A UNIFORM ESTIMATE FOR THE RATE OF CONVERGENCE IN THE MULTIDIMENSIONAL CENTRAL LIMIT THEOREM FOR HOMOGENEOUS MARKOV CHAINS

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ABSTRACT. In this paper a uniform estimate is obtained for the remainder term in the central limit theorem (CLT) for a sequence of random vectors forming a homogeneous Markov chain with arbitrary set of states. The result makes it possible to estimate the rate of convergence in the CLT without assuming the finiteness of the absolute third moment of the transition probabilities. Some consequences are also proved

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1. INTRODUCTION.

Let $x_1, x_2, \ldots$ be a homogeneous Markov process with an arbitrary set of states X, defined by the transition probability function $P(\xi, A)$, $\xi \in X$, $A \in \Im_X$ (σ-algebra of subsets of X), and the initial distribution:

$$P(x_1 \in A) = \pi(A), \qquad A \in \Im_X.$$

Suppose that $P(.,.)$ satisfies the following condition of uniform ergodicity:

$$\sup_{\xi, \eta \in X, A \in \Im_X} |P(\xi, A) - P(\eta, A)| = \rho < 1. \tag{1.1}$$

It is well known that (see [13]) if (1.1) is fulfilled, then a stationary distribution $P(A)$, $A \in \Im_X$ exists.

Let

$$f(\xi) = (f_1(\xi), \ldots, f_k(\xi)),$$

where $f_i(.)$, $(i = \overline{1,k})$ are real $\Im_X$-measurable functions defined on X. Without loss of generality, in the next we shall suppose that

$$\int_X f_i(\eta)P(d\eta) = 0, \qquad i = \overline{1,k}.$$

Suppose that the matrix $\Lambda = \|\sigma_{ij}\|_1^k$, where

$$\sigma_{ij} = E_p[f_i(x_1)f_j(x_1)] + \sum_{r=1}^{\infty} E_p[f_i(x_1)f_j(x_{r+1})] + \sum_{r=1}^{\infty} E_p[f_j(x_1)f_i(x_{r+1})]$$

is positive definite Let,

$$S_n = \frac{1}{\sqrt{n}} \sum_{r=1}^{n+1} Bf(x_r), \qquad P_n(A) = P(S_n \in A) ,$$

$$\triangle_n(A) = P_n(A) - \Phi(A) , \qquad A \in B^k ,$$

where the matrix B is such that $B'B = \Lambda^{-1}$, $\Phi(.)$ is the standard normal measure in $\Re^k$, and B^k is the class of all Borel sets in $\Re^k$.

In many articles (see for example, [2],[5],[6],[7], and [8]) uniform estimates (in particular, the Berry-Essen bounds) are obtained for the quantity:

$$\sup_{A \in B_0^k} |\triangle_n(A)| , \tag{1.2}$$

under the condition that the transition probability function $P(.,.)$ has a finite absolute moment of order not less than three. (Here B_0^k is the class of all convex measurable sets in $\Re^k$).

In the present work a uniform estimate is obtained for (1.2) without assuming the finiteness of:

$$m_3 = \sup_{\xi \in X} \int_X |f(\eta)|^3 P(\xi, d\eta) .$$

This estimate extends the results pertaining to the case of the existence of absolute moments for the transition probabilities of order not less than the third for a sequence of random vectors that are either independent and identically distributed ($P(.,.) = P(.) = \pi(.)$), or linked in a homogeneous Markov chain (see, for example, [1], [19], and [20]).

Some other results, concerning (1.2), are also given.

We introduce the following "truncated" moments:

$$\beta_n = \sup_{\xi \in X} \left[\int_{X_n} |f(\eta)|^3 P(\xi, d\eta) + \sqrt{n} \int_{\overline{X}_n} |f(\eta)|^2 P(\xi, d\eta) \right] ,$$

$$\delta_n = \int_X |f(\xi)| P(d\xi) \int_{\overline{X}_n} |f(\eta)| \, \| S(\xi, d\eta)| + \int_{\overline{X}_n} |f(\xi)| P(d\xi) \int_X |f(\eta)| \, \| S(\xi, d\eta)| ,$$

