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NON-HOMOGENEOUS MARKOV CHAINS WITH A FINITE STATE SPACE AND A DOEBLIN TYPE THEOREM

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ABSTRACT. Doeblin [1] considered some classes of finite state nonhomogeneous Markov chains and studied their asymptotic behavior. Later Cohn [2] considered another class of such Markov chains (not covered earlier) and obtained Doeblin type results. Though this paper does not present the "best possible" results, the method of proof will be of interest to the reader. It is elementary and based on Hajnal's results on products of nonnegative matrices.

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1. INTRODUCTION.

Let $\{X_n: x \geq 0\}$ be a non-homogeneous Markov chain with finite state space $E = \{1, 2, \cdot, S\}$ defined on some probability space (Ω, Σ, P). Let (P_n) be the sequence of transition probability (s by s) matrices such that $(P_n)_{ij} =$ the entry on the ith row and jth column of $P = P(X_{n+1} = j \mid X_n = i), (P_{m,n})_{ij} = (P_{m+1}P_{m+2} \cdots P_n)_{ij} = P(X_{n+1} = j \mid X_{m+1} = i), 0 \leq m < n$. [It will be assumed that the matrices P_n are all stochastic, i.e., every row sum is one; this means that when $P(X_n = i) = 0$, the ith row of P_n can be defined in any way as long as it is nonnegative and has sum 1.] In [1], Doeblin considered classes of non-homogeneous Markov chains satisfying condition (A): $\exists$ a positive number $\delta \ni \forall (i, j) \in E \times E$, either $(P_n)_{ij} > \delta \ \forall\, n$ or $(P_n)_{ij} = 0 \ \forall\, n$. He also studied more general chains:

CONDITION (B). $\exists$m $\delta > 0$ and some positive integer $N \ni \forall (i, j) \in E \times E$, either $(P_n)_{ij} > \delta$ for $n > N$ or $lim (P_n)_{ij} = 0$ as $n \to \infty$.

Cohn [2] made a detailed study of Doeblin's paper [1] and these conditions in the context of Doeblin type results. Cohn [2] also studied chains satisfying conditions even more general than Doeblin's. The most general condition studied in Cohn's paper is:

CONDITION (B*). $\exists\, \delta > 0 \ni lim\ max\{(P_n)_{ij} \mid i, j \ni (P_n)_{ij} < \delta\} = 0$ as $n \to \infty$.

The aim of this paper is to study non-homogeneous Markov chains satisfying conditions essentially different from the above conditions (where one does not require any kind of limit for the sequence $(P_n)_{ij}$ or the sequence $max\ ((P_n)_{ij}: (i, j)\ \varepsilon\ A_n)$, A_n in E) in the context of Doeblin theory. For example, if one considers a non-homogeneous Markov chain where the transition matrices (P_n) satisfy for some $(i, j) \in E \times E$ the condition: $(P_{k(n)})_{ij} > \varepsilon_k > 0$, $k(n) = k^n$, $n > 0$, where k is a prime integer and $lim\ \varepsilon_k = 0$ as $k \to \infty$, then this chain does not belong to the classes of chains studied in [2,3]. As one will see shortly, these chains (for $\varepsilon_k = 1/log\ k$)) are a type of chains that will satisfy the condition $(*)$ below that define the chains studied in this paper.

In this paper, Doeblin type results are obtained for non-homogeneous Markov chains satisfying the following condition:

CONDITION (*). For any $(i, j) \in E \times E$, either $(P_n)_{ij} = 0 \ \forall\, n$, or for n sufficiently large, $(P_n)_{ij} \geq 1/(log\ n)$.

As will be clear from the proof, results of this paper actual holds under conditions more general than $(*)$. The present method of proof is different, and will be of interest to the reader.

