Internat. J. Math. & Math. Sci.
VOL. 19 NO. 3 (1996) 549-554

MAXIMAL IDEALS IN ALGEBRAS OF VECTOR-VALUED FUNCTIONS

J. W. KITCHEN

Department of Mathematics
Duke University
Durham, NC 27706 USA

and

D. A. ROBBINS

Department of Mathematics
Trinity College
Hartford, CT 06106 USA

(Received October 6, 1994 and in revised form December 21, 1994)

ABSTRACT. Subsuming recent results of the authors [6,7] and J Arhippainen [1], we investigate further the structure and properties of the maximal ideal spaces of algebras of vector-valued functions

KEY WORDS AND PHRASES: lmc algebras, maximal ideals, bundles of topological vector spaces
1991 AMS SUBJECT CLASSIFICATION CODES: 46H10, 46H20, 46J10

1. INTRODUCTION

One way to create new topological algebras from old is to look at algebras $\mathcal{A}$ of functions from a space X which take their values in topological algebras $A_x (x \in X)$. If X is itself a topological space (or sometimes even if it is not), these algebras $\mathcal{A}$ can be topologized in various ways. It is natural to ask how the ideal structure of $\mathcal{A}$ is related to the ideal structures of the A_x The history of this question dates back at least to 1960 and C. Rickart's book [9] and to 1961 and the paper of J M. G Fell [2]. Among many other results, this latter paper identified the space of irreducible *-representations of section spaces of bundles of C^*-algebras The topological algebras of these sources were commutative Banach algebras with identities and C^*-algebras, respectively. Among the more recent studies examining the relationships between the ideal structure of $\mathcal{A}$ and the ideal structures of the A_x are the papers by J. Arhippainen [1], who looked at commutative locally multiplicatively convex A_x, and by the authors ([6] and [7]), for whom the A_x were commutative Banach algebras and arbitrary Banach algebras, respectively The references in these papers provide a guide to some of the record.

The purpose of this note is to investigate further the structure and properties of the maximal ideal spaces of algebras of vector-valued functions In it, we subsume results of our own and of J Arhippainen in the works noted above by using the theory of bundles of locally convex topological vector spaces

2. IDENTIFICATION OF MAXIMAL IDEALS

Consider the following situation let X be a completely regular Hausdorff topological space, and denote by $C_b(X)$ the space of bounded and continuous complex-valued functions on X Let $\{A_x : x \in X\}$ be a family of non-trivial commutative locally multiplicatively convex (lmc) algebras indexed by X Let A be the disjoint union $\dot{\bigcup}\{A_x : x \in X\}$ of algebras (which can, if we like, be thought of as the set $\bigcup_{x \in X}(\{x\} \times A_x)$), and let $\pi : A \to X$ be the natural surjection Assume further that we have on the fibered space A a family of seminorms $\{\nu_i : i \in \mathcal{I}\}$ such that, for each $x \in X, \{\nu_i^x : i \in \mathcal{I}\}$ (where ν_i^x is the restriction of ν_i to A_x) is a family of submultiplicative seminorms which generates the topology on A_x Assume, finally, that we have an algebra $\mathcal{A}$ of selections (= choice functions) $\sigma : X \to A$ such that

1) for each $x \in X$, $ev_x(\mathcal{A}) = \{\sigma(x) : \sigma \in \mathcal{A}\} = A_x$ (in this case, $\mathcal{A}$ is said to be full),
2) $\mathcal{A}$ is a $C_b(X)$-module, and
3) for each $\sigma \in \mathcal{A}$ and for each $i \in \mathcal{I}$, the numerical function $x \mapsto \nu_i^x(\sigma(x))$ is upper semicontinuous on X

Before going farther, we point out two special cases of this situation If X is compact, and if each A_x is a commutative Banach algebra (and the set $\mathcal{I}$ is a singleton), then we have the situation in [6] On the other hand, if B is a commutative lmc algebra, and if $\mathcal{A} = C(X, B)$ is the algebra of all continuous B-valued functions on X (so that $A_x = B$ for all $x \in X$), then we have the situation described in [1]

Returning now to the general situation, we make $\mathcal{A}$ into a commutative lmc algebra First, we select a compact cover $\mathcal{K}$ of X which is closed under finite unions For each $K \in \mathcal{K}$ and $i \in \mathcal{I}$ we define a seminorm $\rho_{K,i}$ on $\mathcal{A}$ by $\rho_{K,i}(\sigma) = \sup_{x \in K} \nu_i^x(\sigma(x))$ Then the $\rho_{K,i}$ are easily seen to be submultiplicative, so that they generate an lmc topology on $\mathcal{A}$ The sets

$$V(\sigma, K, i, \epsilon) = \{\tau \in \mathcal{A} : \rho_{K,i}(\sigma - \tau) < \epsilon\}$$

form a subbasic system of neighborhoods of $\sigma \in \mathcal{A}$ as $K \in \mathcal{K}, i \in \mathcal{I}$, and every $\epsilon > 0$ vary