$$m_{\pi,1} = \int_{X_n} |f(\xi)| \pi(d\xi) ,$$

and put,

$$L_{\beta_n} = \frac{1}{\sqrt{n}} \sum_{r=1}^{k} \left[\lambda_r^{-3} \sup_{\xi \in X} \int_{X_n} |(\theta_r, f(\eta))|^3 P(\xi, d\eta) + \sqrt{n} \, \lambda_r^{-2} \sup_{\xi \in X} \int_{\overline{X}_n} |(\theta_r, f(\eta))|^2 P(\xi, d\eta) \right] ,$$

$$L_{\delta_n} = \sum_{i=1}^{k} \sum_{j=1}^{k} (\lambda_i \lambda_j)^{-1} \left[\int_X |(\theta_i, f(\xi))| P(d\xi) \int_{\overline{X}_n} |(\theta_j, f(\eta))| \, |S(\xi, d\eta)| + \right.$$
$$\left. + \int_{\overline{X}_n} |(\theta_i, f(\xi))| P(d\xi) \int_X |(\theta_j, f(\eta))| \, |S(\xi, d\eta)| \right] ,$$

and,

$$L_{\pi,n} = \frac{1}{\sqrt{n}} \sum_{r=1}^{k} \lambda_r^{-1} \int_{X_n} |(\theta_r, f(\xi))| \pi(d\xi) ,$$

where θ_ν, $\nu = 1, 2, ..., k$ are orthonormal eigenvectors of Λ and λ_ν^2 are the corresponding eigenvalues,

$$X_n = \left\{ \xi : |f(\xi)| \le \sqrt{n} \right\} , \qquad \overline{X}_n = X \backslash X_n ,$$

$|S(., A)|$ is the complete variation of the measure,

$$S(.,A) = \sum_{r=1}^{\infty} \left(P^{(r)}(.,A) - P(A) \right), \qquad A \in \Im_X$$

and $P^{(r)}(.,.)$ is the transition probability function after r-steps   The measure $S(.,A)$, $A \in \Im_X$ exists in accordance with condition (1 1)

2. FORMULATION OF THE RESULTS

THEOREM 1. The inequality

$$\sup_{A \in B_0^k} |\triangle_n(A)| \le C(k,\rho) \left[L_{\beta_n} + L_{\delta_n} + L_{\pi,n} + \left(\frac{1}{3} + \frac{2}{3}\rho \right)^n \right], \tag{2.1}$$

holds true. Where here and in what follows, $C(.)$ denote a positive constant, not the same at different places, which depends only on the arguments in the parentheses

As it is shown in [1], [11], and [15] the expression L_{β_n} characterizes the corresponding result in the case where summation of independent identically distributed random vectors $(p(.,.) = p(.) = \pi(.))$ is considered. The expression L_{δ_n} characterizes, in some sense, the dependence structure of the random vectors

$$f(x_1), \qquad f(x_2), \quad ... \tag{2.2}$$

($L_{\delta_n} = 0$ for independent identically distributed random vectors).

REMARK. If $g(x)$ is an increasing function, with positive argument, defined on $\Re^k$, such that $\frac{x}{g(x)}$ is non-decreasing, and $\lim_{x \to \infty} g(x) = \infty$, and if

$$m_g = \sup_{\xi \in X} \int_X |f(\eta)|^2 g(|f(\eta)|) P(\xi, d\eta) < \infty,$$

then (2.1) remains true, but with the replacement of $(L_{\beta_n} + L_{\delta_n})$ by $(L_g + Q_g)/g(k\sqrt{n})$, where

$$L_g = \sum_{r=1}^{k} (\lambda_r^{-3} + \lambda_r^{-2}) \sup_{\xi \in X} \int_X |(\theta_r, f(\eta)|^2 g(|(\theta_r, f(\eta))|) P(\xi, d\eta), \qquad \text{and}$$

$$Q_g = \sum_{\iota=1}^{k} \sum_{j=1}^{k} \left[\int_X |(\theta_\iota, f(\xi))| P(d\xi) \int_X |(\theta_j, f(\eta))| g(|(\theta_j, f(\eta))|) |S(\xi, d\eta)| \right.$$

$$\left. + \int_X |(\theta_\iota, f(\xi))| g(|(\theta_\iota, f(\xi))|) P(d\xi) \int_X |(\theta_j, f(\eta))| |S(\xi, d\eta)| \right].$$

If $\pi(.) = p(.)$, then the sequence (2.2) forms a stationary sequence of random vectors and in this case Theorem 1 has the following simple form:

THEOREM 2. If $\pi(.) = p(.)$, then

$$\sup_{A \in B_0^k} |\triangle_n(A)| \le C(k,\rho)[L_{\beta_n} + L_{\delta_n}]. \tag{2.3}$$

Repeating the arguments given in [3], and [4], one can see that the estimate (2.3) has a universal character. In particular, from (2.3) one can derive an analogy of the generalized Berry-Esseen inequality adduced in [12], and [16] for independent random vectors.

