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RESEARCH NOTES

ON THE RICCI TENSOR OF REAL HYPERSURFACES OF QUATERNIONIC PROJECTIVE SPACE

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ABSTRACT. We study some conditions on the Ricci tensor of real hypersurfaces of quaternionic projective space obtaining among other results an improvement of the main theorem in [9].

KEY WORDS AND PHRASES. Quaternionic projective space, real hypersurface, Ricci tensor.
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1. INTRODUCTION.

Let M be a real hypersurface, which in the following we shall always consider connected, of a quaternionic projective space QP^m, $m \geq 2$, with metric g of constant quaternionic sectional curvature 4. Let ζ be the unit normal vector field on M and $\{J_1, J_2, J_3\}$ a local basis of the quaternionic structure of QP^m, see [2]. Then $U_i = -J_i\zeta, i = 1,2,3,$ are tangent to M. Let S be the Ricci tensor of M.

In [6] we studied pseudo-Einstein real hypersurfaces of QP^m. These are real hypersurfaces satisfying

$$SX = aX + b\Sigma_{i=1}^{3} g(X,U_i)U_i \tag{1.1}$$

for any X tangent to M, where a and b are constant. If $m \geq 3$ we obtained that M is pseudo-Einstein if it is an open subset of either a geodesic hypersphere or of a tube of radius r over QP^k, $0 < k < m-1, 0 < r < \Pi/2$ and $cot^2 r = (4k+2)/(4m-4k-2)$.

As a corollary we also obtained that the unique Einstein real hypersurfaces of $QP^m, m \geq 2$, are open subsets of geodesic hyperspheres of QP^m of radius r such that $cot^2 r = 1/2m$.

The purpose of the present paper is to study several conditions on the Ricci tensor of M. Concretely in 3 we prove the following result: if X is tangent to M we shall write $J_iX = \Phi_iX + f_i(X)\zeta, i = 1,2,3,$ where Φ_iX denotes the tangent component of J_iX and $f_i(X) = g(X,U_i)$. Then

THEOREM 1. Let M be a real hypersurface of QP^m, $m \geq 3$, such that $\Phi_iS = S\Phi_i$, $i = 1,2,3$. Then M is an open subset of a tube of radius r, $0 < r < \Pi/2$, over $QP^k, k \in \{0, \cdots, m-1\}$.

This theorem generalizes results obtained by Pak in [7].

In [9] we studied real hypersurfaces of QP^m with harmonic curvature for which $U_i, i = 1,2,3,$ are eigenvectors of the Weingarten endomorphism of M with the same principal curvature. A real hypersurface has harmonic curvature if

$$(\nabla_X S)Y = (\nabla_Y S)X \tag{1.2}$$

for any X, Y tangent to M, where ∇ denotes the covariant differentiation of M In 4 we shall improve the result of [9] showing that the condition about principality of $U_i, i = 1,2,3,$ is unnecessary Concretely we obtain

THEOREM 2. A real hypersurface of $QP^m, m \geq 2$, has harmonic curvature if and only if it is Einstein

As a consequence we can classify Ricci-parallel real hypersurfaces of QP^m, that is, real hypersurfaces such that $\nabla_X S = 0$ for any X tangent to M. We get

COROLLARY 3. The unique Ricci-parallel real hypersurfaces of QP^m, $m \geq 2$, are open subsets of geodesic hyperspheres of radius r, $0 < r < \pi/2$, such that $cot^2 r = 1/2m$.

From this result we introduce in 5 a condition that generalize Ricci-parallel real hypersurfaces We shall say that a real hypersurface of QP^m is pseudo Ricci-parallel if it satisfies

$$(\nabla_X S)Y = c\,\Sigma_{i=1}^3\{g(\Phi_i X, Y)U_i + f_i(Y)\Phi_i X\} \tag{1.3}$$

for any X, Y tangent to M, c being a nonnull constant We obtain

THEOREM 4. M is a pseudo Ricci-parallel real hypersurface of QP^m, $m \geq 2$, if and only if it is an open subset of a geodesic hypersphere.

