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THE T-STATISTICALLY CONVERGENT SEQUENCES ARE NOT AN FK-SPACE

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Abstract In this note we show that under certain restrictions on a nonnegative regular summability matrix T, the space of T-statistically convergent sequences cannot be endowed with a locally convex FK topology.

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1. INTRODUCTION

Statistical convergence, introduced by Fast [5], has most recently been studied by Fridy and Orhan [7] [8], and Kolk [9], among others [3] [4] [6] [11]. In [3], it is shown that the space of statistically convergent sequences cannot be endowed with a locally convex FK topology. In this note, we establish that under certain restrictions on a nonnegative regular summability matrix T, the space of T-statistically convergent sequences cannot be endowed with a locally convex FK topology.

An infinite matrix $T = (t_{nk})$ is *nonnegative* if $t_{nk} \geq 0$ for all n and k, and *regular* if, for a convergent sequence x with limit l, $\lim_n \sum_{k=1}^{\infty} t_{nk}x_k = l$. Throughout this note T denotes a nonnegative regular matrix. We say that *the rows of T spread* if $\lim_n \max_k t_{nk} = 0$. We let ω denote the space of all real valued sequences, φ denote the finitely nonzero elements of ω and N denote the positive integers. For $\epsilon > 0$ and a scalar l, we let $A_{\epsilon,l} = \{k : |x_k - l| < \epsilon\}$. A sequence $x \in \omega$ is *T-statistically convergent* to l provided that for all $\epsilon > 0$,

$$\lim_n \sum_{k=1}^{\infty} t_{nk}\,\chi_{A_{\epsilon,l}}(k) = 1, \tag{1.1}$$

(where χ_A is the characteristic function of A). The space of T-statistically convergent sequences is denoted by S_T. Note that for $T = C_1$, the Cesàro matrix, this definition concurs with the definition of statistical convergence [6].

2. THE MAIN RESULT

A common theme in summability is the quest for "soft" methods to apply to classical type problems. An example of this is the "FK program," in which a summability space is given an FK topology ([13], pg. 54). An FK space X is a subspace of ω with a complete locally convex Fréchet topology such that the inclusion map from X into ω is continuous [13]. Our result shows that we cannot apply the FK-program to T-statistical convergence and improves Theorem 3.3 of [3].

THEOREM 1. *For a nonnegative regular summability matrix T whose rows spread, the space of T-statistically convergent sequences cannot be endowed with a locally convex FK topology.*

The proof of this theorem depends upon the following result of Bennett and Kalton [**2**].

THEOREM 2. *Let S be a dense subspace of ω. Then the following are equivalent:*

1) S is barrelled.

2) If E is a locally convex FK space that contains S, then $E = \omega$.

Note that the above statement is a restricted version of the result in [**2**], and an exposition can be found in ([**12**], pg. 253).

The proof of the main result follows that of Theorem 3.3 in [**3**].

PROOF OF THEOREM 1:

We show that S_T is a dense barrelled subspace of ω. Recall that S_T is barrelled if and only if every $\sigma(\varphi, S_T)$-bounded subset of φ is $\sigma(\varphi,\omega)$-bounded ([**12**], pg.248). Thus, to show S_T is barrelled it suffices to show that if E is not $\sigma(\varphi,\omega)$-bounded, then E is not $\sigma(\varphi, S_T)$-bounded.

We may assume that E is $\sigma(\varphi,\varphi)$-bounded, since otherwise E is not $\sigma(\varphi, S_T)$-bounded and we are done. Thus, there exists a sequence of integers $< B_n >$ such that for x an element of E with $spt(x) = \{k \in \mathbf{N} : x_k \neq 0\} \subseteq \{1,2,\dots,n\}$, we have $\sup_{1\leq i\leq n} |x_i|$ is less than or equal to B_n. Since E is not $\sigma(\varphi,\omega)$-bounded, we can choose s in ω such that $\sup_{x\in E} |\sum_{i=1}^{\infty} x_i s_i|$ is infinite. Note that for all x in E, we have $|\sum_{i=1}^{n} x_i s_i| \leq nB_n \sup_{i\leq n} |s_i|$.