Note that different choices of covers $\mathcal{K}$ may lead to different topologies on $\mathcal{A}$ In the constant fiber case $\mathcal{A} = C(X, B)$, described above, we can let $\mathcal{K}$ be the family of all compact subsets of X, in which case $\mathcal{K}$ has the compact-open topology (the topology of uniform convergence on compact subsets of X) If, at the other extreme, we let $\mathcal{K}$ be the family of finite subsets of X, then $\mathcal{A}$ has the topology of pointwise convergence on X

In the general case, we note further that since $\mathcal{A}$ with the given topology is an lmc algebra, the multiplication on $\mathcal{A}$ is (jointly) continuous in the topology given by the seminorms $\rho_{K,i}$ (see [8]) Moreover, if we endow $C_b(X)$ with the sup norm topology, it is easily seen that the module multiplication $(f, \sigma) \mapsto f\sigma$ from $C_b(X) \times \mathcal{A}$ to $\mathcal{A}$ is also jointly continuous, so that $\mathcal{A}$ is in fact a topological $C_b(X)$-module

For a subset $J \subset \mathcal{A}$ and $K \in \mathcal{K}$, let $J|K = \{\sigma|K : \sigma \in J\}$, where $\sigma|K$ denotes the restriction of σ to K. Denote the restriction map by $rest_K : \mathcal{A} \mapsto \mathcal{A}|K$

PROPOSITION 1. *Suppose that $J \subset \mathcal{A}$ is an ideal in $\mathcal{A}$ which is also a $C_b(X)$-module of $\mathcal{A}$. Then $J|K$ is an ideal in $\mathcal{A}|K$ which is also a $C(K)$-module.*

PROOF. Evidently, $J|K$ is an ideal in $\mathcal{A}|K$

Let $\sigma \in J$, and let $f \in C(K)$ We may extend f to $f^* \in C_b(X)$, see [4, p 90] Then

$$rest_K(f^*\sigma) = rest_K(f^*) \cdot rest_K(\sigma) = f \cdot (\sigma|K) \in J|K,$$

since $f^*\sigma \in J$ □□□

PROPOSITION 2. *Suppose that $J \subset \mathcal{A}$ is a $C_b(X)$-submodule and a closed proper ideal. Then there exists $x \in X$ such that $\overline{ev_x(J)} = \overline{J_x}$ is a closed proper ideal in A_x.*

PROOF. Fix $K \in \mathcal{K}$ and consider $\mathcal{A}|K$ This is a space of choice functions over K, whose seminorm functions $x \mapsto \nu_i^x(\sigma(x))(\sigma \in \mathcal{A}, i \in \mathcal{I})$ are then upper semicontinuous over K by restriction, and hence bounded on K By [3, Theorem 5 9, p 49], there is a bundle $\pi_K : A_K \to K$ of lmc topological algebras such that $\Gamma(\pi_K) \simeq \mathcal{A}|K$, the topology on $\mathcal{A}|K$ is generated by the $\rho_{K,i} (i \in \mathcal{I})$

Suppose now that for each $x \in X$, we have $\overline{J_x} = A_x$, and let $\sigma \in \mathcal{A}$ We will show that every neighborhood V of σ contains an element $\tau \in J$ Since J is closed, this will show that $\sigma \in J$, contrary to the assumption that J is a proper ideal in $\mathcal{A}$ We may assume that V is of the form

$$V = \bigcap_{p=1}^{n} V(\sigma, K, i_p, \epsilon),$$

where the i's are indices in $\mathcal{I}$ From the preceding, $J|K$ is a $C(K)$-submodule of $\mathcal{A}|K \simeq \Gamma(\pi_K)$ such that $ev_x(J|K)$ is dense in each $A_x (x \in K)$ Then, using [3, Theorem 4 2, p 39], $J|K$ is dense in $\mathcal{A}|K$ By the definition of the topology on $\mathcal{A}|K$, this means that there is a $\tau \in J$ such that $\rho_{K,i_p}(\sigma - \tau) < \epsilon$ for $p = 1, ..., n$ But this says precisely that $\tau \in V$ □□□