Finally, we characterize pseudo-Einstein real hypersurfaces of QP^m by the following

THEOREM 5. Let M be a real hypersurface of QP^m, $m \geq 3$, then

$$\| S \|^2 \geq \Sigma_{i=1}^3 (f_i(SU_i))^2 + \left(\rho - \Sigma_{i=1}^3 f_i(SU_i))^2\right)/4(m-1) \tag{1.4}$$

where ρ denotes the scalar curvature of M. The equality holds if and only if M is pseudo-Einstein.

2. PRELIMINARIES.

Let us call $\mathbb{D}^\perp = \mathrm{Span}\{U_1, U_2, U_3\}$ and $\mathbb{D}$ its orthogonal complement in TM. Let X, Y be vector fields tangent to M. Then, [6], we have

$$\Phi_i^2 X = -X + f_i(X)U_i \tag{2.1}$$

$$g(\Phi_i X, Y) + g(X, \Phi_i Y) = 0, \Phi_i U_i = 0, \Phi_j U_k = -\Phi_k U_j = U_t \tag{2.2}$$

where $i = 1,2,3$ and (j,k,t) is a circular permutation of (1,2,3).

From the expression of the curvature tensor of QP^m, [2], the Ricci tensor of M is given by

$$SX = (4m+7)X - 3\Sigma_{i=1}^3 f_i(X)U_i + hAX - A^2X \tag{2.3}$$

for any X tangent to M, where $h = trace(A)$. Moreover, [6],

$$\nabla_X U_i = q_k(X)U_j - q_j(X)U_k + \phi_i AX \tag{2.4}$$

for any X tangent to M, (i,j,k) being a circular permutation of $(1,2,3)$ and $q_i, i = 1,2,3,$ certain local 1-forms on M (see [2]). Finally the equation of Codazzi is given by

$$(\nabla_X A)Y - (\nabla_Y A)X = \Sigma_{i=1}^3\{f_i(X)\Phi_i Y - f_i(Y)\Phi_i X + 2g(X, \Phi_i Y)U_i\} \tag{2.5}$$

for any X, Y tangent to M

3. PROOF OF THEOREM 1.

Let us call $H = A^2 - fA$, f being a differentiable function on M

If we suppose that $H\Phi_i = \Phi_i H, i = 1, 2, 3,$ from (2 2) $H\Phi_1 U_1 = 0 = \Phi_1 HU_1$ This implies $0 = \Phi_1^2 HU_1 = -HU_1 + f_1(HU_1)U_1$. That is, U_1 is an eigenvector of H Similarly, U_2 and U_3 are also eigenvectors of H Let us consider $T_x M = H(\alpha_1) \oplus H(\alpha_2) \oplus \cdots \oplus H(\alpha_p)$, where $H(\alpha_j) = \{X \in T_x M / HX = \alpha_j X\}$. Suppose that $U_i \in H(\alpha_i), i = 1, 2, 3$.

If $X \in \mathbb{D}$ is such that $X \in H(\alpha_i), H\Phi_j X = \Phi_j HX = \alpha_i \Phi_j X$, that is, $\Phi_j X \in H(\alpha_i)$, $j = 1, 2, 3$. Moreover $H\Phi_j U_1 = \Phi_j HU_1 = \alpha_1 \Phi_j U_1, j = 1, 2, 3$. If $j = 2$, we obtain that $HU_3 = \alpha_1 U_3$. If $j = 3$ we obtain $HU_2 = \alpha_1 U_2$. Thus $\alpha_1 = \alpha_2 = \alpha_3$. Then $H(\alpha_1)$ is odd-dimensional and from (2 5) the proof of Theorem 6 1 in [6] implies that $U_i, i = 1, 2, 3,$ are eigenvectors of A