2. PRELIMINARIES.

Throughout this and the next section, we will assume that the P_n's have the same skeleton, i.e., either $(P_n)_{ij} = 0\ \forall\ n \geq 1$ or $(P_n)_{ij} > 0\ \forall\ n \geq 1$. Define that $i \rightarrow j$ if $P(X_n = j \mid X_0 = i) > 0$ for some $n \geq 1$. If $i \rightarrow i$, i is self-communicating and define the period of $i, d(i) = g.c.d$ $\{n \mid (P_{k,k+n})_{ii} > 0$ for some $k \geq 0\}$. In the parenthesis above the phrase "for some $k \geq 0$" can be replaced by "$\forall\ k > 0$" without changing the definition since the P_n's have the same skeleton. Note that it is easily proven that the set $F = \{i \in E \mid i \rightarrow i\}$ is a nonempty subset of E(since E is finite). A state i, as usual, is called essential if $i \rightarrow j \Rightarrow j \rightarrow i$. A state which is not essential is called unessential. All states in $E - F$ are unessential. As in the homogeneous case, F is partitioned into equivalence classes with respect to the equivalence relation irj iff $i \rightarrow j$ and $j \rightarrow i$. Then it is easily verified that all states within the same class have the same period. Also, in class G_α with period d_α and any two states $i, j \in G_\alpha$, $\exists! r_{ij} \ni 0 \leq r_{ij} < d_\alpha$ and $(P_{m,n})_{ij} > 0 \Rightarrow n - m = r_{ij}(mod$ $d_\alpha)$. [Recall: $P_{m,n} = P_{m+1}P_{m+2} \cdots P_n, m < n$]. Also, each class G_α with period d_α can be partitioned into sub-classes $C_j, j = 1, 2, \cdots d_\alpha, \ni$ if $i \in C_{t1}$ and $j \in C_{t2}$ then $(p_{m,n})_{ij} > 0 \Rightarrow n - m = t_2 - t_1(mod d_\alpha)$. [The proofs of the above assertions are the same as the homogeneous case in Chung's book [3]]. In the proof of our theorem, we need to apply Hajnal's weak ergodicity result in [4]. We explain what it is. A nonnegative square matrix is called *allowable* if $\exists$ at least one positive entry in each row and each column. For an allowable matrix P, Hajnal [4] defined $\Phi(P)$ as :

$$\Phi(P) = min\ \frac{P_{ik}P_{jl}}{P_{jk}P_{il}},\ \forall i, j, k, \ell \text{ if } P \text{ has all entries positive,}$$

$$= 0, \text{ otherwise}$$

A sequence of sxs nonnegative matrices is called weakly ergodic if for each $m \geq 0$ and any i, j, k in the state space $\frac{(P_{m,n})_{ij}}{(P_{m,n})_{kj}} \rightarrow V^{(m)}{}_{ik}$ as $n \rightarrow \infty$, where the $V^{(m)}{}_{ik}$'s are independent of j. We need the following theorem: Theorem (Hajnal). A sequence of allowable matrices is weakly ergodic if $\exists$ a strictly increasing sequence of integers $(r_m) \ni \Sigma_{m=1} \sqrt{\Phi(P_{rm,rm+1})} = \infty$.

3. MAIN RESULTS.

We now state the main theorem:

THEOREM 3.1. Let (P_n) be a sequence of $s \times s$ stochastic matrices with state space S such that they all have the same skeleton. Let us assume the following condition: "For each $i \in S$, let $E_i = \{j \in S : i \leftrightarrow j\}$. Then for any two states $u, v \in E_i$, either $(P_n)_{uv} = 0$ for all n or for sufficiently large n, $(P_n)_{uv} \geq 1/(log\ n)$. "Then the following results hold: the state space S can be partitioned as $S = T_0 U(\underset{\alpha}{U} E_\alpha) U(\underset{\beta}{U} I_\beta)$, where T_0 contains all the non-self communicating elements, E_α's the essential self-communicating classes and the I_β's the inessential self-communicating classes. Each E_α can further be partitioned into cyclical subclasses $E_\alpha = \bigcup_{u=1}^{d(\alpha)} E_{\alpha u}, d(\alpha)$ being the period of E_α. Similarly, each I_β can be partitioned as $I_\beta = \bigcup_{v=1}^{\alpha(\beta)} I_{\beta v}$, where $d(\beta)$ is the period of I_β. Also

(i) $lim(P_{m,n})_{ij} =$ for $m \geq 0$ for all i, if j in T_0 as $n \rightarrow \infty$

(ii) $(P_{m,n})_{ij} = 0$ for $m < n$ if i in E_α and j not in E_α.