The next result follows from Theorem 1.

THEOREM 3. If

$$m_2 = \sup_{\xi \in X} \int_X |f(\eta)|^2 P(\xi, d\eta) < \infty,$$

then the CLT holds for the sequence (2.2), i.e. for any initial distribution $\pi(.)$:

$$\sup_{A \in B_0^k} |\triangle_n(A)| \to 0 \qquad \text{as} \qquad n \to \infty .$$

3.   AUXILIARY RESULTS

Assume that

$$\triangle = J \tag{3 1}$$

where J is the k-dimensional unit matrix Consequently $B = J$ Assume also that

$$\frac{\beta_n}{\sqrt{n}} + \delta_n \leq C(k, \rho) . \tag{3.2}$$

Through the proof of Theorem 1, it is shown that assumptions (3.1) and (3.2) do not bound the generality

Let

$$f_n(.) = (f_{n1}(.), f_{n2}(.), ..., f_{nk}(.))$$
$$= \begin{bmatrix} f(.), & |f(.)| \leq \sqrt{n} \\ 0 , & |f(.)| > \sqrt{n} \end{bmatrix}$$

$$\overline{S}_n = \frac{1}{\sqrt{n}} \sum_{r=1}^{n+1} f_n(x_r) , \qquad \Psi_n(t) = E\left[e^{i(t,\overline{S}_n)}\right] .$$

Put

$$|\hat{t}| = \max(1, |t|) , \qquad \text{and} \qquad T_n = \frac{\sqrt{n}}{50 \overline{\beta}_n} ,$$

where

$$\overline{\beta}_n = 20 M^3(\rho) \beta_n , \qquad \text{and} \qquad M(\rho) = (9 + 12\rho)/(1 - \rho) .$$

LEMMA 3.1. For $|t| < T_n$

$$\left| \Psi_n(t) - e^{-|t|^2/2} \right| \leq C(k, \rho) \left[\left(\frac{\beta_n}{\sqrt{n}} + \delta_n \right) \max(|t|, |t|^3) + m_{\pi,1} \frac{|t|}{\sqrt{n}} \right] e^{-t^2/20}$$
$$+ C(1)\beta_n \left(\frac{1}{3} + \frac{2}{3}\rho \right)^n \frac{|t|}{\sqrt{n}} .$$

PROOF. Put

$$\sigma_{ij}^{(n)} = E_p[f_{ni}(x_1)f_{nj}(x_1)] + \sum_{r=1}^{\infty} E_p[f_{ni}(x_1)f_{nj}(x_{r+1}) - E_p[f_{ni}(x_1)f_{nj}(x_{r+1})]]$$
$$+ \sum_{r=1}^{\infty} E_p[f_{nj}(x_1)f_{ni}(x_{r+1}) - E_p[f_{nj}(x_1)f_{ni}(x_{r+1})]] .$$

It is clear that, for $i = \overline{1, k}$, $r = 1, 2, ...$

$$E_p[f_{ni}(x_1)] = E_p[f_{ni}(x_{r+1})] . \tag{3.3}$$

Therefore,

$$|E_p[f_{ni}(x_1)]| = \left| \int_{\overline{X}_n} f_i(x)P(dx) \right| \leq \frac{\beta_n}{\sqrt{n}} . \tag{3.4}$$

Similarly, it is easy to see that, for any $i, j = \overline{1, k}$

$$\left| E_p[f_{ni}(x_1)f_{nj}(x_1)] - E_p[f_i(x_1)f_j(x_1)] \right| \leq \frac{\beta_n}{\sqrt{n}} . \tag{3 5}$$

