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GENERALIZED TOTALLY DISCONNECTEDNESS

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ABSTRACT In this paper totally disconnectedness is generalized to maximal disconnectedness, which is investigated, and additional properties of totally disconnectedness and 0-dimensional are given.

KEY WORDS AND PHRASES Totally disconnectedness, 0-dimensional, and $T_0 - identification$ spaces
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1. INTRODUCTION

Totally disconnected spaces were considered as early as 1921 by Knaster and Kuratowski [1] and by Sierpinski [2]. A space (X,T) is totally disconnected iff the components of (X,T) are the points. Questions about possible generalizations of totally disconnected spaces led to the following discovery.

THEOREM 1. Let (X,T) be a totally disconnected space. Then (X,T) is T_1.

PROOF: Since components are closed, then the singleton sets are closed, which implies (X,T) is T_1.

Thus for T_1 spaces, totally disconnectedness is maximal disconnectedness in the sense that the components are the smallest possible sets, which motivated the introduction and investigation of maximal disconnected spaces in this paper.

2. MAXIMAL DISCONNECTEDNESS.

DEFINITION 1. Let (X,T) be a space and let C be a component of (X,T). Then C is a minimal component of (X,T) iff C does not contain a nonempty proper closed connected subset. The space (X,T) is maximal disconnected iff the components of (X,T) are minimal components.

Note that for a space (X,T), $Cl(\{x\})$ is connected for each $x \in X$ and that a component C of (X,T) is a minimal component of (X,T) iff $C = Cl(\{x\})$ for each $x \in X$.

In 1961 A. Davis [3] was interested in properties R_{i-1} weaker than T_i, which together with T_{i-1}, would be equivalent to $T_i, i = 1,2$. In the 1961 investigation R_0 and R_1 spaces were defined. A space (X,T) is R_0 iff one of the following equivalent conditions is satisfied: (a) if $O \in T$ and $x \in O$, then $Cl(\{x\}) \subset O$, and (b) $\{Cl(\{x\}) \mid x \in X\}$ is a decomposition of X. A space (X,T) is R_1 iff for $x, y \in X$ such that $Cl(\{x\}) \neq Cl(\{y\})$, there exist disjoint open sets U and V such that $Cl(\{x\}) \subset U$ and $Cl(\{y\}) \subset V$. The 1961 paper [3] was a continuation of work done by N. Shanin in 1943 [4], in which R_0 spaces were called weak regular spaces. Combining this information with the note above in a straightforward proof, which is omitted, gives the following result.

THEOREM 2. Let (X,T) be a space. Then (X,T) is maximal disconnected iff (X,T) is R_0 and the components of (X,T) are closures of singleton sets.

The results above can be combined to obtain the following result.

COROLLARY 3. Let (X,T) be a space. Then the following are equivalent: (a) (X,T) is totally disconnected,

(b) (X,T) is T_1 maximal disconnected, and (c) (X,T) is T_0 maximal disconnected

THEOREM 4. Let (X,T) be a space. Then (X,T) is maximal disconnected iff every homeomorphic image of (X,T) is maximal disconnected.

The straightforward proof is omitted

Combining Theorem 4 with Corollary 3 and the fact that T_1 is a topological property gives the following result

COROLLARY 5 Totally disconnectedness is a topological property.

In 1977 [5] T_0 – *identification* spaces were used to further investigate and better understand R_0 spaces Let (X,T) be a space and let R be the equivalence relation on X defined by xRy iff $Cl(\{x\}) = Cl(\{y\})$ Then the T_0 – *identification* space of (X,T) is $(X(TO),Q(TO))$, where $X(TO)$ is the set of equivalence classes of R and $Q(TO)$ is the decomposition topology on $X(TO)$ [6] The space (X,T) is R_0 iff $(X(TO),Q(TO))$ is T_1 [6]. Let $P(TO)$ $(X,T) \to (X(TO),Q(TO))$ be the natural map. Below, T_0 – *identification* spaces are used to further investigate maximal disconnected spaces

THEOREM 6. Let (X,T) be a space. Then (X,T) is maximal disconnected iff $(X(TO),Q(TO))$ is totally disconnected