(iii) $(P_{m,m})_{ij} = 0$ of $n - m \neq v - u(mod\ d(\alpha))$, whenever j in $E_{\alpha u}, i$ in $E_{\alpha v}$.

A similar result holds when i in $I_{\beta v}$ and j in $I_{\beta u}$.

(iv) If $i \in E_{\alpha u}$, $j \in E_{\alpha v}$, and $n - m = v - u(mod\ d(\alpha))$, then $(P_{mn})_{ij} = (P_n)_j + (\varepsilon_{m,n})_{ij}$, where $lim(\varepsilon_{m,n})_{ij} = 0$ and $lim\ \Sigma_{j \epsilon E_{\alpha v}}(P_n)_{ij} = 1$ as $n \rightarrow \infty$.

(v) If $i, k \in I_{\beta u}$ and $j \in I_{\beta v}, n-m = v-u(mod\ d(\beta))$, then $lim[(P_{m,n})_{ij}/(P_{m,n})_{kj}] = v^{(m)}{}_{ik}$ as $n \to \infty$.

(vi) Let $j \in E_{\alpha u}$, $1 \leq u \leq d(\alpha)$. Then for $i \in S$, $(P_{m,n})_{ij} = (P_n)_j \cdot \Sigma_{k \in E_{\alpha u}}(P_{m,n})_{ik} + (\varepsilon_{m,n})_{ik} + (\varepsilon_{m,n})_{ij}$ and $lim(\varepsilon_{m,n})_{ij} = 0$ as $n \to \infty$.

The idea of the proof is the following. First, to find a useful estimate of the integer N (and this is one of the crucial steps in my proof) with the following property: $(P_{k,k+nd})_{ii} > 0$ whenever $n \geq N$ where d is the period of the element $i, i \notin T_0$. (The estimate is in terms of d and the number a which is the number of elements in the class containing i). The second step is to consider restrictions of the sequence of blocks (each block is a product of length $d(\alpha)$) $P_{m,m+d(\alpha)}$ to an essential class (with period $d(\alpha)$); these restrictions are allowable nonnegative matrices and then use Hajnal's theorem to this sequence after estimating the Φ function (given in Hanjal's theorem) based on the estimate that I have obtained in the first step. The third step is consider a similar procedure for the unessential classes.

PROOF. We discuss the proof in several parts.

(1) Let a, b be positive integers and $0 < a < b$, $g.c.d\ \{a, b\} = (a,b) = d$. Then there exist integers u and v such that $ua + vb = d$ and $|v| \leq u \leq b$.

PROOF of (1). With no loss of generality, we can assume that $d = 1$. It is known that there are integers s and t such that

$$sa + tb = 1. \tag{3.1}$$

Let x be the greatest integer less than or equal to $\frac{b-s}{b}$. We claim that

$$|t - ax| \leq s + bx \leq b. \tag{3.2}$$

Notice that (3.2), once established, will complete the proof of (1), for

$$(s+bx)a + (t-ax)b = 1. \tag{3.3}$$

To establish (3.2), note first that

$$\frac{-s-t}{b-a} \leq \frac{-s}{b} \leq \frac{t-s}{a+b} \leq \frac{b-s}{b}. \tag{3.4}$$

Write, $|s| = bq + r, 0 \leq r < b$, where q and r are integers. Let $s > 0$. Then

$$\frac{b-s}{b} = 1 - q - \frac{r}{b} \text{ so that } \frac{b-s}{b} - x \leq 1 - \frac{r}{b} \leq 1 - \frac{1}{b(a+b)} = \frac{b-s}{b} - \frac{t-s}{a+b}. \Rightarrow$$

$$\frac{t-s}{a+b} \leq x \leq \frac{b-s}{b}. \tag{3.5}$$

If $s < 0$, then $\frac{b-s}{b} = 1 + q + \frac{r}{b} \Rightarrow \frac{b-s}{b} - x = \frac{r}{b} \leq 1 - \frac{1}{b(a+b)}$. (Since $r(a+b) < b(a+b) \Rightarrow r(a+b) \leq -1 + b(a+b)$). This means (3.5) holds. Note that (3.5) implies that

$$t - ax \leq s + bx \leq b. \tag{3.6}$$

Also, (3.4) implies that $\frac{-s-t}{b-a} \leq x$ or

$$ax - t \leq s + bx. \tag{3.7}$$

This establishes (3.2) and (1) is proven.

