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TOTALLY REAL SUBMANIFOLDS OF A COMPLEX SPACE FORM

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ABSTRACT. Totally real submanifolds of a complex space form are studied. In particular, totally real submanifolds of a complex number space with parallel mean curvature vector are classified.

KEY WORDS AND PHRASES. Totally real submanifolds, isoperimetric section and complex space form.

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0. INTRODUCTION.

Totally real submanifolds of a Kaehler manifold are very typical submanifolds of a Kaehler manifold introduced by Chen and Ogiue [2] and Yau [9]. In particular Chen, Houh and Lue [1] pointed out that it is interesting to study totally real submanifolds of the complex number space C^m with parallel isoperimetric section and they classified compact totally real submanifolds with nonnegative sectional curvature in C^m. In 1987, Urbano [7] studied compact totally real submanifold with non-vanishing parallel mean curvature vector.

In this paper, we shall study m-dimensional complete totally real submanifolds of a complex space form $M^m(c)$ and obtain some classification theorems.

1. PRELIMINARIES.

Let $\widetilde{M}$ be a Kaehler manifold of real dimension $2m$ with almost complex structure J and metric tensor g. We then have $J^2 = -I$ and $g(JX, JY) = g(X,Y)$ for any vector fields. X and Y on $\widetilde{M}$, where I denotes the identity transformation on the tangent bundle. Let $\widetilde{\nabla}$ be the Levi-Civita connection of $\widetilde{M}$ satisfying $\widetilde{\nabla} J = 0$. Let M be an n-dimensional Riemannian manifold isometrically immersed in $\widetilde{M}$ by the immersion $i: M \to \widetilde{M}$. We then obtain the induced metric on M which will be represented the same notation g. We also identify X with $i_*(X)$ and M with $i(M)$.

Let ∇ be the induced Levi-Civita connection on M. Then the equations of Gauss and Weingarten are respectively given by $\widetilde{\nabla}_X Y = \nabla_X Y + h(X,Y)$ and $\widetilde{\nabla}_X \xi = -A_\xi X + \nabla^\perp_X \xi$, where h is the second fundamental form, A_ξ the Weingarten map associated to the normal vector field ξ satisfying $g(h(X,Y),\xi) = g(A_\xi X, Y)$ and $\nabla^\perp$ the connection in the normal bundle $T^\perp M$ of M. The mean curvature vector H is then given by $H = \frac{1}{n}\, Trh$. An n-dimensional submanifold M in a Kaehler manifold $\widetilde{M}$ is called *totally real* if $J(T_PM) \subset T_P^\perp M$ for each P in M, where T_PM is the tangent space of M at P and $T_P^\perp M$ the normal space of M at P.

Since J has the maximal rank, $m \geq n$. Let $N_P(M)$ be the orthogonal complement of $J(T_PM)$ in $T_P^\perp M$. Then we get the decomposition $T_P^\perp M = J(T_PM) \oplus N_P(M)$. It follows that the space $N_P(M)$ is invariant under the action of J.

We now consider an m-dimensional totally real submanifold M of $2m$-dimensional Kaehler manifold $\widetilde{M}$. Then we may set

$$JX = \theta(X), \tag{1.1}$$

$$J\xi = -U_\xi, \tag{1.2}$$

where X is a vector field tangent to M, $\theta(X)$ a normal vector valued 1-form, ξ a normal vector field and U_ξ a vector field on M satisfying $g(U_\xi, X) = g(\theta(X), \xi)$. Applying J to (1.1) and (1.2), we have

$$X = U_{\theta(X)} \text{ and } \theta(U_\xi) = \xi. \tag{1.3}$$

Differentiating (1 1) and (1.2) covariantly and making use of the equations of Gauss and Weingarten, we get

$$U_{h(X,Y)} = A_{\theta(X)}Y, \tag{1.4}$$

$$\theta(\nabla_X Y) = \nabla^\perp_X \theta(X), \tag{1.5}$$

$$\nabla_X U_\xi = U_{\nabla^\perp_X \xi}, \tag{1.6}$$

$$\theta(A_\xi X) = h(X, U_\xi), \tag{1.7}$$

where X and Y are vector fields tangent to M and ξ a vector field normal to M.

