Internat. J. Math. & Math. Sci.
VOL. 18 NO. 2 (1995) 317-322

AN APPLICATION OF THE SAKAI'S THEOREM TO THE CHARACTERIZATION OF H^*-ALGEBRAS

BORUT ZALAR

University of Ljubljana
Institute of Mathematics, Jadranska 19
61000 Ljubljana, SLOVENIJA

(Received May 11, 1993)

ABSTRACT. The well-known Sakai's theorem, which states that every derivation acting on a von Neumann algebra is inner, is used to obtain a new elegant proof of the Saworotnow's characterization theorem for associative H^*-algebras via two-sided H^*-algebras. This proof completely avoids structure theory.

KEY WORDS AND PHRASES. H^*-algebra, involution, automorphism, derivation, centralizer, von Neumann algebra.

1991 AMS SUBJECT CLASSIFICATION CODES. 16K15, 46H20, 46L10, 46L57.

1. PRELIMINARIES.

One of the important (and very deep) results in the theory of von Neumann algebras states that all derivations of von Neumann algebras are inner. This result was established by Sakai in [7].

SAKAI'S THEOREM. *Let $\mathcal{R}$ be a von Neumann algebra and D derivation of $\mathcal{R}$. Then there exists some $a \in \mathcal{R}$ such that $D(x) = ax - xa$ for all $x \in \mathcal{R}$.*

This result has many applications and we present another one. We show how the Sakai's theorem can be used to give a new proof of the characterization of H^*-algebras in the terms of two-sided H^*-algebras. This characterization was established by Saworotnov in [8] who described the structure of two-sided H^*-algebras and then compare it with the structure of (one-sided) H^*-algebras obtained by Ambrose in [1].

The theory of associative H^*-algebras started in 1945 when Ambrose introduced this concept in order to obtain an abstract characterization of the Hilbert-Schmidt operators. Later various generalizations appeared such as two-sided H^*-algebras, complemented algebras and Hilbert modules. We refer to [8], [9], [10], [11] and [12]. Nonassociative H^*-algebras were also studied. Some recent papers on this subject are [3], [4], [5], [6] and [13]. Ambrose introduced the following

DEFINITION. A complex algebra $\mathcal{A}$ which is also a Hilbert space with the inner product $\langle\, ,\, \rangle$ and an algebra involution $* : \mathcal{A} \longrightarrow \mathcal{A}$ is called an *H^*-algebra* if the following H^*-conditions

$$\langle xy, z\rangle = \langle x, zy^*\rangle = \langle y, x^* z\rangle$$

are satisfied for all $x, y, z \in \mathcal{A}$. A model for such algebra is the algebra $\mathbf{HS}(\mathcal{H})$ of all Hilbert-Schmidt operators acting on a Hilbert space $\mathcal{H}$ together with the inner product $\langle T, S\rangle = \mathrm{trace}(TS^*)$.

If we take a look at the H^*-conditions, we can ask what happens if the algebra $\mathcal{A}$ would have two involutions: one left and one right. This leaded to the following:

DEFINITION. A complex algebra $\mathcal{A}$ which is also a Hilbert space with an inner product $\langle\, ,\, \rangle$ is called a *two-sided H^*-algebra* if for every $x \in \mathcal{A}$ there exist elements $x^r, x^l \in \mathcal{A}$ called a right and a left adjoint of the element x such that

$$\langle xy, z\rangle = \langle y, x^l z\rangle, \qquad \langle yx, z\rangle = \langle y, zx^r\rangle$$

holds for all $y, z \in \mathcal{A}$.

A model for a two-sided H^*-algebra which is not an H^*-algebra can be found in [8]. However Saworotnow proved that two-sided H^*-algebras are very close to H^*-algebra. Explicitly he proved

THEOREM. *Let $(\mathcal{A}, \langle\, ,\, \rangle, l, r)$ be a two-sided associative H^*-algebra with zero annihilator. There exists another inner product $\langle\, ,\, \rangle_1$ on $\mathcal{A}$ such that $(\mathcal{A}, \langle\, ,\, \rangle_1, r)$ becomes an H^*-algebra.*

Before we start our new proof of this result we introduce the following notation for the annihilators of $\mathcal{A}$:

left annihilator : $\mathrm{Lann}(\mathcal{A}) = \{x \in \mathcal{A} : x\mathcal{A} = (0)\}$,

right annihilator : $\mathrm{Rann}(\mathcal{A}) = \{x \in \mathcal{A} ; \mathcal{A}x = (0)\}$,

annihilator : $\mathrm{Ann}(\mathcal{A}) = \mathrm{Lann}(\mathcal{A}) \cap \mathrm{Rann}(\mathcal{A})$.