PROPOSITION 3. *Suppose that $H : \mathcal{A} \mapsto \mathbb{C}$ is a non-trivial continuous multiplicative homomorphism; set $J = \ker H$. Then there exists $x \in X$ such that $\overline{J_x}$ is a proper ideal in A_x.*

PROOF. It suffices to show that J is a $C_b(X)$-submodule of $\mathcal{A}$ If it is not, we may choose $\sigma \in J$ and $f \in C_b(X)$ such that $f\sigma \notin J$ Since J is in any event an ideal, we have $(f\sigma)^2 = (f^2\sigma)\sigma \in J$ But $H\big((f\sigma)^2\big) = [H(f\sigma)]^2 \neq 0$, a contradiction □□□

PROPOSITION 4. *Let $\triangle(\mathcal{A})$ be the Gelfand space of $\mathcal{A}$ (= space of non-trivial continuous homomorphisms $H: \mathcal{A} \to \mathbb{C}$). If $H \in \triangle(\mathcal{A})$, then there exist $x \in X$, $h \in \triangle(A_x)$ such that $H = h \circ ev_x$.*

PROOF. Let $H \in \triangle(A_x)$, set $J = \ker H$, and choose $x \in X$ such that $\overline{J_x}$ is a proper ideal in A_x Thus, $\frac{A_x}{\overline{J_x}} \neq 0$ Since $ev_x : \mathcal{A} \to A_x$ maps J into $\overline{J_x}$, there is a unique linear map $\phi : \frac{\mathcal{A}}{J} \to \frac{A_x}{\overline{J_x}}$ which makes the diagram

$$\begin{array}{ccc} \mathcal{A} & \xrightarrow{ev_x} & A_x \\ \pi \downarrow & & \downarrow \pi_x \\ \frac{\mathcal{A}}{J} & \xrightarrow{\phi} & \frac{A_x}{\overline{J_x}} \end{array}$$

commute, where π and π_x are the natural surjections Since $ev_x : \mathcal{A} \to A_x$ is surjective, the induced map $\phi : \frac{\mathcal{A}}{J} \to \frac{A_x}{\overline{J_x}}$ is also surjective Thus, ϕ maps the one-dimensional space $\frac{\mathcal{A}}{J}$ surjectively onto the non-zero space $\frac{A_x}{\overline{J_x}}$ It follows that $\frac{A_x}{\overline{J_x}}$ is one-dimensional, which means that $\overline{J_x}$ is a closed regular maximal ideal in A_x Hence, $\overline{J_x} = \ker h$ for some $h \in \triangle(A_x)$. The map $h \circ ev_x : \mathcal{A} \to \mathbb{C}$ is clearly a non-trivial algebra homomorphism. If $\sigma \in J$, then $ev_x(\sigma) \in \overline{J_x} = \ker h$, so $(h \circ ev_x)(\sigma) = 0$ Hence ker $H = J \subset \ker(h \circ ev_x)$. Because ker H and $\ker(h \circ ev_x)$ are closed maximal ideals, it follows that ker $H = \ker(h \circ ev_x)$, and hence that $H = h \circ ev_x$ □□□

COROLLARY 5. *Under the situation as described, we may identify $\triangle(\mathcal{A})$ as a point set with the disjoint union of the $\triangle(A_x)$. (For bookkeeping purposes, we may also write $\triangle(\mathcal{A}) = \bigcup_{x \in X}(\{x\} \times \triangle(A_x))$.)*

PROOF. Since $ev_x : \mathcal{A} \to A_x$ is continuous, it follows that, if $x \in X$ and $h \in \triangle(A_x)$, then $h \circ ev_x \in \triangle(\mathcal{A})$ By using the same method as in the proof of [6, Proposition 6], it may be shown that the map

$$\phi : \bigcup_{x \in X}(\{x\} \times \triangle(A_x)) \to \triangle(\mathcal{A}), (x, h) \mapsto h \circ ev_x = H$$

is a bijection □□□

In all the above, we need to call on the result for lmc algebras which corresponds to that for Banach algebras: namely, in a commutative lmc algebra B, there is a one-to-one correspondence between the set of continuous non-trivial homomorphisms from B to $\mathbb{C}$ and the set of closed regular maximal ideals in B, see [8, Corollaries 7.1, 7.2, pp. 71-72].

3. TOPOLOGICAL CONSIDERATIONS

So, under the circumstances described, we have a fibering of $\triangle(\mathcal{A})$ by X. For $H \in \triangle(\mathcal{A})$, we may write $h \circ ev_x$ for some (unique) $x \in X$ and $h \in \triangle(A_x)$. Let $p : \triangle(\mathcal{A}) \to X$ be the obvious projection map, $H = h \circ ev_x \mapsto x$.