We now assume that the ambient manifold $\widetilde{M}$ is of constant holomorphic sectional curvature $4c$, which is called a complex space form and it is denoted by $M(c)$. Then the Riemann Christoffel curvature tensor $\widetilde{R}$ of $M(c)$ has the form

$$g(\widetilde{R}(X,Y)Z,W) = c(g(X,W)g(Y,Z) - g(Y,W)g(X,Z) + g(JX,W)g(JY,Z) \\ - g(JY,W)g(JX,Z) - 2g(JX,Y)g(JZ,W)).$$

Since the manifold M is totally real, it follows from equations(1.1)-(1.7) that the equations of Gauss, Codazzi and Ricci for M are respectively obtained

$$g(R(X,Y)Z,W) = c(g(X,W)g(Y,Z) - g(Y,W)g(X,Z)) \\ + g(h(X,W),h(Y,Z)) - g(h(Y,W),h(X,Z)), \tag{1.8}$$

$$(\overline{\nabla}_X h)(Y,Z) = (\overline{\nabla}_Y h)(X,Z), \tag{1.9}$$

$$g(R^\perp(X,Y)\xi,\eta) = c(g(\theta(X),\eta)g(\theta(Y),\xi) - g(\theta(Y),\eta)g(\theta(X),\xi)) \\ + g([A_\xi, A_\eta]X,Y),$$

where $\overline{\nabla}$ is the covariant derivative on $T(M) \oplus T^\perp(M)$ defined by $(\overline{\nabla}_X h)(Y,Z) = \nabla^\perp_X h(Y,Z) - h(\nabla_X Y, Z) - h(Y, \nabla_X Z)$, R and $R^\perp$ are the Riemann curvature tensor of M and that in the normal bundle respectively and $[A_\xi, A_\eta] = A_\xi A_\eta - A_\eta A_\xi$.

2. FUNDAMENTAL LEMMAS.

In this section, we assume that M is an m-dimensional totally real submanifold of a complex space form $M(c)$ of real dimension $2m$ A normal vector field ξ is said to be *parallel* if $\nabla^{\perp}_X \xi = 0$ for any vector field X on M and ξ is called an *isoperimetric section* if $Tr\ A_\xi$ is non-zero constant

LEMMA 1. Let M be an m-dimensional totally real submanifold of $M(c)$ with parallel isoperimetric section ξ If A_ξ has no simple eigenvalues, then $M(c)$ is flat

PROOF. Since A_ξ is self-adjoint with respect to g, there exists an orthonormal basis $\{e_1, e_2, \cdots, e_m\}$ for $T_P M$ such that $g(A_\xi e_i, e_i) = \lambda_i \delta_{ij}$, where $\lambda_1, \lambda_2, \cdots, \lambda_m$ are eigenvalues of A_ξ. Since ξ is parallel, we see that

$$\begin{aligned} g([A_\xi, A_\eta]e_i, e_j) &= (\lambda_i - \lambda_j) g(A_\eta e_i, e_j) \\ &= c(g(\theta(e_i), \eta) g(\theta(e_j), \xi) - g(\theta(e_j), \eta) g(\theta(e_i), \xi)) \end{aligned}$$

for any normal vector field η because of (1.10). Since A_ξ has no simple eigenvalues, for each $i \in \{1, 2, \cdots, m\}$ there is $j \neq i$ such that

$$c(g(\theta(e_i), \eta) g(\theta(e_j), \xi) - g(\theta(e_j), \eta) g(\theta(e_i), \xi)) = 0.$$

Choosing η as $\theta(e_i)$, we get $cg(\theta(e_i), \xi) = 0$ By (1 1), we see that $\{\theta(e_i) \mid i = 1, 2, \cdots, m\}$ forms an orthonormal basis for $T_P^{\perp} M$. It follows that $M(c)$ is flat. (Q.E.D.)

REMARK 1. Let M be an m-dimensional totally real submanifold of $M(c) (c \neq 0)$. If M has an isoperimetric section ξ, then A_ξ has simple eigenvalues

Let H be the mean curvature vector field defined by $H = \frac{1}{n}\ Trh$. We now assume that H is nonvanishing parallel in the normal bundle. We choose an orthonormal frame $\{\xi_1, \xi_2, \cdots, \xi_m\}$ in the normal bundle in such a way that $\xi_1 = H / \parallel H \parallel$. It follows that $TrA_i = 0$ for $i \geq 2$, where $A_i = A_{\xi_i}$ and $U_1, U_2, \cdots U_m$ form an orthonormal basis for $T_P M$ because of (1.2), where $U_i = U_{\xi_i}$. Then (1.3) and (1.4) imply

$$A_i U_j = U_{h(U_i, U_j)}, \tag{2.1}$$

which shows that

$$A_i U_j = A_j U_i.$$

Taking the scalar product with ξ_k and making use of (1.3), (1.7) and (2.1), we may set

$$A_i U_j = \sum_k P_{ijk} U_k, \tag{2.2}$$

where $P_{ijk} = g(\theta(A_i U_j), \xi_k)$. Because A_i is a symmetric operator and h is a symmetric bilinear form, P_{ijk} is symmetric with respect to all indices i, j and k.