2. RESULTS.

First we shall collect some simple observations on two-sided H^*-algebras in the following:

LEMMA. *Let $(\mathcal{A}, \langle\, ,\, \rangle, l, r)$ be a two-sided associative H^*-algebra with zero annihilator. Then the following holds:*

(i) $\mathrm{Lann}(\mathcal{A}) = \mathrm{Rann}(\mathcal{A}) = (0)$.

(ii) A left and right adjoint of any element are unique and the mappings $x \longmapsto x^r$, $x \longmapsto x^l$ *are both algebraic involutions.*

(iii) The product of $\mathcal{A}$ *is continuous.*

(iv) $\mathcal{A}^2$ *is dense in* $\mathcal{A}$.

(v) Both involutions $x \longmapsto x^r$ *and* $x \mapsto x^l$ *are continuous.*

(vi) $\langle x, y\rangle = \langle y^r, x^l\rangle = \langle y^l, x^r\rangle$ *holds for all* $x, y \in \mathcal{A}$.

Proof. (i) Suppose that $a \in \mathrm{Lann}(\mathcal{A})$. Take any $x, y \in \mathcal{A}$. Then we have

$$\langle xa, y\rangle = \langle a, x^l y\rangle = \langle ay^r, x^l\rangle = 0$$

where x^l is some left adjoint of x and y^r some right adjoint of y (note that we don't need the uniqueness of the left and right adjoint in this moment). This implies $a \in \mathrm{Rann}(\mathcal{A})$ and therefore $\mathrm{Lann}(\mathcal{A}) \subset \mathrm{Rann}(\mathcal{A})$. In a similar way we get $\mathrm{Rann}(\mathcal{A}) \subset \mathrm{Lann}(\mathcal{A})$ and therefore left and right annihilator of $\mathcal{A}$ are both equal to the annihilator of $\mathcal{A}$ which is zero by assumption.

(ii) Let a^{l_1} and a^{l_2} be two left adjoints of the element a. Take any $x, y \in \mathcal{A}$. Then we have

$$\langle x, (a^{l_1} - a^{l_2})y\rangle = \langle x, a^{l_1} y\rangle - \langle x, a^{l_2} y\rangle = \langle ax, y\rangle - \langle ax, y\rangle = 0.$$

Using (i) we get $a^{l_1} = a^{l_2} = a^l$. In a similar way we can prove that the right adjoint is unique and

$$a^{ll} = a^{rr} = a, \quad (\lambda a)^r = \overline{\lambda} a^l, \quad (\lambda a)^r = \overline{\lambda} a^r$$

holds for all $a \in \mathcal{A}$ and all complex λ. Also we have for all $a, b, x, y \in \mathcal{A}$

$$\langle ((ab)^l - b^l a^l)x, y\rangle = \langle (ab)^l x, y\rangle - \langle b^l a^l x, y\rangle =$$

$$= \langle x, aby\rangle - \langle a^l x, by\rangle = \langle x, aby\rangle - \langle x, aby\rangle = 0.$$

(iii) The operators $L_a(x) = ax$ and $R_a(x) = xa$ both have adjoints (as operators acting on a Hilbert space $\mathcal{A}$) and are therefore continuous. By standard arguments we can now prove the continuity of the product.

(iv) Take any $a \in \mathcal{A}^{2\perp}$. For all $x, y \in \mathcal{A}$ we have

$$\langle ax, y\rangle = \langle a, yx^r\rangle = 0$$

and using (i) we get $a = 0$.

(v) An easy application of the closed graph theorem.

(vi) First we take $y = ab$. Then we have

$$\langle x, y\rangle = \langle x, ab\rangle = \langle a^l x, b\rangle =$$

$$= \langle a^l, bx^r\rangle = \langle b^l a^l, x^r\rangle = \langle (ab)^l, x^r\rangle = \langle y^l, x^r\rangle.$$

By (iv), (v) and the continuity of the inner product the above identity holds for all $x, y \in \mathcal{A}$.

The following proposition contains an application of the Sakai's theorem and will be crucial in the proof of Theorem.