PROOF Suppose (X,T) is maximal disconnected. Let $\mathcal{C}$ be a component of $(X(TO),Q(TO))$. Since (X,T) is R_0, then $X(TO) = \{Cl(\{x\}) \mid x \in X\}$ [5]. Let $Cl(\{x\}) \in \mathcal{C}$. Then $P(TO)^{-1}(\mathcal{C})$ is a closed and connected [7] subset of (X,T) containing x, which implies $P(TO)^{-1}(\mathcal{C}) = Cl(\{x\})$ and $\mathcal{C} = \{Cl(\{x\})\}$. Thus components of $(X(TO),Q(TO))$ are singleton sets, which implies $(X(TO),Q(TO))$ is totally disconnected.

Conversely, suppose $(X(TO),Q(TO))$ is totally disconnected. Let C be a component of (X,T). Let $x \in C$. Since $(X(TO),Q(TO))$ is totally disconnected, then $(X(TO),Q(TO))$ is T_1, which implies (X,T) is R_0 and $X(TO) = \{Cl(\{x\}) \mid x \in X\}$. Since $P(TO)$ is continuous and closed [7], then $P(TO)(C)$ is closed connected, with $Cl(\{x\}) \in P(TO)(C)$, which implies $P(TO)(C) = \{Cl(\{x\})\}$ and thus $C = Cl(\{x\})$. Hence (X,T) is maximal disconnected.

Combining the results above with the fact that for a space (X,T), $(X(TO),Q(TO))$ is T_0 gives the following result.

COROLLARY 7. Let (X,T) be a space. Then (X,T) is maximal disconnected iff $(X(TO),Q(TO))$ is maximal disconnected.

THEOREM 8. Let (X,T) be a space, let $Y \subset X$, and let T_Y be the relative T topology on Y. Then $(Y(T_YO),Q(T_YO))$ is homeomorphic to $(P(TO)(Y),Q(TO)_{P(TO)(Y)})$.

PROOF: For each $y \in Y$ let $K_y \in Y(T_YO)$ containing y and let $C_y \in P(TO)(Y)$ containing y. Let $f = \{(K_y,C_y) \mid y \in Y\}$. If $K_y = K_z$, then $Cl_{T_Y}(\{y\}) = Cl_{T_Y}(\{z\})$, which implies $Cl_T(\{y\}) = Cl_T(\{z\})$ and $C_y = C_z$. Thus f is a function. Clearly f is onto. If $C_y = C_z$, where $y,z \in Y$, then $Cl_T(\{y\}) = Cl_T(\{z\})$, which implies $Cl_{T_Y}(\{y\}) = Cl_{T_Y}(\{z\})$ and $K_y = K_z$. Thus f is one-to-one. Let $\mathcal{O} \in Q(TO)_{P(TO)(Y)}$. Let $\mathcal{U} \in Q(TO)$ such that $\mathcal{O} = \mathcal{U} \cap P(TO)(Y)$. Then $f^{-1}(\mathcal{O}) = P(T_YO)(P(TO)^{-1}(\mathcal{U}) \cap Y)$ and since $P(TO)^{-1}(\mathcal{U}) \cap Y \in T_Y$ and $P(T_YO)$ is open [7],

then $f^{-1}(\mathcal{O}) \in Q(T_YO)$. Thus f is continuous. If $\mathcal{V} \in Q(T_YO)$, then $P(T_YO)^{-1}(\mathcal{V}) = W \cap Y$, where $W \in T$, and $f(\mathcal{V}) = P(TO)(W) \cap P(TO)(Y) \in Q(TO)_{P(TO)(Y)}$. Thus f is open and hence f is a homeomorphism from $(Y(T_YO),Q(T_YO))$ onto $(P(TO)(Y),Q(TO)_{P(TO)(Y)})$.

THEOREM 9. Let (X,T) be a space. Then (X,T) is maximal disconnected iff every subspace of (X,T) is maximal disconnected.