PROPOSITION 6. *The projection map p is continuous when $\triangle(\mathcal{A})$ is given its weak-* topology.*

PROOF. It suffices to show that whenever $\{H_\alpha\} = \{h_\alpha \circ ev_{x_\alpha}\}$ is a net in $\triangle(\mathcal{A})$ such that $H_\alpha = h_\alpha \circ ev_{x_\alpha} \to H = h \circ ev_x$, we have $f(x_\alpha) \to f(x)$ for each $f \in C_b(X)$, because when X is completely regular and Hausdorff is topology is determined by $C_b(X)$, see [4, p. 40]. Suppose now that $f \in C_b(X)$ and that $\sigma \in \mathcal{A}$, with $H(\sigma) = h(\sigma(x)) \neq 0$. Since $f\sigma \in \mathcal{A}$, and since $h_\alpha \circ ev_{x_\alpha} \to h \circ ev_x$ weak-* in $\triangle(\mathcal{A})$, we have

$$h_\alpha([f\sigma](x_\alpha)) = h_\alpha(f(x_\alpha)\sigma(x_\alpha)) = f(x_\alpha)h(\sigma(x_\alpha)) \to h([f\sigma](x)) = f(x)h(\sigma(x)).$$

Since $h_\alpha(\sigma(x_\alpha)) \to h(\sigma(x)) \neq 0$, it follows that $f(x_\alpha) \to f(x)$. Since $f \in C_b(X)$ was arbitrary, we have the desired result. □□□

On the other hand, we can look at how $\triangle(A_x)$ embeds into $\triangle(\mathcal{A})$.

PROPOSITION 7. *Give $\triangle(\mathcal{A})$ its weak-* topology and, for each $x \in X$, give $\triangle(A_x)$ its weak-* topology. Then $\triangle(A_x)$ embeds homeomorphically into $\triangle(\mathcal{A})$.*

PROOF. Fix $x \in X$. Evidently, the map $\gamma_x : \triangle(A_x) \to \triangle(\mathcal{A})$, $h \mapsto h \circ ev_x$, is one-to-one: if $h_1 \neq h_2$, then we may choose $a \in A_x$ such that $h_1(a) \neq h_2(a)$, and use the fullness of $\mathcal{A}$ to choose $\sigma \in \mathcal{A}$ such that $\sigma(x) = a$. It is then clear that $(h_1 \circ ev_x)(\sigma) \neq (h_2 \circ ev_x)(\sigma)$.

Now, suppose that we have a net $\{h_\alpha\} \subset \triangle(A_x)$ such that $h_\alpha \to h \in \triangle(A_x)$ when $\triangle(A_x)$ is given its weak-* topology. Let $\sigma \in \mathcal{A}$. We then have $(h_\alpha \circ ev_x)(\sigma) = h_\alpha(\sigma(x)) \to h(\sigma(x)) = (h \circ ev_x)(\sigma)$, i.e. $\gamma_x(h_\alpha) \to h \circ ev_x$ in $\triangle(\mathcal{A})$. It is likewise easy to show that if $\{h_\alpha \circ ev_x\}$ is a net in $\gamma_x(\triangle(A_x))$ which converges weak-* to $h \circ ev_x \in \gamma_x(\triangle(A_x))$, then $h_\alpha \to h$ weak-* in $\triangle(A_x)$. □□□

Previous work of the authors [6] has provided examples which demonstrate that the projection map need not be closed, even when each fiber A_x is a Banach algebra with identity. Moreover, the projection need not be open, even when each fiber A_x is a Banach algebra with identity and $\mathcal{A}$ satisfies the even stronger condition that it contain the identity selection. Both of these examples use the weak-* topologies.

Suppose now that we re-examine the situation when each A_x is a commutative Banach algebra and X is compact. Under these special conditions, $\mathcal{A}$ is the space of sections of a bundle of Banach algebras $\pi : A \to X$. We may look at the Seda topology on $\mathcal{M} = \bigcup_{x\in X}(\{x\} \times \triangle(A_x)) = \dot{\bigcup}_{x\in X}\triangle(A_x)$. Recall from the Banach bundle case that the Seda topology is the weak topology on $\mathfrak{S} = \bigcup_{x\in X}(\{x\} \times B((A_x)^*))$ (where $B(Z)$ denotes the closed unit ball of a Banach space Z) which is generated by the conditions $(x_\alpha, F_\alpha) \to (x, F) \in \mathcal{M}$ iff $x_\alpha \to x \in X$ and $F_\alpha(\sigma(x_\alpha)) \to F(\sigma(x))$ for each $\sigma \in \mathcal{A}$. It is shown elsewhere that $\mathfrak{S}$ is compact in the Seda topology. (See [10] and [5] for more information about this topology.)