PROPOSITION. *Let $\mathcal{A}$ be as above and $D : \mathcal{A} \longrightarrow \mathcal{A}$ a continuous derivation. Then there exist continuous linear operators T_0 and S_0 such that the following holds:*

(i) $D = T_0 - S_0$,
(ii) $T_0(xy) = T_0(x)y$ for all $x, y \in \mathcal{A}$,
(iii) $S_0(xy) = xS_0(y)$ for all $x, y \in \mathcal{A}$,
(iv) $xT_0(y) = S_0(x)y$ for all $x, y \in \mathcal{A}$.

If D is self-adjoint (as an operator acting on a Hilbert space), T_0 and S_0 can also be taken self-adjoint.

REMARK. In the theory of C^*-algebras the pair (T_0, S_0) is usually called a *double centralizer*.

PROOF. Define $L(\mathcal{A})$ and $R(\mathcal{A})$ to be the algebras of continuous left and right multipliers i.e.

$$L(\mathcal{A}) = \{\ T \in B(\mathcal{A})\ ;\ T(xy) = T(x)y \ \text{ for all } \ x, y \in \mathcal{A}\ \},$$

$$R(\mathcal{A}) = \{\ S \in B(\mathcal{A})\ ;\ S(xy) = xS(y) \ \text{ for all } \ x, y \in \mathcal{A}\ \}.$$

Then $L(\mathcal{A})$ and $R(\mathcal{A})$ are von Neumann algebras and $L(\mathcal{A})' = R(\mathcal{A})$ holds.

It is easy to see that $L(\mathcal{A})$ an $R(\mathcal{A})$ are algebras with identity. From Lemma(iii) we get that $L(\mathcal{A})$ and $R(\mathcal{A})$ are closed in the strong operator topology. We must see that they are also selfadjoint as subalgebras of $B(\mathcal{A})$. Take any continuous left multiplier T and compute

$$\langle T^*(xy), z\rangle = \langle xy, T(z)\rangle = \langle x, T(z)y^r\rangle =$$

$$= \langle x, T(zy^r)\rangle = \langle T^*(x), zy^r\rangle = \langle T(x)y, z\rangle.$$

Hence T^* is a left multiplier and so $L(\mathcal{A})$ is a von Neumann algebra. In a similar way we can prove that $R(\mathcal{A})$ is also a von Neumann algebra.

Take some continuous left multiplier T and some continuous right multiplier S. By

$$TS(xy) = T(xS(y)) = T(x)S(y) = S(T(x)y) = ST(xy)$$

and Lemma(iv), T and S commute. This shows that $R(\mathcal{A}) \subset L(\mathcal{A})'$. To prove the converse observe that operators $L_x : y \mapsto xy$ are continuous left multipliers for all $x \in \mathcal{A}$. Take any $S \in L(\mathcal{A})'$. Then

$$S(xy) = SL_x(y) = L_xS(y) = xS(y)$$

shows that S is a right multiplier.

Now define $\Delta : \mathcal{L}(\mathcal{A}) \longrightarrow \mathcal{L}(\mathcal{A})$ by $\Delta(T) = DT - TD$. We must show that $\Delta(T)$ is in fact left multiplier:

$$\Delta(T)(xy) = DT(xy) - TD(xy) = D(T(x)y) - T(D(x)y + xD(y)) =$$

$$= (DT)(x)y + T(x)D(y) - (TD)(x)y - T(x)D(y) = \Delta(T)(x)y.$$

Straightforward verification shows that Δ is indeed a derivation. Now we use a well-known result that every derivation of a von Neumann algebra is inner. There exists a left multiplier T_0 such that

$$\Delta(T) = T_0T - TT_0 = DT - TD.$$

Thus for all $T \in \mathcal{L}(\mathcal{A})$ we have

$$(T_0 - D)T = T(T_0 - D).$$

So $S_0 = T_0 - D \in \mathcal{L}(\mathcal{A})' = \mathcal{R}(\mathcal{A})$. We must see that $xT_0(y) = S_0(x)y$ holds for all $x, y \in \mathcal{A}$:

$$xT_0(y) - S_0(x)y = xT_0(y) - xS_0(y) + xS_0(y) - T_0(x)y + T_0(x)y - S_0(x)y =$$

$$= xD(y) + S_0(xy) - T_0(xy) + D(x)y = xD(y) - D(xy) + D(x)y = 0.$$

Finally suppose that D is self-adjoint. From the fact that left and right multipliers are closed for adjoints (the second paragraph of this proof), it follows that

$$D = D^* = \frac{1}{2}(T_0 + T_0^*) - \frac{1}{2}(S_0 + S_0^*)$$

is a self-adjoint decomposition with properties (i)-(iv).