Select $x^1 \in E$ such that $|\sum_{i=1}^{\infty} x_i^1 s_i| > B_1|s_1|$, and select $j_1 > 1$ such that $x_{j_1}^1 \neq 0$ (such a j_1 exists since E is not $\sigma(\varphi,\omega)$-bounded and since $|x_1| \leq B_1$). Assume that $\{x^1, x^2, \dots, x^n\}$ and $j_1 < j_2 < \dots < j_n$ have been chosen so that $x_{j_n}^n \neq 0$ and $j_n > \max\{k \in \mathbf{N} : k \in spt(x^{n-1})\}$. Set $t = \max\{j_n, \max\{k \in \mathbf{N} : k \in spt(x^n)\}$, and select x^{n+1} such that

$$|\sum_{i=1}^{\infty} x_i^{n+1} s_i| > tB_t \sup_{k\leq t} |s_k|. \tag{2.1}$$

Now select j_{n+1} such that $t < j_{n+1}$ and $x_{j_{n+1}}^{n+1} \neq 0$, and proceed inductively.

By [**10**], since the rows of T spread there is a subsequence $< j_{p_m} >$ of $< j_n >$ such that $\lim_n \sum_{k=1}^{\infty} t_{nk}\chi_{\{j_{p_m}, m\in\mathbf{N}\}}(k) = 0$. Since $x_{j_{p_m}}^{p_m} \neq 0$ for all m, it is possible to construct a sequence $\alpha = (\alpha_k)$ such that $\sum_{k=1}^{\infty} \alpha_k x_{j_{p_k}}^{p_m} \to \infty$ as $m \to \infty$. Then we set

$$z = (z_r) = \begin{cases} \alpha_k \text{ if } r = j_{p_k}, \text{ for } k = 1,2,\dots \\ 0 \text{ else.} \end{cases} \tag{2.2}$$

Now, z is T-statistically convergent to 0 (because the non-zero entries of z occur on the subsequence $< j_{p_m} >$ and by the definition of T-statistical convergence). Note also that $\sum_{k=1}^{\infty} x_k^{p_m} z_k = \sum_{k=1}^{\infty} x_{j_{p_k}}^{p_m} \alpha_k$. Since the right hand side of this equation tends to infinity as m does, it follows that E is not $\sigma(\varphi, S_T)$-bounded. Since $\varphi \subseteq S_T$, S_T is a dense barrelled subspace of ω. Now, by the result of Bennett and Kalton, we have that S_T cannot be endowed with a locally convex FK topology.

The following examples illustrate the necessity of the hypothesis that the rows of T spread. Consider the identity matrix $I = (i_{nk})$, where $i_{nn} = 1$ and $i_{nk} = 0$ for all $k \neq n$. The space of I-statistically convergent sequences is the space of convergent sequences c, a well-known FK space. For a less trivial example, consider the matrix $T = (t_{nk})$ where $t_{11} = 1$, $t_{1k} = 0$ for $k \geq 2$, and for $n \geq 2$,

$$t_{nk} = \begin{cases} \frac{1}{2} \text{ if } k = n \text{ or } k = n-1, \\ 0 \text{ else.} \end{cases} \tag{2.3}$$

Note that the rows of T spread, and that this method is regular. In fact, T is stronger than convergence since the sequence $x =< (-1)^n >$ has $\lim_n \sum_{k=1}^{\infty} t_{nk} x_k = 0$. As in the case of the identity matrix, the T-statistically convergent sequences are again the FK space c of convergent sequences.

A consequence of this result is that we cannot employ the FK program when studying T-statistical convergence for a matrix T whose rows spread. Instead, the Stone-Čech compactification of the integers has been used [4][1] as an avenue for "soft" methods for treating T-statistical convergence of bounded sequences. This result also includes Corollary 4.4 of [9], where it is shown that under certain restrictions, a matrix B maps the space of T-statistically convergent sequences into a sequence space Y if and only if B has at most finitely many non-zero columns which belong to Y.

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