On the other hand, (2.2) implies

$$h(U_i, U_j) = \theta(A_i U_j) = \sum_k P_{ijk} \xi_k.$$

Since any vector field X on M can be expressed as $X = \sum_k g(X, U_k) U_k$, h can be written by

$$h(X, Y) = \sum_{i,j,k} P_{ijk} g(\theta(X), \xi_i) g(\theta(Y), \xi_j) \xi_k, \tag{2.3}$$

which implies

$$Trh = \sum_k P_k \xi_k, \tag{2.4}$$

where $P_k = \sum_i P_{iik}$. Since ξ_1 is parallel in the normal bundle, (1 10) gives

$$g([A_i, A_1]X, Y) = c(g(\theta(Y), \xi_1) g(\theta(X), \xi_i) - g(\theta(X), \xi_1) g(\theta(Y), \xi_i) \tag{2.5}$$

for all vector fields X and Y on M. (2.5) together with (2.3) yields

$$\sum_{i,j} P_{kji} P_{1ji} - (TrA_1) P_{11k} = c(m-1)\delta_{1k} \tag{2.6}$$

and hence

$$\sum_{i,j} (P_{1ji})^2 = (TrA_1) P + c(m-1), \tag{2.7}$$

where $P = P_{111}$.

We now prove

LEMMA 2. Let M be an m-dimensional totally real submanifold of a complex space form $M(c)$ with nonvanishing parallel mean curvature vector H. Then A_H is parallel.

PROOF. Let $\{e_1, e_2, \cdots, e_m, \xi_1, \xi_2, \cdots, \xi_m)$ be an orthonormal frame of $M(c)$ at a point P of M such that $e_1, e_2, \cdots, e_m$ are tangent to M and $\xi_1, \xi_2, \cdots, \xi_m$ are normal to M, where $\xi_1 = H / \| H \|$. Then we get

$$\tfrac{1}{2} \Delta Tr A_1^2 = g(\Delta' A_1, A_1) + \| \nabla A_1 \|^2, \tag{2.8}$$

where Δ is the Laplacian operator and $\Delta' A_1$ denotes the restricted Laplacian Δ' of A_1 is given by

$$(\Delta' A_1) X = \sum_i [R(e_i, X), A_1] e_i$$

(see [6] for detail). Making use of (1.8) of Gauss and the fact that M is totally real, we have

$$\begin{aligned}\Delta' A_1 = c(m-1)A_1 - c(TrA_1)(I - U_1 \otimes U_1) + (TrA_1) \sum_{i,j,k} P_{ij1} P_{jk1} U_j \otimes U_k \\ - \sum_{i,j,k} P_{ijk} P_{ij1} A_k\end{aligned} \tag{2.9}$$

with the help of (2.3), (2.4) and (2.5). If we use (2.5) and (2.6), we obtain

$$g(\Delta' A_1, A_1) = 0. \tag{2.10}$$

On the other hand, we can put

$$A_1 X = \sum_{i,j} P_{ij1} g(U_i, X) U_j \tag{2.11}$$

because of (2.3). We now extend $\xi_1, \xi_2, \cdots, \xi_m$ to differentiable orthonormal normal vector fields defined on a normal neighborhood O of P by parallel translation with respect to normal connection along geodesics in M. Then we get

$$(\nabla_Y A_1) X = \sum_{i,j} (\nabla Y P_{ij1}) g(U_i, X) U_j \text{ at } P \tag{2.12}$$

because of (1.6). Therefore, $\Delta' A_1$ is reduced to

$$\Delta' A_1 = \sum_{i,j} (\nabla_Y P_{ij1}) U_i \otimes U_j\,. \tag{2.13}$$

If we use (2.9), then we have

$$g((\Delta' A_1) U_1, U_1) = c(m-1)P + (TrA_1) \sum_i (P_{i11})^2 - \sum_{i,j,k} P_{ijk} P_{ij1} P_{k11}.$$

Making use of (2.6), we obtain

$$g((\Delta' A_1)U_1, U_1) = 0.$$

Thus (2 13) implies

$$\Delta P = 0. \tag{2.14}$$

Since $TrA_1^2 = \sum_i g(A_1U_i, A_1U_i) = \sum_{i,j}(P_{ij1})^2 = (TrA_1)P + c(m-1)$, we see that

$$\tfrac{1}{2}\Delta(TrA_1^2) = (TrA_1)\Delta P = 0.$$

Combining (2.8), (2.10) and the last equation, we get the result (Q.E.D)

3. MAIN THEOREMS.

Let M be an m-dimensional totally real submanifold of a complex space form $M(c)$ with nonvanishing parallel mean curvature vector. By lemma 2, we know that A_H is parallel. We now define a function h_n for any integer $n \geq 1$ by $h_n = Tr(A_H^n)$. Then h_n is constant on M for any integer n since A_H is parallel. This implies that each eigenvalue λ_j of A_H is constant on M. Let $\mu_1, \mu_2, \cdots, \mu_\alpha$ be mutually distinct eigenvalues of A_H and $n_1, n_2, \cdots, n_\alpha$ their multiplicities. So the smooth distributions T_β consisting of all eigenvectors corresponding to μ_β are defined and orthogonal each other.