Further, using (3 3), for any i and j

$$\left| E_p[f_i(x_1)f_j(x_{r+1})] - E_p\Big[f_{ni}(x_1)f_{nj}(x_{r+1}) - E_p[f_{ni}(x_1)]E_p[f_{nj}(x_{r+1})] \Big] \right|$$
$$\leq \int_X |f(\xi)|P(d\xi) \int_{\overline{X}_n} |f(\eta)|V_r(\xi, d\eta)$$
$$+ \int_{\overline{X}_n} |f(\xi)|P(d\xi) \int_X |f(\eta)|V_r(\xi, d\eta) \tag{3 6}$$

where $V_r(x, .)$ is the complete variation of the measure $(P^{(r)}(x, .) - P(.))$ From (3.4)-(3 6), it follows that

$$\left| \sigma_{ij}^{(n)} - \delta_{ij} \right| \leq C(\rho)\left(\frac{\beta_n}{\sqrt{n}} + \delta_n \right) \tag{3 7}$$

where δ_{ij} is the kronker's symbol

Consider the $\Im_X$-measurable function

$$f_{t,n}(x) = f_t(x) = (t, f_n(x)), \qquad x \in X ,$$

and let $m_t = E_p[f_t(x_1)]$,

$$\sigma_t^2 = E_p\big[(f_t(x_1) - m_t)^2\big] + 2\sum_{r=1}^{\infty} [E_p[f_t(x_1)f_t(x_{r+1})] - m_t^2]$$

and

$$\gamma_t^3 = \sup_{\xi \in X} \int_X |f_t(\eta) - m_t|^3 P(\xi, d\eta) .$$

It is easy to see that

$$|m_t| \leq \sum_{j=1}^{k} |t_j| \left| E_p[f_{nj}(x_1)] \right| \leq k|t|\frac{\beta_n}{n} , \tag{3.8}$$

and that

$$\sigma_t^2 = (t, \sigma_n t) , \tag{3.9}$$

where

$$\sigma_n = \left\| \sigma_{ij}^{(n)} \right\|_1^k .$$

It follows from (3.1) and (3.7)-(3.9) that,

$$\left| \sigma_t^2 - |t|^2 \right| \leq C(k, \rho)\left(\frac{\beta_n}{\sqrt{n}} + \delta_n \right)|t|^2 , \tag{3.10}$$

Consider now the sum of one-dimensional random variables

$$\widehat{S}_n = \frac{1}{\sigma_t \sqrt{n}} \sum_{r=1}^{n+1} (f_t(x_r) - m_t) .$$

and put

$$\overline{\Psi}_n(\tau) = E(e^{i\tau\widehat{S}_n}) , \qquad \overline{T}_n = \frac{\sigma_t^3 \sqrt{n}}{100 M^3(\rho)\gamma_t^3} .$$

According to the results of [14], we have for $|\tau| < \overline{T}_n$,

$$\left|\overline{\Psi}_n(\tau) - e^{-\frac{1}{2}\tau^2}\right| \leq \left[\frac{2\gamma_t^3}{\sigma_t^3 \sqrt{n}} \left(|\tau|^3 + |\tau|\right) + \frac{m_{11}}{\sigma_t \sqrt{n}} |\tau|\right] e^{-\frac{1}{10}\tau^2} + \frac{2\gamma_t^3}{\sigma_t^3 \sqrt{n}} |\tau|\left(\frac{1}{3} + \frac{2}{3}\rho\right)^n , \qquad (3\ 11)$$

where

$$m_{11} = \int_X |f_t(\xi)|\pi(d\xi) .$$

In accordance with (3.2) and (3.10), we have

$$\left|\sigma_t^2 - |t|^2\right| \leq 1/2 |t|^2 ,$$

and consequently,

$$1/2 \leq \frac{\sigma_t^2}{|t|^2} \leq 3/2 , \qquad (3.12)$$

As a consequence of (3.8), it is easy to see that

$$\gamma_t^3 \leq 5|t|^3 \beta_n . \qquad (3.13)$$

On putting (see [14]) $\overline{\beta}_n = 20 M^3(\rho)\beta_n$, we get from (3.12) and (3.13), that

$$\frac{\sigma_t}{\overline{T}_n} = \frac{100 M^3(\rho)\gamma_t^3}{\sigma_t^2 \sqrt{n}} \leq \frac{50\overline{\beta}_n |t|}{\sqrt{n}} \leq 1, \qquad \text{for} \qquad |t| \leq \frac{\sqrt{n}}{50\overline{\beta}_n} = T_n .$$

