha)$ is relatively open in X.

Then there exists a point $\bar{x} \in D = \prod_{\alpha \in I} D_\alpha$ such that $\bar{x} \in T(x) = \prod_{\alpha \in I} co\ S_\alpha(x)$, i.e., $\bar{x}_\alpha \in co\ S_\alpha(x)$ for

$$\bar{u}(y)=\begin{cases} h'(y)u(h^{-1}(y)) & \text{for } y\in]c,d[, \\ 0 & \text{for } y\in\mathbb{R}-]c,d[. \end{cases}$$

To assure that each orbit of any flow from $Fl(f)$ crosses the set of branched points of f, we assume additionally that there exist a', b' such that

$$\int_{a'}^{b'} \alpha(s)ds > \sigma^{-1}(1) \tag{4.3}$$

where σ is defined in Example 3.3.

DEFINITION 4.5

We say that a vector field v_1 is a *modification* of v_0 and write v_1=MOD$(U,\eta)v_0$ if there exist α, h, λ, Ω as above such that $v_1=v_0$ outside U and $v_1=f$ in U in local coordinates given by λ.

To analyze properties of v_1 note that f has Fc-property: for any $\Phi\in Fl(\bar{u})$, it generates the flow

$$\varphi(t,(x,y))=(x+t,\ \Phi(\int_0^t \alpha(x+s)ds, y)) . \tag{4.4}$$

Vector fields $\langle 1,0\rangle$ and f are different only inside the rectangle Ω. It is easy to see that $\mathcal{R}$ = $]a,b[\times\mathcal{R}(\bar{u})$ is the set of f-strong branched points. Condition (4.3) implies that the orbits of any flow generated by f, which pass across Ω have nonempty intersection with the set $\mathcal{R}$. It is easy to see that flows generated by f which are of the form (4.4) are conjugate with the unit flow $\mathbf{1}_2$ by the following homeomorphism Λ:

$$\Lambda(x,y) = (x,\ \Phi(\int_{-\alpha}^{x} \alpha(x+s)ds,\ y)),\ \text{for } (x,y)\in\mathbb{R}^2.$$

DEFINITION 4.6

We say that a vector field V has *α-property* if for any point $p\in\overline{\mathcal{R}(v)}$ there exists a connected neighborhood $U(p)$ and a local map $\lambda:U(p)\to\lambda(U(p))\subseteq\mathbb{R}^2$ such that V has the coordinates $\langle 1,0\rangle$ in the map λ.

Observe that if a vector field V has the α-property, p and U(p) are as in the above definition, then MOD$(U(p),\eta)$V has the α-property. The operation MOD$(U(p),\eta)$ depends on the choice of the local map λ. We can choose λ in such a way that $p\in\mathcal{R}$(MOD$(U(p),\eta)$V). In the following we shall always choose such a λ.

STEP 2

We choose a countable dense set $P=\{p_n:\ n\in\mathbb{N}\}$ in $\mathbb{R}^2$ and start from the vector field V_1 with coordinates $\langle 1,0\rangle$ which obviously has α-property. Let V_n denote the vector field obtained in n-th iteration. As the next iteration we take V_{n+1}=MOD$(U(p_n),\eta_n)V_n$ if $p_n\notin\overline{\mathcal{R}(V_n)}$ and $V_{n+1}=V_n$ otherwise.

LEMMA 4.7

Parameters η_n of the MOD-operation can be chosen small enough, so that

a) the sequence (V_n) converges uniformly on $\mathbb{R}^2$ (in the sense of the uniform convergence of coordinates in the canonical map in $\mathbb{R}^2$).

b) each of V_n has the Fc-property.

$$= \left[P_\alpha^{-1}(y_\alpha) \cup F_\alpha\right] \cap A_\alpha^{-1}(y_\alpha).$$