(2) Let $d = g.c.d.\{n_1, n_2, \cdots, n_k\}$, where $1 \leq n_1 < n_2 < \cdots < n_k$ are positive integers. Then $\exists$ positive integers $c_1, c_2, \cdots, c_k$ such that

(i) $c_1 \geq c_2 \geq \cdots \geq c_k$

(ii) $c_1 n_1 - c_2 n_2 - \cdots - c_k n_k = d$

(iii) If $d_i = g.c.d.\{n_1, n_2, \cdots, n_i\}, 1 \leq i \leq k$, then $c_k \leq \frac{n_k}{d_k}$, $c_{k-1} \leq \frac{n_k n_{k-1}}{d_k d_{k-1}}$ etc. $c_i \leq \frac{n_i n_{i+1} \cdots n_k}{d_i d_{i+1} \cdots d_k}$.

PROOF OF (2). The proof follows easily using induction on k and (1).

(3) Let d be the period of a self-communicating class F and let a be the number of elements in this class. For a state i in this class, define the set $A(i) = \{n \in z^+ : (P_{k,k+n})_{ii} > 0$ for all $K\}$. Also, let $A(a) = \{n \in z^+ : n \leq a$ and $n \in A(j)$ for j in $F\}$. Then, $d = g.c.d.\ A(a)$.

PROOF OF (3). Notice that $d = g.c.d.\ A(j)$ for each $j \in F$. Hence, $d \mid d_0$, where $d_0 = g.c.d. A(a)$. Now, let $n \in A(i)$. Then, $(P_{k,k+n})_{ii} > 0$. If $n \leq a$, then $n \in A(a)$ and $d_0 \mid n$. Let $n > a$. Since i cannot lead to a state j outside F (the class containing i), which can then lead to a state in F, it is clear that one can write $n = n_1 + n_2 + \cdots + n_{\underline{1}}$, where each $n_t, 1 \leq t \leq \underline{1}$, is in $A(a)$. To see this, let $i = j_n$; then notice that $(P_{k,k+n})_{ii} = \Sigma (P_{k+1})_{ij_1} (P_{k+2})_{j_1 j_2} \cdots (P_{k+n}) j_{n-1} j_n$. If m is the smallest integer such that j_m appears at least twice and $j_m = j_{m+p}$, then $a \geq p$ and $n = p + (n-p)$, where$(P_{k+m,k+m+p})_{j_m j_m} > 0$ and $(P_{k,n-p+k})_{ii} > 0$. This process is repeated. So $d_0 \mid n$ since $d_0 \mid n_t, 1 \leq t \leq \underline{1}$.

(4) Let d be the period of a self-communicating class with a elements and $N = \left\{\left[\frac{a}{d}\right]! a\right\}^2$. $\forall n \geq N, (P_{k,k+nd})_{ii} > 0, \forall k$ and states i in this class.

PROOF of (4). Let i be a state in this class. First, consider the shortest path from i to i through all the other states in this class which can be described as follows: $j_0 = \underbrace{i \text{ to } j_1}_{s_1\text{-steps}}\ \underbrace{\text{to } j_2}_{s_2\text{-steps}}$ to etc.

$$\cdots \cdot \underbrace{j_v \text{ to } i}_{s_{v+1}\text{-steps}} = j_0$$

where all the $j_{\underline{1}}$'s are distinct and each $s_{\underline{1}} \leq a$. If the length of this shortest path is b, then $d \mid b$ and $b \leq a^2$. Note that the corresponding shortest path for any other state j in this class has also length b, since, for example, if $j = j_1$, then:

$$\underbrace{j_1 \text{ to } j_2}_{s_2\text{-steps}}\ \underbrace{\text{to } j_3}_{s_3\text{-steps}} \quad \text{to} \cdots \cdot \text{etc.} \quad \cdots \text{to } \underbrace{i \text{ to } j_1}_{s_1\text{-steps}}.$$