PROOF: Suppose (X,T) is maximal disconnected. Let $Y \subset X$. Then $(X(TO),Q(TO))$ is totally disconnected, which implies $(P(TO)(Y),Q(TO)_{P(TO)(Y)})$ is totally disconnected [6]. Since $(P(TO)(Y),Q(TO)_{P(TO)(Y)})$ is homeomorphic to $(Y(T_YO),Q(T_YO))$, then $(Y(T_YO),Q(T_YO))$ is totally disconnected, which implies (Y,T_Y) is maximal disconnected

Clearly, the converse is true.

THEOREM 10. Let $\{(X_\alpha, T_\alpha) \mid \alpha \in A\}$ be a nonempty collection of nonempty spaces. Then (X_α, T_α) is maximal disconnected for each $\alpha \in A$ iff $(\prod_{\alpha \in A} X_\alpha, W)$ is maximal disconnected, where W is the product topology on $\prod_{\alpha \in A} X_\alpha$.

PROOF: Suppose (X_α, T_α) is maximal disconnected for each $\alpha \in A$. Then $(X_\alpha(T_\alpha O), Q(T_\alpha O))$ is totally disconnected for each $\alpha \in A$ and $(\prod_{\alpha \in A} X_\alpha(T_\alpha O), \mathcal{W})$, where $\mathcal{W}$ is the product topology on $\prod_{\alpha \in A} X_\alpha(T_\alpha O)$, is totally disconnected [6]. Since $(\prod_{\alpha \in A} X_\alpha(T_\alpha O), \mathcal{W})$ is homeomorphic to $((\prod_{\alpha \in A} X_\alpha)(WO), Q(WO))$ [8], then $((\prod_{\alpha \in A} X_\alpha)(WO), Q(WO))$ is totally disconnected, which implies $(\prod_{\alpha \in A} X_\alpha, W)$ is maximal disconnected.

Conversely, suppose $(\prod_{\alpha \in A} X_\alpha, W)$ is maximal disconnected. Let $\beta \in A$. Let C_β be a component of (X_β, T_β). Let $x_\beta \in C_\beta$. For each $\alpha \in A$, $\alpha \neq \beta$, let C_α be a component of (X_α, T_α) and let $x_\alpha \in C_\alpha$. Then $\prod_{\alpha \in A} C_\alpha$ is a closed connected subset of $\prod_{\alpha \in A} X_\alpha$ containing $x = \prod_{\alpha \in A}\{x_\alpha\}$, which implies $\prod_{\alpha \in A} C_\alpha = Cl(\{x\}) = \prod_{\alpha \in A} Cl(\{x_\alpha\})$ and $C_\beta = Cl(\{x_\beta\})$. Thus (X_β, T_β) is maximal disconnected.

Combining the results above with the fact that the product space of a nonempty collection of nonempty spaces is T_1 iff each factor space is T_1 gives the next result.

COROLLARY 11. Let $\{(X_\alpha, T_\alpha) \mid \alpha \in A\}$ be a nonempty collection of nonempty spaces. Then (X_α, T_α) is totally disconnected for each $\alpha \in A$ iff $(\prod_{\alpha \in A} X_\alpha, W)$ is totally disconnected.

In S. Willard's 1970 book [6] relationships between totally disconnected and 0-dimensional were examined. A space (X, T) is 0-dimensional iff each point of X has a neighborhood base consisting of closed open sets. Below, results in Willard's book are used not only to further investigate maximal disconnectedness, but also, to further investigate totally disconnectedness.

Since every 0-dimensional space is completely regular, then every 0-dimensional space is regular, and thus R_1 [9] and R_0 [3].

THEOREM 12. Let (X, T) be a space. Then (X, T) is 0-dimensional iff $(X(TO), Q(TO))$ is 0-dimensional.

PROOF: Suppose (X, T) is 0-dimensional. Let $C \in X(TO)$. Let $x \in C$. Let $\mathcal{O}$ be a neighborhood base of x consisting of closed open sets. Since $P(TO)$ is closed, open, and continuous [7], then $P(TO)(\mathcal{O})$ is a neighborhood base of C consisting of closed open sets. Thus $(X(TO), Q(TO))$ is 0-dimensional.