We note that the first inequality follows from the fact that for each $y_\alpha \in D_\alpha$, $P_\alpha^{-1}(y_\alpha) \subset (coP_\alpha)^{-1}(y_\alpha)$ because $P_\alpha(x) \subset (coP_\alpha)(x)$ for each $x \in X$. Furthermore, by virtue of (iv), for each $y_\alpha \in D_\alpha, T_\alpha^{-1}(y_\alpha)$ contains a relatively open set O_{y_α} of X such that $\bigcup_{y_\alpha \in D_\alpha} O_{y_\alpha} = co\ D$. Hence by Theorem 2.2 there exists a point $\bar{x} = \{\bar{x}_\alpha\}$ such that $\bar{x}_\alpha \in T_\alpha(\bar{x})$ for each $\alpha \in I$. By condition (v) and the definition of T_α, it now easily follows that $\bar{x} \in X$ is an equilibrium point of $\mathcal{E}$.

COROLLARY 3.1. Let $\mathcal{E} = \{X_\alpha, P_\alpha, A_\alpha; \alpha \in I\}$ be an abstract economy such that for each $\alpha \in I$, the following conditions hold:

(i) X_α is convex;

(ii) D_α is a nonempty subset of X_α;

(iii) for each $x \in X, A_\alpha(x)$ is a nonempty convex subset of D_α;

(iv) the set $G_\alpha = \{x \in X: P_\alpha(x) \cap A_\alpha(x) \neq \emptyset\}$ is a closed subset of X;

(v) for each $y_\alpha \in D_\alpha, P_\alpha^{-1}(y_\alpha)$ is a relatively open subset in G_α and $A_\alpha^{-1}(y_\alpha)$ is a relatively open subset in X;

(vi) for each $x = \{x_\alpha\} \in X$, $x_\alpha \notin co\ P_\alpha(x)$ for each $\alpha \in I$.

There there is an equilibrium point of the economy $\mathcal{E}$.

PROOF. Since $P_\alpha^{-1}(y_\alpha)$ is relatively open in G_α, there is an open subset U_α of X with $P_\alpha^{-1}(y_\alpha) = G_\alpha \cap U_\alpha$. Hence for $y_\alpha \in D_\alpha, P_\alpha^{-1}(y_\alpha) \cup F_\alpha = (G_\alpha \cap U_\alpha) \cup F_\alpha = X \cap (U_\alpha \cup F_\alpha)$. Thus

$$\{P_\alpha^{-1}(y_\alpha) \cup F_\alpha\} \cap A_\alpha^{-1}(y_\alpha) = (U_\alpha \cup F_\alpha) \cap A_\alpha^{-1}(y_\alpha) = Oy_\alpha, \text{ Say,}$$

is a relatively open subset of X for each $y_\alpha \in D_\alpha$, since U_α, F_α and $A_\alpha^{-1}(y_\alpha)$ are open subsets of X. Now it follows (e.g., see Remark 3.1 in Tarafdar [7]) that $\bigcup_{y_\alpha \in D_\alpha} O_{y_\alpha} = co\ D$. The corollary is thus a consequence of Theorem 3.1.

THEOREM 3.2. Let $\Gamma = \{X_\alpha, P_\alpha : \alpha \in I\}$ be a qualitative game such that for each $\alpha \in I$, the following conditions hold:

(i) X_α is convex;

(ii) D_α is a nonempty compact convex subset of X_α;

(iii) for each $x_\alpha \in D_\alpha, \{P_\alpha^{-1}(x_\alpha) \cup F_\alpha\}$ contains a relatively open subset O_{x_α} of $co\ D$ such that $\bigcup_{x_\alpha \in D_\alpha} O_{x_\alpha} = co\ D$, where

$$F_\alpha = \{x \in X: P_\alpha(x) = \emptyset\};$$

(iv) for each $x = \{x_\alpha\} \in X, x_\alpha \notin co\ P_\alpha(x)$.

Then there is a maximal element of the game Γ.

PROOF. For each $\alpha \in I$, we define the set valued map $A_\alpha : X \to 2^{D_\alpha}$ by $A_\alpha(x) = D_\alpha$ for each $x \in X$. Now Theorem 3.1 applies.

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