If we now consider a real hypersurface of $QP^m, m \geq 3$, such that $\Phi_i S = S\Phi_i$, $i = 1, 2, 3$, from (2 3) we obtain that $\Phi_i H = H\Phi_i, i = 1, 2, 3,$ for $f = -h$. Thus U_i, $i = 1, 2, 3,$ are eigenvectors of A. Thus, [1], M is an open subset of either a tube of radius r, $0 < r < \Pi/2$, over $QP^k, k \in \{0, \cdots, m-1\}$ or of a tube of radius r, $0 < r < \Pi/4$, over CP^m

Let us consider the second case The eigenvalues of A are $cot(r)$ with multiplicity $2(m-1)$, $-tan(r)$ with multiplicity $2(m-1)$, $2cot(2r)$ with multiplicity 1 and $-2tan(2r)$ with multiplicity 2 Let X be a unit vector field such that $AX = cot(r)X$ Then $\Phi_2 SX = (4m + 7 + hcot(r) - cot^2(r))\Phi_2 X$ and $S\Phi_2 X = (4m + 7 - htan(r) - tan^2(r))\Phi_2 X$ From this we have $h(cot(r) + tan(r)) + tan^2(r) - cot^2(r) = 0$. Thus either $cot(r) + tan(r) = 0$ and this implies $cot^2(r) = -1$ which is impossible or $h + tan(r) - cot(r) = 0$. As $h = 2(m-1)(cot(r) - tan(r)) + 2cot(2r) - 4tan(2r)$ it is easy to see that $tan^2(2r) = m - 1$.

On the other hand, $\Phi_2 SU_1 = (4m + 4 + 2hcot(2r) - 4cot^2(2r))U_3$ and $S\Phi_2 U_1 = -SU_3 = 4m + 4 - 2htan(2r) - 4tan^2(2r))U_3$. This implies $h(cot(2r) + tan(2r)) - 2(cot^2(2r) - tan^2(2r)) = 0$. Thus either $cot(2r) + tan(2r) = 0$ which implies $cot^2(2r) = -1$ which is impossible or $h - 2(cot(2r) - tan(2r)) = 0$. This implies $tan^2(2r) = 2(m-1)$. Thus $m - 1 = 2(m-1)$. Then $m = 1$ which is impossible. This finishes the proof

4. PROOF OF THEOREM 2.

As M has harmonic curvature for any X, Y tangent to M we get

$$\nabla_X SY - \nabla_Y SX = S([X, Y]) \tag{4.1}$$

Then for any X, Y, Z tangent to M we obtain

$$\begin{aligned} R(Z,X)SY &= \nabla_Z \nabla_X SY - \nabla_X \nabla_Z SY - \nabla_{[Z,X]} SY = \\ &= S(R(Z,X)Y) + \nabla_Z(\nabla_Y S)X + (\nabla_Z S)(\nabla_X Y) - \\ &- \nabla_X(\nabla_Y S)Z - (\nabla_X S)(\nabla_Z Y) - (\nabla_{[Z,X]} S)Y \end{aligned} \tag{4.2}$$

where R denotes the curvature tensor of M.

From (4.2), (1.2) and the first identity of Bianchi we get

$$\sigma(R(X,Y)SZ) = 0 \tag{4.3}$$

for any X, Y, Z tangent to M, where σ denotes the cyclic sum. The result now follows from the main theorem of [8].