Since A_H is parallel, T_β are parallel and completely integrable. By the de Rham decomposition theorem [4], the submanifold M is a product manifold $M_1 \times M_2 \times \cdots \times M_\alpha$, where the tangent bundle of M_β corresponds to T_β. We now assume that the ambient manifold is flat, that is, a complex number space C^m and M is embedded in C^m. Then as in [1] we can choose an orthonormal basis $e_1, e_2, \cdots, e_m$ for T_pM as eigenvectors of A_H and $J_{e_1}, J_{e_2}, \cdots, J_{e_m}$ for $J(T_PM)$ in such a way that $h^k_{ji} = h^i_{jk} = h^j_{ik}$, where $h^k_{ji} = g(A_{J_{e_k}}e_i, e_j)$ and $h^k_{ji} = 0$ for $e_j \in [\mu_\beta], e_i \in [\mu_\gamma], \beta \neq \gamma$, where $[\mu_\beta]$ is the eigenspace corresponding to the eigenvalue μ_β.

Let $\pi_\beta(H)$ be the component of H in the subspace $C^{v\beta}$. Then $\pi_\beta(H)$ is a parallel normal section of M_β in $C^{v\beta}$ and M_β is umbilical with respect to $\pi_\beta(H)$. Therefore, M_β is a minimal submanifold of a hypersphere in $C^{v\beta}$. Hence M is a product submanifold $M_1 \times M_2 \times \cdots \times M_\alpha$ embedded in $C_m = C^{v1} \times C^{v2} \times \cdots \times C^{v\alpha}$, where M_β is a totally real submanifold embedded in some $C^{v\beta}$. Thus we have

THEOREM 1. Let M be an m-dimensional complete totally real submanifold embedded in a complex number space C^m. If M has parallel mean curvature vector H, then M is either a minimal submanifold or a product submanifold $M_1 \times M_2 \times \cdots \times M_\alpha$ embedded in $C^m = C^{v1} \times C^{v2} \times \cdots \times C^{v\alpha}$, where M_β is a totally real submanifold embedded in some $C^{v\beta}$ and M_β is also a minimal submanifold of a hypersphere of $C^{v\beta}$

THEOREM 2. Let M be an m-dimensional complete totally real submanifold embedded in a complex number space C^m. If M has the nonvanishing parallel mean curvature vector and A_H has mutually distinct eigenvalues, then M is a product submanifold of circles $S^1 \times S^1 \times \cdots \times S^1$.

PROOF. By a lemma of Moore [5], $M = M_1 \times M_2 \times \cdots \times M_m$ is a product immersion embedded in C^m, and M_i is a totally real submanifold in C^{n_i} and contained in a hypersphere in C^{n_i}. Since $n_1 + n_2 + \cdots + n_m = m, n_i$ must be 1. Hence $M_i = S^1$, a circle in a complex space C. (Q.E.D.)

THEOREM 3. Let M be an m-dimensional totally real submanifold of a complex space form $M(c)$ with nonvanishing parallel mean curvature vector H If A_H has mutually distinct eigenvalues, then M is flat.

PROOF. Let $e_1, e_2, \cdots e_m$ be eigenvectors of A_H corresponding to eigenvalues $\lambda_1, \lambda_2, \cdots, \lambda_m$ respectively. Since A_H is parallel by Lemma 2, we have

$$A_H R(X,Y)e_i = \lambda_i R(X,Y)e_i$$

for any vector fields X and Y on M, that is $R(X,Y)e_i$ is an eigenvector of A_H corresponding to λ_i. Taking the inner product with e_j, we obtain

$$(\lambda_i - \lambda_j) g(R(X,Y)e_i, e_j) = 0$$

because A_H is a symmetric operator. Thus M is flat if A_H has mutually distinct eigenvalues. (Q.E.D.)

REMARK. Let M be a totally real submanifold of complex space form $M(c)$ with nonvanishing parallel mean curvature vector H. Considering Lemma 1, we see that $M(c)$ is flat if the sectional curvatures defined by principal vectors of H are nonzero.

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