Conversely, suppose $(X(TO), Q(TO))$ is 0-dimensional. Let $x \in X$. Let $C \in X(TO)$ such that $x \in C$. Let $\mathcal{O}$ be a neighborhood base of C consisting of closed open sets. Since $P(TO)^{-1}(P(TO)(O)) = O$ for each $O \in T$ [7], $P(TO)^{-1}(\mathcal{O})$ is a neighborhood base of x consisting of closed open sets. Thus (X, T) is 0-dimensional.

THEOREM 13. Let (X, T) be a 0-dimensional space. Then (X, T) is maximal disconnected.

PROOF: Since (X, T) is 0-dimensional, then (X, T) is R_0 and $(X(TO), Q(TO))$ is 0-dimensional T_1, which implies $(X(TO), Q(TO))$ is totally disconnected [6] and (X, T) is maximal disconnected.

COROLLARY 14. Every 0-dimensional T_0 space is totally disconnected.

DEFINITION 2. A space is rim-compact iff each of its points has a base of neighborhoods with compact frontier [6].

THEOREM 15. Let (X, T) be a space. Then (X, T) is rim-compact iff $(X(TO), Q(TO))$ is rim-compact.

PROOF: Suppose (X, T) is rim-compact. Let $C \in X(TO)$. Let $x \in C$. Let $\mathcal{O}$ be a neighborhood base of x consisting of neighborhoods with compact frontiers. Let $O \in \mathcal{O}$. Then $Fr(Int(O))$ is a closed subset of the compact set $Fr(O)$, which implies $Fr(Int(O))$ is compact. Then $P(TO)(Int(O)) \in Q(TO)$ and $P(TO)^{-1}(Fr(P(TO)(Int(O))))$ $= Fr(P(TO)^{-1}(P(TO)(Int(O))))$[7] $= Fr(Int(O))$ is compact, which implies $Fr(P(TO)(Int(O)))$ is compact [7]. Thus $\{P(TO)(Int(O)) \mid O \in \mathcal{O}\}$ is a neighborhood base of C consisting of neighborhoods with compact frontiers.

Conversely, suppose $(X(TO), Q(TO))$ is rim-compact. Let $x \in X$. Let $C \in X(TO)$ such that $x \in C$. Let $\mathcal{O}$ be a neighborhood base of C consisting of neighborhoods with compact frontiers. Then $\{Int(\mathcal{U}) \mid \mathcal{U} \in \mathcal{O}\}$ is a neighborhood base of C consisting of neighborhoods with compact frontiers and since for each $\mathcal{U} \in \mathcal{O}$, $Fr(P(TO)^{-1}(Int(\mathcal{U}))) =$

$P(TO)^{-1}(Fr(Int(U)))$, which is compact [7], then $\{P(TO)^{-1}(Int(U)) \mid U \in \mathcal{O}\}$ is a neighborhood base of x consisting of neighborhoods with compact frontiers.

THEOREM 16. Let (X,T) be rim-compact. Then (X,T) is 0-dimensional iff (X,T) is maximal disconnected.

PROOF: Suppose (X,T) is 0-dimensional. Then $(X(TO),Q(TO))$ is rim-compact 0-dimensional T_1, which implies $(X(TO),Q(TO))$ is totally disconnected [6] and thus (X,T) is maximal disconnected.

Conversely, suppose (X,T) is maximal disconnected. Then $(X(TO),Q(TO))$ is rim-compact totally disconnected T_1, which implies $(X(TO),Q(TO))$ is 0-dimensional [6] and thus (X,T) is 0-dimensional.

COROLLARY 17. Let (X,T) be rim-compact. Then (X,T) is 0-dimensional T_0 iff (X,T) is totally disconnected.

The last result in this section follows immediately from the fact that for T_1 spaces, metrizability and pseudometrizability are equivalent.

COROLLARY 18. Let (X,T) be totally disconnected. Then (X,T) is metrizable iff (X,T) is pseudometrizable.

Thus in results known for totally disconnected metric spaces, metric can be replaced by pseudometric.

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