This information will be used later. Now, by step (3) $d = \text{g.c.d. } \{n_1, n_2, \cdots, n_t\}$, $n_{\underline{1}}$'s being distinct, each $n_{\underline{1}} \leq a$ and for each $n_{\underline{1}}$, $\underline{1} \leq 1 \leq t$, there is some state i in the given class (equivalent) $\ni (P_k, k + n_1)_{ii} > 0$. By part (2), $\exists$ positive integers $c_1 \geq c_2 \geq \cdots \geq c_t \ni d = c_1 n_1 - c_2 n_2 - \cdots - c_t n_t$. Let $N_0 d = c_1 n_2 + \cdots + c_t n_t$. Let $n \geq N_0(N_0 - 1)$. Then $n = a_1 N_0 (N_0 - 1) + a_2 N_0 + a_3$ where $a_1 \geq 1, a_2 \geq 0, 0 \leq a_3 < N_0$. Thus,

$$nd = a_1(N_0 - 1) \sum_{\underline{1}=2}^{t} c_{\underline{1}} n_{\underline{1}} + a_2 \sum_{\underline{1}=2}^{t} c_{\underline{1}} n_{\underline{1}} + a_3 \left(c_1 n_1 - \sum_{\underline{1}=2}^{t} c_{\underline{1}} n_{\underline{1}} \right)$$

$$= \sum_{\underline{1}=2}^{t} \{a_1(N_0 - 1) + a_2 - a_3\} c_{\underline{1}} n_{\underline{1}} + a_3 c_1 n_1.$$

Note that by part (2),

$$c_1 \leq \frac{n_1}{d} \cdot \frac{n_2}{d} \cdot \cdots \cdot \frac{n_t}{d} \leq \frac{2d.3d...\left[\frac{a}{d}\right]d}{d^t} \leq \left[\frac{a}{d}\right]!$$

$$\ni N_0 d(N_0 - 1) = (c_1 n_1 - d)(c_1 n_1 - 2d)\frac{1}{d} \leq \left(\left[\frac{a}{d}\right]! a - d\right)^2 \cdot \frac{1}{d}.$$

Note that if $md = \sum_{\underline{1}=1}^{t} c_{\underline{1}}{}^{(m)} n_{\underline{1}}$, $c_{\underline{1}}{}^{(m)} \geq 0$, then $(P_{k,k+md+b})_{ii} > 0 \forall$ states i in the class. [The reason is the following: Considering the shortest path of length b from i to i through all the states in the class i to j_1 to j_2 to $\cdots$etc$\cdots$to j_v to i. Attach to this path an extra $m.d$ steps in the most obvious manner, i.e., for each $j_{\underline{1}}$, there is an $n_{\underline{1}}$ such that $(P_{k_2, k+c_{\underline{1}}^{(m)}n_{\underline{1}}})_{j_{\underline{1}} j_{\underline{1}}} > 0$, so that the new path looks like

$$i \text{ to } \underbrace{j_1 \text{ to } j_1}_{c_1^{(m)}n_1\text{-steps}} \text{ to } \underbrace{j_2 \text{ to } j_2}_{c_2 n_2\text{-steps}} \text{ to } \cdots \text{to } j_v \text{ to } i].$$

Since, $b \leq a^2$ and $d \mid b, N_0(N_0-1)+\frac{b}{d} \leq \frac{1}{d^2}\left([\frac{a}{d}]!a-d\right)^2+\frac{b}{d} \leq \left([\frac{a}{d}]!a\right)^2$.

Therefore, if $n \geq \left([\frac{a}{d}]!a\right)^2$, then $n.d = m.d+b, m \geq N_0(N_0-1), \forall$ i in the class, $(P_{k,k+nd})_{ii} > 0$.