Now we are ready for the final step.

PROOF OF THE THEOREM. Define a linear operator Φ on $\mathcal{A}$ with $\Phi(x) = (x^r)^l$. By Lemma(v) Φ is continuous and by Lemma(iii) Φ is an automorphism of $\mathcal{A}$. Using Lemma(vi) we obtain

$$\langle \Phi(x), x\rangle = \langle x^{rl}, x\rangle = \langle x^r, x^{rll}\rangle = \|x^r\|^2 \geq 0$$

which shows that Φ is selfadjoint and that its spectrum is contained in $\mathbb{R}^+$. By [2, Theorem 15, page 91] $\Phi = \exp(D)$ for some continuous (obviously self-adjoint) derivation D of $\mathcal{A}$.

According to the Proposition, we may decompose $D = T_0 - S_0$. Since D is selfadjoint, we can also choose T_0 and S_0 both selfadjoint. Define $U = \exp(T_0)$ and $V = \exp(S_0)$. Then U and V are positive invertible operators and $\Phi = UV^{-1}$ holds. U is a left multiplier, V is a right multiplier and therefore they commute. It is easy to see that $xU(y) = V(x)y$ holds for all $x, y \in \mathcal{A}$ (compare Proposition(iv)).

Now define $\langle x, y\rangle_1 = \langle U(x), y\rangle$. It remains to verify that $(\mathcal{A}, \langle\ ,\ \rangle_1, r)$ is an H^*-algebra. This is done in the following computation:

$$\langle x, zy^r\rangle_1 = \langle U(x), zy^r\rangle = \langle U(x)y, z\rangle =$$

$$= \langle U(xy), z\rangle = \langle xy, z\rangle_1,$$

$$\langle y, x^r z\rangle_1 = \langle U(y), x^r z\rangle = \langle x^{rr}U(y), z\rangle =$$

$$= \langle \Phi(x)U(y), z\rangle = \langle UV^{-1}(x)U(y), z\rangle = \langle VUV^{-1}(x)y, z\rangle =$$

$$= \langle U(x)y, z\rangle = \langle U(xy), z\rangle = \langle xy, z\rangle_1.$$

The proof is thus completed.

REMARK. It is not known whether the nonassociative H^*-algebras can be characterized in the same way. Saworotnow's original proof is entirely associative. In our proof the associativity is used once in the proof of the Proposition namely when we proved that $L(\mathcal{A})$ is a commutant of $R(\mathcal{A})$. This is true if and only if $\mathcal{A}$ is associative.

REFERENCES

1. W. AMBROSE, Structure theorems for a special class of Banach algebras, Trans. Amer. Math. Soc. 57 (1945), 364-386.

2. F.F. BONSAL and J. DUNCAN, Complete normed algebras, Ergeb. Math. Grenzgeb. 80 (1973)

3. M. CABRERA, J. MARTINEZ and A. RODRIGUEZ, Nonassociative real H^*-algebras, Publ. Mat. 32 (1988), 267-274

4. M. CABRERA, J. MARTINEZ and A. RODRIGUEZ, Mal'cev H^*-algebras, Math. Proc. Camb. Phil. Soc. 103 (1988), 463-471

5. M. CABRERA, J. MARTINEZ and A. RODRIGUEZ, Structurable H^*-algebras, J. Algebra 147 (1992), 19-62

6. J.A. CUENCA, A. GARCIA and C. MARTIN, Structure theory for L^*-algebras, Math. Proc. Camb. Phil. Soc. 107 (1990), 361-365

7. S. SAKAI, Derivations of W^*-algebras, Ann. Math. 83 (1966), 273-279

8. P.P. SAWOROTNOW, On generalization of notition of H^*-algebra, Proc. Amer. Math. Soc. 8 (1957), 56-62

9. P.P. SAWOROTNOW, On two-sided H^*-algebras, Pac. J. Math 16 (1966), 365-370

10. P.P. SAWOROTNOW, A generalized Hilbert space, Duke Math. J. 35 (1968), 191-197

11. P.P. SAWOROTNOW, Representation of a complemented algebra on a locally compact space, Illinois J. Math. 14 (1970), 674-677

12. J.F. SMITH and P.P. SAWOROTNOW, On some classes of scalar-product algebras, Pac. J. Math. 11 (1961), 739-750

13. B. ZALAR, Continuity of derivations on Mal'cev H^*-algebras, Math. Proc. Camb. Phil. Soc. 110 (1991), 455-459