It is not hard to see that

$$\Psi_n(t) = e^{i\sqrt{n} m_t} \overline{\Psi}_n(\sigma_t) . \qquad (3.14)$$

From (3.11) and (3.14), on noticing that $m_{11} \leq |t|m_{\pi,1}$, and $\gamma_t^3/\sigma_t^3 \leq 10\sqrt{2}$, it follows that

$$\left|\Psi_n(t) - e^{i\sqrt{n} m_t - \frac{1}{2}\sigma_t^2}\right| \leq \left[50\sqrt{3} \frac{\beta_n}{\sqrt{n}} |\widehat{t}|^3 + \frac{|t|}{\sqrt{n}} m_{\pi,1}\right] e^{-\frac{1}{20}|t|^2} + 20\frac{\beta_n}{\sqrt{n}} |t|\left(\frac{1}{3} + \frac{2}{3}\rho\right)^n . \quad (3.15)$$

Further, using (3.8), (3.10), and (3.12), consistent with (3.2), it is easy to see that

$$\left|e^{i\sqrt{n} m_t - \frac{1}{2}\sigma_t^2} - e^{-\frac{1}{2}|t|^2}\right| \leq \left[K|t| \frac{\beta_n}{\sqrt{n}} + C(k,\rho)\left(\frac{\beta_n}{\sqrt{n}} + \delta_n\right)|t|^2\right] e^{-\frac{1}{4}|t|^2} . \qquad (3.16)$$

Finally, the assertion of the lemma follows from (3.15) and (3.16).

Through the proof of Theorem 1, we shall need estimates for the derivatives of the characteristic function $\Psi_n(.)$. Let us fix a direction θ and let t_θ be the projection onto θ.

LEMMA 3.2. a. For any $\nu \geq 0$, for $|t| \leq C_0(k,\rho)T_n^{1/2}$

$$\sup_\theta \left|\frac{\partial^\nu}{\partial t_\theta^\nu}\left(\Psi_n(t) - e^{-\frac{1}{2}|t|^2}\right)\right| \leq C(k,\rho,\nu)\left[|\widehat{t}|^{\nu^*} e^{-\frac{1}{20}|t|^2}\left(\frac{\beta_n}{\sqrt{n}} + \frac{m_{\pi,1}}{\sqrt{n}}\right) + \frac{m_1}{\sqrt{n}} (1/3 + 2/3\rho)^n |t|\right]$$

where

$$\nu^* = \max(2\nu, \nu+3), \qquad m_1 = \sup_{\xi \in X} \int_X |f(\eta)| P(\xi, d\eta) .$$

b For any $\nu > 0$, and for $|t| \leq T_n$

$$\sup_\theta \left| \frac{\partial^\nu}{\partial t_\theta^\nu} \Psi_n(t) \right| \leq c(k, \rho, \nu) \left[|\hat{t}|^\nu e^{-\frac{1}{20}|t|^2} + \frac{m_1}{\sqrt{n}} |t| \left(\frac{1}{3} + \frac{2}{3}\rho \right)^n \right] .$$

The proof of this lemma is easily obtained by repeating, word for word, the arguments of the proof of Lemma 2 1 in [8] with the use of the general spectral theory of linear operators acting in the Banach space of all bounded functions with the uniform norm Through the proof of part b, Lemma 3 of Chapter 6 in [16] is used

Let

$$\triangle(A) = \Phi(A) - P(A), \qquad A \in B^k ,$$

where $P(.)$ is a distribution function in $\Re^k$. Let $\eta^{(1)}$ be such a vector that

$$P_1'(A) = P(\eta^{(1)} \in A), \qquad \text{and} \qquad P(|\eta^{(1)}| > h) \leq \chi_1 h^{-1} ,$$

where $\chi_1 = \chi_1(k)$ depends only on k.

For any $T > 0$, put

$$\eta^{(T)} = T^{-1}\eta , \qquad P_T'(A) = P(\eta^{(T)} \in A), \qquad \text{and} \qquad \triangle_T(A) = (\triangle * P_T')(A)$$

where $*$ denotes the composition operation.

The following lemma is proved in [18] and [20].