(5) Let $G_0 = \{j \in S\}: lim(P_{m,n})_{ij} = 0, \forall m \geq 0, \forall$ states i as $n \to \infty\}$.
Since S is finite, $G_0^c \neq \Phi$. Let $k \in G^c{}_0$. $\exists\ i_1 \in S, \ni\ lim(P_{m,n_t})_{i_1 k} > 0$ for some m as $t \to \infty$. Or $g_{i_1 k}^{(m)} > 0$. Since

$$g^{(m)}{}_{i_1 k} = \sum_{r=m+1}^{\infty} f^{m,r}{}_{i_1,k} g^{(r)}{}_{kk} \exists r > m \in g_{kk}^{(r)} > 0.$$

Thus k is recurrent and $k \to k$.

(6) Let $T_0 = \{j \text{ in } s: j \not\to j\}$. Then T_0 in G_0. The set T_0^c can be partitioned into equivalence classes with respect to the equivalence relation "$\leftrightarrow$". The equivalence classes in T_0^c has either all essential or all unessential states.

(7) Let $\{E_1, E_2, \cdot, E_e\}$ be all the equivalence classes of T_0^c consisting of only essential states. Each E_α can also be partitioned into subclasses $\{E_{\alpha 1}, E_{\alpha 2}, \cdots, E_{\alpha d(\alpha)}\}$, where $d(\alpha)$ is the period of the class E_α as follows: For a fixed i in $E_\alpha, E_{\alpha r} = \{j \varepsilon E_\alpha: (P_{m,m+n})_{ij} > 0) \Rightarrow n = r(mod\, d(\alpha)))\}$, $1 \leq r \leq d(\alpha)$. Clearly, for j in $E_{\alpha \underline{1}}, k$ in $E_{\alpha \underline{1}'}, (P_{m,m+n})_{jk} > 0$ implies that $n = \underline{1}' - \underline{1}\ (mod\ d(\alpha))$. Note that the restriction of $P_{m,m+d(\alpha)}$ to $E_{\alpha d(\alpha)}$, i.e., $P_{m,m+d(\alpha)} \mid E_{\alpha d(\alpha)}$ is an allowable non-negative matrix because $(P_{m,m+d(\alpha)})_{ij} = 0\ \forall\ j$ in $E_{\alpha d(\alpha)}$, for some $i \in E_{\alpha d(\alpha)}$ and for some m, so that $(P_{m,m+d(\alpha)})_{ii} = 0\ \forall\ n$, which is a contradiction. Similarly, no column of $P_{m,m+d(\alpha) \mid E\alpha d(\alpha)}$ can be a zero column. For $i, j \in E_{\alpha d(\alpha)}$, $\exists$ a positive integer $k_{ij} \leq \left[\frac{a}{d(\alpha)}\right] \ni \forall m$, $(P_{m,m+k_{ij} d(\alpha)})_{ij} > 0$. This means that $n \geq N$ (where N is as in (5))$\Rightarrow (P_{m,m+nd(\alpha)+k_{ij}d(\alpha)})_{ij} > 0$. Let $M = max\{N + k_{ij}: i, j \text{ in } E_{\alpha d(\alpha)}\}$. Then $M \leq N + \left[\frac{a}{d(\alpha)}\right]$. By the assumption in the theorem, when $(P_n)_{ij} > 0$ for $i, j \in E_\alpha$, and n sufficiently large,

$$(*)(P_n)_{ij} \geq \frac{1}{n^{\theta(a,d(\alpha))}}, \theta(a, d(\alpha)) = \frac{1}{\left(N + \left[\frac{a}{d(\alpha)}\right]\right) d(\alpha)}.$$

{Notice that for $i, j \in E_{\alpha d(\alpha)}\}$ $(P_{m,m+Md(\alpha)})_{ij} = (P_{m,[N+(M-N-k_{ij})]d(\alpha)+K_{ij}d(\alpha)})_{ij} > 0$, and $\forall k \geq 1$ and n sufficiently large, by condition $(*)$, we have

$$(P_{n+kMd(\alpha), n+(k+1)Md(\alpha)})_{ij} \geq \frac{1}{n+(k+1)Md(\alpha)}.$$

It is clear that for n sufficiently large $\Phi(P_{n,n+Md(\alpha)} \mid E_{\alpha d(\alpha)}) \geq \frac{1}{(n+Md(\alpha))^2}$.