5. PROOFS OF THEOREMS 4 AND 5.

Firstly, let us suppose that M is pseudo Ricci-parallel Then applying (1 3) and (2 4) we have

$$\nabla_W(\nabla_X S))Y - (\nabla_{\nabla_W X} S)Y = c\Sigma_{i=1}^3\{g(\Phi_i X, Y)\Phi_i AW + g(Y, \Phi_i AW)\Phi_i X +$$

$$+ f_i(X)g(AW, Y)U_i - 2f_i(Y)g(AX, W)U_i + f_i(Y)f_i(X)AW\} \quad (5\ 1)$$

for any X, Y, W tangent to M If in (5 1) we exchange X and W we get

$$(R(W, X)S)Y = c\Sigma_{i=1}^3\{f_i(X)g(AW, Y)U_i - f_i(W)g(AX, Y)U_i + g(\Phi_i X, Y)\Phi_i AW - \quad (5\ 2)$$

$$- g(\Phi_i W, Y)\Phi_i AX + g(\Phi_i AW, Y)\Phi_i X - g(\Phi_i AX, Y)\Phi_i W + f_i(Y)f_i(X)AW - f_i(Y)f_i(W)AX\}$$

Taking a local orthonormal frame $\{E_1, \cdots, E_{4m-1}\}$ of TM, from (5 2), (2 1) and (2 2) we have

$$\Sigma_{j=1}^{4m-1}g((R(E_j, X)S)Y, E_j) = c\Sigma_{i=1}^3\{f_i(X)f_i(AY) - g(\Phi_i X, Y)trace(A\Phi_i) - 2f_i(Y)f_i(AX) -$$

$$- g(A\Phi_i Y, \Phi_i X) + hf_i(Y)f_i(X)\} \quad (5\ 3)$$

Now the left hand side of (5 3) is symmetric with respect to X, Y (see [4]) Thus (5 3) gives

$$3c\Sigma_{i=1}^3 f_i(X)f_i(AY) = 3c\Sigma_{i=1}^3 f_i(Y)f_i(AX) - 2c\Sigma_{i=1}^3 trace(A\Phi_i)g(\Phi_i Y, X) \quad (5\ 4)$$

But $trace(A\Phi_i)$ is easily seen to be 0 and bearing in mind that c is nonzero, (5.4) can be written as

$$\Sigma_{i=1}^3 f_i(X)f_i(AY) = \Sigma_{i=1}^3 f_i(Y)f_i(AX) \quad (5.5)$$

for any X, Y tangent to M.

We know, [1], that if $g(A\mathbb{D}, \mathbb{D}^\perp) = \{0\}, U_i,\ i = 1, 2, 3$ are principal for A. Let us suppose that $g(A\mathbb{D}, D^\perp) \neq \{0\}$ We shall distinguish the following cases where $X^{\mathbb{D}}$ denotes the $\mathbb{D}$-component of X.

(i) $(AU_2)^{\mathbb{D}} = (AU_3)^{\mathbb{D}} = 0$ and $(AU_1)^{\mathbb{D}} \neq 0$. Then we write $AU_1 = \alpha X_1 + \beta Y_1$ where $X_1 \in \mathbb{D}$ and $Y_1 \in \mathbb{D}^\perp$ are unit. If we take in (5.5) $X = X_1$ and $Y = U_1$ we have $0 = \Sigma_{i=1}^3 f_i(Y_1)f_i(AX_1) = g(AU_1, X_1) = \alpha$. Then $g(A\mathbb{D}, \mathbb{D}^\perp) = \{0\}$.

(ii) $(AU_3)^{\mathbb{D}} = 0$ and $(AU_1)^{\mathbb{D}}, (AU_2)^{\mathbb{D}}$ are linearly dependent. We write $AU_1 = \alpha_1 X_1 + \beta_1 U_1 + \beta_2 U_2 + \beta_3 U_3$ and $AU_2 = \alpha_2 X_1 + \beta_2 U_1 + \gamma_2 U_2 + \gamma_3 U_3$ where $X_1 \in \mathbb{D}$ is unit. If in (5.5) we take $X = X_1, Y = U_1$ we obtain $0 = g(AU_1, X_1) = \alpha_1$. Now we have case (i).