LEMMA 3.3. For any $T > 0$

$$\sup_{A \in B_0^k} |\triangle_n(A)| \leq 2 \sup_{A \in B^k} |\triangle_T(A)| + \frac{12\sqrt{2}}{\sqrt{2\pi}} \frac{\chi_1}{T} k^{3/2} .$$

If $\eta^{(1)}$ has the distribution with density

$$u(x) = \alpha_{k,r} \left[J_{k/2}(|x|/2r)/|x|^{k/2} \right]^{2r} , \qquad (3.17)$$

where $J_{k/2}(.)$ is the Bessel function of order $k/2$, the constant $\alpha_{k,r}$ is such that $\int_{\Re^k} u(x)dx = 1$, and r is a positive integer, then for all $\nu \leq k + q$

$$E[|\eta^{(1)}|^\nu] \leq \chi_\nu(k, q) < \infty , \qquad \text{if} \qquad r > 2 + \frac{q-1}{k+1} \qquad (3.18)$$

and for $|t| > 1$

$$\int_{\Re^k} e^{i(t,x)} P_1'(dx) = 0 . \qquad (3.19)$$

4. PROOFS OF THE THEOREMS
PROOF OF THEOREM 1. Let

$$\overline{P}_n(A) = P(\overline{S}_n \in A), \qquad \text{and} \qquad \overline{\triangle}_n(A) = \overline{P}_n(A) - \Phi(A), \qquad A \in B^k .$$

It is not difficult to see that for any $A \in B^k$

$$|P_n(A) - \overline{P}_n(A)| \leq \sum_{r=1}^{n+1} P(|f(x_r)| > \sqrt{n}) . \qquad (4.1)$$

Further

$$P(|f(x_1)| \geq \sqrt{n}\,) = \int_{\overline{X}_n} \pi(dx) \leq \frac{m_{\pi,1}}{\sqrt{n}} \tag{4 2}$$

and for $r \geq 2$

$$P(|f(x_r)| \geq \sqrt{n}\,) = \int_X P^{(r-1)}(x, \overline{X}_n)\pi(dx) \leq \sup_x P^{(r-1)}(x, \overline{X}_n) \leq \sup_x P(x, \overline{X}_n)$$

$$\leq \frac{1}{(\sqrt{n})^3} \sqrt{n} \sup_{\xi \,\in\, X} \int_{\overline{X}_n} |f(\eta)|^2 P(\xi, d\eta) \leq \frac{\beta_n}{(\sqrt{n})^3} \; . \tag{4.3}$$

It follows from (4 1)-(4 3) that

$$|P_n(A) - \overline{P}_n(A)| \leq \frac{m_{\pi,1}}{\sqrt{n}} + \frac{\beta_n}{\sqrt{n}} \; . \tag{4.4}$$

Therefore, for any $A \in B^k$

$$|\triangle_n(A)| \leq |\overline{\triangle}_n(A)| + \frac{m_{\pi,1}}{\sqrt{n}} + \frac{\beta_n}{\sqrt{n}} \; . \tag{4.5}$$

Let $T > 0$ and $\overline{\triangle}_{n,T}(A) = (\overline{\triangle}_n * P_T')(A)$, where P_T' is the distribution with the density defined by (3.17). It follows from Lemma 3.3 that

$$\sup_{A \,\in\, B_0^k} |\overline{\triangle}_n(A)| \leq 2 \sup_{A \,\in\, B^k} |\overline{\triangle}_{n,T}(A)| + \frac{C(k)}{T} \; . \tag{4.6}$$

Let

$$V_T(t) = E\left[e^{i(t, \eta^{(T)})}\right] \; .$$

Thus, from (3.18) and (3.19), we have $V_T(t) = 0$, for $|t| > T$, and for all $\nu \leq k + q$

$$\sup_\theta \left|\frac{\partial^\nu}{\partial t_\theta^\nu} V_T(t)\right| \leq E[|\eta^{(T)}|^\nu] \leq \frac{C(k, \nu)}{T^\nu} \; . \tag{4.7}$$

Now, we put $T = T_n$ (defined in Lemma 3.1), and $q = 1$. It is clear that the generalized measure $\overline{\triangle}_{n,T_n}$ with the characteristic function

$$W_n(t) = \left(\Psi_n(t) - e^{-\frac{1}{2}|t|^2}\right) V_{T_n}(t)$$

satisfies all the conditions of Lemma 1 in [17]. Hence, from (4.7), for any $0 < \nu_0 < \infty$, we have

$$\int_{\Re^k} \left|\frac{\partial^{\nu_0}}{\partial t_\theta^{\nu_0}} W_n(t)\right| dt \leq C(T_n, \nu_0) \max_{\nu \,=\, 0, \nu_0} \int_{|t| \leq T_n} \left|\frac{\partial^\nu}{\partial t_\theta^\nu}\left(\Psi_n(t) - e^{-\frac{1}{2}|t|^2}\right)\right| dt \tag{4.8}$$

where

$$C(T_n, \nu_0) = \sum_{\nu=0}^{\nu_0} \binom{\nu}{\nu_0} C(\nu, k)/T_n^\nu \; .$$