Also, $P_{m,m+Md(\alpha)} \mid E_{\alpha d(\alpha)} = P_{n,n+Md(\alpha)} \mid E_{\alpha d(\alpha)} \cdots P_{m+(n-1)Md(\alpha), m+nMd(\alpha)} \mid E_{\alpha d(\alpha)}$. Thus, using Hajnal's theorem observe that the chain $P_{m,m+nMd(\alpha)} \mid E_{\alpha d(\alpha)}$ is weakly ergodic. That is the chain $P_{m,m+nd(\alpha)} \mid E_{\alpha d(\alpha)}$ is also weakly ergodic, because for $n >> n'$, $P_{m,m+nd(\alpha)} \mid E_{\alpha d(\alpha)} = P_{m,m+n'd(\alpha).M \mid E_{\alpha d(\alpha)}}$ times. Due to weak ergodicity, $i, j \in E_{\alpha d(\alpha)}$, $\mid (P_{m,n})_{ij} - (P_{m,n})_{jj}) \mid \to 0$ as $n \to \infty$. If $n = r(n)(mod\ d(\alpha)), 0 \leq r(n) < d(\alpha)$, then for $m \geq d(\alpha) \ni n = m(mod\ d(\alpha))(P_{r(n),n})_{jj} = \sum_{\underline{1} \text{ in } E_{\alpha d(\alpha)}} (P_{r(n),m})_{j\underline{1}}(P_{m,n})_{\underline{1}j} \ni$ for $n = m(mod\ d(\alpha))$ and as $n \to \infty$ $\mid (P_{r(n),n})_{jj} - (P_{m,n})_{jj} \mid \to 0$. For $i, j \in E_{\alpha d(\alpha)}$ and $n = m(mod\ d(\alpha))(P_{m,n})_{ij} = (P_{r(n),n})_{jj} + (\varepsilon_{m,n})_{ij}$, where $lim(\varepsilon_{m,n})_{ij} = 0$ as $n \to \infty$. Writing $(P'_n)_j = (P_{r(n),n})_{jj}$ then $lim \sum_{j \varepsilon E_{\alpha d(\alpha)}} (P_n)_j = 1$ as $n \to \infty$. Let $i \in E_{\alpha \underline{1}}, j \in E_{\alpha \underline{1}'}$,

$m < n, \ni n - m = \underline{1}' - \underline{1}\ (mod\ d(\alpha)), 1 \leq \underline{1} \leq \underline{1}' \leq d(\alpha)$. Then

$$(P_{m,n})_{ij} = \sum_{s\varepsilon E_{\alpha \underline{1}'}} (P_{m,m+\underline{1}'-\underline{1}})_{is}(P_{m+\underline{1}'-\underline{1},n})_{sj}$$

$$= \sum_{s\varepsilon E_{\alpha \underline{1}'}} (P_{m,m+\underline{1}'-1})_{is}\left[(P_n)_j + (\varepsilon_{m+\underline{1}'-\underline{1},n})_{sj}\right] = (P_n)_j + (\varepsilon'_{m,n})_{ij}$$

where $\quad (\varepsilon'_{m,n})_{ij} = \sum_{s\varepsilon E_{\alpha \underline{1}'}} (P_{m,m+\underline{1}'-\underline{1}})_{is}(\varepsilon_{m+\underline{1}'-\underline{1},n})_{sj} \Rightarrow lim(\varepsilon'_{m,n})_{ij} = 0$ as $n \to \infty$.