It is easy to see that the rest of cases (if $(AU_3)^{\mathbb{D}} = 0$ and $(AU_1)^{\mathbb{D}}$, $(AU_2)^{\mathbb{D}}$ are linear independent or if $(AU_i)\mathbb{D} \neq 0, i = 1, 2, 3$) are similar. That is, $g(A\mathbb{D}, \mathbb{D}^\perp) = \{0\}$. Thus M, [1], is an open subset of a geodesic hypersphere or of a tube of radius $r, 0 < r < \Pi/2$, over $QP^k, k \in \{1, \cdots, m-2\}$ or of a tube of radius $r, 0 < r < \Pi/4$, over CP^m.

In the second case, M has 3 distinct principal curvatures $\lambda_1 = cot(r)$ with multiplicity $4(m-k-1), \lambda_2 = -tan(r)$ with multiplicity $4k$ and $\alpha = 2cot(2r)$ with multiplicity 3

Let us take a unit X such that $AX = \lambda_1 X$. If we develop $g((\nabla_X S)\Phi_1 X, U_1)$ we obtain $c = -(hcot(r) - cot^2(r) + 3 - 2hcot(2r) + 4cot^2(2r))cot(r)$. If we take a unit Y such that $AY = \lambda_2 Y$ and develop $g((\nabla_Y S)\Phi_1 Y, U_1)$ we get $c = (-htan(r) - tan^2(r) + 3 - 2hcot(2r) + 4cot^2(2r))tan(r)$. From this we get $tan^2(r) = -1$ which is impossible

The same result is obtained if M is an open subset of a tube of radius r, $0 < r < \Pi/4$, over CP^m

On the other hand, if M is an open subset of a geodesic hypersphere M has two distinct principal curvatures, $\lambda = cot(r)$ with multiplicity $4(m-1)$ and $\alpha = 2cot(2r)$ with multiplicity 3 Then it is easy to see that such an M satisfies (1 3) and this finishes the proof

Finally, the fact of a real hypersurface M of QP^m, $m \geq 3$, being pseudo-Einstein is equiva – lent to the fact that $g(SX,Y) = ag(X,Y)$ for any $X, Y \in \mathbb{D}$ and that $U_i, i = 1,2,3$, are eigenvectors of S This is equivalent to $g(SX,Y) = \rho_0 g(X,Y)$ for any $X, Y \in \mathbb{D}$ and $\rho_0 = (\rho - \Sigma_{i=1}^3 g(SU_i, U_i))/4(m-1)$ This is equivalent to $SX - \Sigma_{i=1}^3 f_i(X)SU_i - \rho_0 X - \Sigma_{i=1}^3 g(SX,U_i)U_i + \Sigma_{i=1}^3 f_i(X)g(SU_i,U_i)U_i + \rho_0\Sigma_{i=1}^3 f_i(X)U_i = 0$. If we define the tensor P as $P(X,Y) = g(SX,Y) - \rho_0 g(X,Y) + \rho_0\Sigma_{i=1}^3 f_i(X)f_i(Y) + \Sigma_{i=1}^3\{f_i(SU_i)f_i(X)f_i(Y) - f_i(X)f_i(SY) - f_i(SX) f_i(Y)\}$ for any X, Y tangent to M and compute its length we obtain

$$\| P \|^2 = \| S \|^2 - 4(m-1)\rho_0^2 - 2\Sigma_{i=1}^3 \| SU_i \|^2 + \Sigma_{i=1}^3 (f_i(SU_i))^2 \tag{5 6}$$

But it is easy to see that for any real hypersurface M

$$\Sigma_{i=1}^3 g(SU_i, SU_i) \geq \Sigma_{i=1}^3 (g(SU_i, U_i))^2 \tag{5 7}$$

Then (1 4) follows from (5 6), (5 7) and the expression of ρ_0 Moreover if U_i, $i = 1,2,3$, are eigenvectors of S we obtain the equality in (1 4) Thus we have finished the proof of Theorem 5

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