In accordance with (3.2) it follows from Lemmas 3.1 and 3.2, that for any $\nu \leq k + 1$

$$\sup_\theta \int_{|t| \leq T_n} \left|\frac{\partial^\nu}{\partial t_\theta^\nu}\left(\Psi_n(t) - e^{-\frac{1}{2}|t|^2}\right)\right| dt \leq C(k, \rho, \nu)\left[\frac{\beta_n}{\sqrt{n}} + \delta_n + \frac{m_{\pi,1}}{\sqrt{n}} + \left(\frac{1}{3} + \frac{2}{3}\rho\right)^n\right] \; . \tag{4.9}$$

It follows from (4.9), on using Lemma 1 of [17], that

$$\sup_{A \in B^k} |\overline{\triangle}_{n,T_n}(A)| \leq C(k,\rho) \left[\frac{\beta_n}{\sqrt{n}} + \delta_n + \frac{m_{\pi,1}}{\sqrt{n}} + \left(\frac{1}{2} + \frac{2}{3}\rho \right)^n \right]. \tag{4 10}$$

Theorem 1 follows from (4 5), (4 6), and (4 10) if conditions (3 1) and (3 2) are satisfied

It is clear that the generality of Theorem 1 is not lost by condition (3 2) because in the opposite case the obtained estimate is trivial   Finally the proof that condition (3 1) is nonessential is obtained by repeating, word-for-word, the corresponding part in the proof of Theorem 3 1, p. 108 in [19].

PROOF OF THEOREM 2. The proof of Theorem 2 goes along the lines of the proof of Theorem 1, where the condition $\pi(.) = P(.)$ is taken into account.

PROOF OF THEOREM 3. It suffices to prove Theorem 3 under the condition $\pi(.) = P(.)$ (the general case reduces to this partial one)   By Theorem 1 it suffices to show that

$$L_{\beta_n} + L_{\delta_n} \to 0 \qquad \text{as} \qquad n \to \infty . \tag{4.11}$$

Since $m_2 < \infty$, then

$$\sup_{\xi \in X} \int_{\overline{X}_n} |(\theta_r, f(\eta))|^2 P(\xi, d\eta) \to 0, \qquad \text{as} \qquad n \to \infty .$$

It is not hard to see that

$$\frac{1}{\sqrt{n}} \int_{X_n} |(\theta_r, f(\eta))|^3 P(\xi, d\eta) \leq C(k) \left[\frac{1}{\sqrt[4]{n}} \int_{\overline{X}_n} |(\theta_r, f(\eta))|^2 P(\xi, d\eta) \right.$$

$$\left. + \int_{\{\eta \cdot |f(\eta)| > \sqrt[4]{n}\}} |(\theta_r, f(\eta))|^2 P(\xi, d\eta) \right] .$$

Hence,

$$L_{\beta_n} \to 0 , \qquad \text{as} \qquad n \to \infty . \tag{4.12}$$

Also, for L_{δ_n}, one can show, using Holder's inequality, that

$$L_{\delta_n} \leq \frac{4}{1-\rho} \sum_{i=1}^{k} \sum_{j=1}^{k} \left\{ \left[\int_X |(\theta_i, f(\xi)|^2 P(d\xi) \right]^{1/2} \left[\int_{\overline{X}_n} |(\theta_j, f(\eta))|^2 P(\xi, d\eta) \right]^{1/2} \right.$$

$$\left. + \left[\int_{\overline{X}_n} |(\theta_i, f(\xi))|^2 P(d\xi) \right]^{1/2} \left[\int_X |(\theta_j, f(\eta)|^2 P(\xi, d\eta) \right]^{1/2} \right\} . \tag{4.13}$$

It is clear from (4.13) that

$$L_{\delta_n} \to 0 , \qquad \text{as} \qquad n \to \infty . \tag{4.14}$$

Finally, from (4.12) and (4.14), we get the assertion of Theorem 3.

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