(8) Let $\{I_1, I_2, \cdots, I_f\}$ be all the equivalence classes consisting of unessential self-communicating states. Let I_β be a class with period $d(\beta)$. Partitioning it into further subclasses $\{I_{\beta 1}, I_{\beta 2}, \cdots, I_{\beta d(\beta)}\}$ as before, $\forall m \geq 1, P_{m,m+k} \mid I_{\beta k}$ is an allowable non-negative matrix. Also

$$\exists M \ni n \equiv m + (\underline{1}' - 1)(mod\ d(\beta)) \Rightarrow (P_{m,n+Md(\beta)})_{ij} \geq \frac{1}{n + Md(\beta)} \text{ for } i\varepsilon I_{\beta \underline{1}} \text{ and } j\varepsilon I_{\beta 1'}.$$

So
$$\frac{(P_{m,m+nMd(\beta)})_{ij}}{(P_{m,m+nMd(\beta)})_{kj}} \to V^{(m)}{}_{ik}$$

as $n \to \infty$ for $i, j, k\varepsilon I_{\beta d(\beta)}$ [From Hajnal]. $\forall m \geq 0 i, k \in I_{\beta \underline{1}}$ and $j \in I_{\beta \underline{1}'}, n - m = \underline{1}' - \underline{1}\ (mod\ d(\beta))$ one has $\frac{(P_{m,m+n})_{ij}}{(P_{m,m+n})_{kj}} \to V^{(m)}{}_{ik}$ as $n \to \infty$.

(9) Let i be any state and $j \in E_{\alpha d(\alpha)}$. Let $n = r(n)(mod\ d(\alpha))$. Then $(P_{m,n})_{ij} = (P_{r(n),n})_{jj}(P_{m,n})_{iE_{\alpha d(\alpha)}} + (\varepsilon_{m,n})_{ij}$ where $lim(\varepsilon_{m,n})_{ij} = 0$ as $n \to \infty$. A similar statement holds for $j\varepsilon E_{\alpha u'}, 1 \leq u \leq d(\alpha)$. To prove this, assume the opposite. Then $\exists r \ni 1 \leq r \leq d(\alpha)$ and a sequence of positive integers $(n_t) \ni$ if $t \geq 1$,

$$(*)\ 0 < \delta < \mid (P_{m,nt})_{ij} - (P_{r,nt})_{jj}(P_{m,nt})_{iE\alpha d(\alpha)} \mid$$

where each $n_t = r(mod\ d(\alpha))$, and $\forall k \geq 0, P_{k,nt} \to Q_k, Q_k Q = Q_k, Q_{nt} \to Q = Q^2$. Clearly, j is in a C-block of Q. (Note that C-blocks of Q are strictly positive stochastic blocks with identical rows). If not then the j-th column of Q is a zero column, hence a zero column of Q_m and Q_r, and this will contradict (*). Since, $(Q_m)_{i\underline{1}} = 0$ for $\underline{1} \in T$ (=the zero columns of Q), for $t \geq t_0$, $\sum_{\underline{1} \varepsilon T}(P_{m,nt})_{i\underline{1}} < \frac{\delta}{4}$. Also, since each $n_t = r(mod\ d(\alpha)), (P_{nt,nt'})_{j\underline{1}} = 0$ for $\underline{1} \notin E_{\alpha d(\alpha)} \Rightarrow Q_{j\underline{1}} = 0$. If $\underline{1} \notin E_{\alpha d(\alpha)} UT$, then $Q_{j\underline{1}} = 0$ and therefore $Q_{\underline{1}j} = 0 \Rightarrow$ for t large and

$$n_{t'} >> n_t: \sum_{\underline{1} \notin TUE_{\alpha d(\alpha)}} (P_{nt,nt'})_{\underline{1}j} < \frac{\delta}{4}.$$

Thus
$$(P_{m,nt'})_{ij} = \sum_{\underline{1} \in T \setminus E_{\alpha d(\alpha)}} (P_{m,n})_{i\underline{1}}(P_{nt,nt'})_{\underline{1}j} + \sum_{\underline{1} \in E_{\alpha d(\alpha)}} (P_{m,nt})_{i\underline{1}}(p_{nt,nt'})_{\underline{1}j}$$

$$+ \sum_{\underline{1} \notin TUE_{\alpha d(\alpha)}} (P_{m,nt})_{i\underline{1}}(P_{nt,nt'})_{\underline{1}j}.$$

From (weak ergodicity result) (8)

$$\mid (P_{m,nt'})_{ij} - (P_{r,nt'})_{jj}(P_{m,nt'})_{iE_{\alpha d(\alpha)}} \mid < \delta.$$

This is a contradiction.

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