PROPOSITION 8. *Let X be a compact Hausdorff space, and suppose that $\mathcal{A} = \Gamma(\pi)$ is the space of sections of the bundle of commutative Banach algebras $\pi : A \to X$. Then the weak-* topology on $\triangle(\mathcal{A})$ and the (relative) Seda topology on $\mathcal{M}$ are homeomorphic.*

PROOF. As above, for $H \in \triangle(\mathcal{A})$, write $H = h \circ ev_x$ for some $x \in X$ and $h \in \triangle(A_x)$ The map $H \mapsto (x, h)$ is a bijection If $H_\alpha = h_\alpha \circ ev_{x_\alpha} \to H = h \circ ev_x$ weak-* in $\triangle(\mathcal{A})$, this says precisely that $H_\alpha(\sigma) = h_\alpha(\sigma(x_\alpha)) \to H(\sigma) = h(\sigma(x))$ for each $\sigma \in \mathcal{A}$, above we have shown that $x_\alpha \to x$ Thus, $(x_\alpha, h_\alpha) \to (x, h)$ in the Seda topology The other direction is clear □□□

We may also consider the continuity of the projection map and the embeddings when $\triangle(\mathcal{A})$ and $\triangle(A_x)$ are endowed with their hull-kernel topologies

PROPOSITION 9. *Under the given general circumstances, suppose that $\triangle(\mathcal{A})$ is given its hull-kernel topology, and that each $\triangle(A_x)(x \in X)$ is given its hull-kernel topology. Then the projection map $p : \triangle(\mathcal{A}) \to X$ and the embeddings of the $\triangle(A_x)$ into $\triangle(\mathcal{A})$ are continuous.*

PROOF. To show that the natural projection $p : \triangle(\mathcal{A}) \to X$ is continuous in the hull-kernel topology, let $\{H_\alpha\} = \{h_\alpha \circ ev_{x_\alpha}\}$ be a net in $\triangle(\mathcal{A})$ with $h_\alpha \circ ev_{x_\alpha} \to h \circ ev_x = H \in \triangle(\mathcal{A})$ in the hull-kernel topology We claim that $x_\alpha \to x$

If not, we may then choose an open neighborhood N of x and a subnet $\{x_{\alpha'}\}$ of $\{x_\alpha\}$ such that $x_{\alpha'} \notin N$ Choose $a \in A_x$ such that $h(a) \neq 0$, and choose $\sigma' \in \mathcal{A}$ such that $\sigma'(x) = ev_x(\sigma') = a$ Since X is completely regular, we may choose a function $f \in C_b(X)$ with $f(X) \subset [0,1]$ and with $f(x) = 1$ and $f(X \backslash N) = 0$ Set $\sigma = f\sigma'$ Since $h_{\alpha'} \circ ev_{x_{\alpha'}} \to h \circ ev_x$, we have $P = \bigcap_{\alpha'} \ker(h_{\alpha'} \circ ev_{x_{\alpha'}}) \subset \ker(h \circ ev_x)$ Since $\sigma(x_{\alpha'}) = 0$ for all α', we have $\sigma \in P \subset \ker(h \circ ev_x)$ But this is a contradiction, since $(h \circ ev_x)(\sigma) = h(\sigma(x)) = h(a) \neq 0$ Hence, $x_\alpha \to x$

Now, fix $x \in X$ For the second part, it suffices to show that for a set $W \subset \triangle(A_x)$, and for $h \in \triangle(A_x)$, we have h in the hull-kernel closure of W iff $H = h \circ ev_x$ is in the hull-kernel closure of $\gamma_x(W) = \{h' \circ ev_x : h' \in W\}$

Suppose, then, that h is in the hull-kernel closure of W in $\triangle(A_x)$ Then $\bigcap\{\ker h' : h' \in W\} \subset \ker h$, we claim that $\bigcap \ker\{h' \circ ev_x : h' \in W\} \subset \ker(h \circ ev_x)$ So, let $\sigma \in \mathcal{A}$ be such that $\sigma \in ker(h' \circ ev_x)$ for each $h' \in W$ Then $h'(\sigma(x)) = 0$ for each $h' \in W$, i e $\sigma(x) \in \ker h'$ for all $h' \in W$, so that $\sigma(x) \in \ker h$ Hence, $\sigma \in \ker(h \circ ev_x)$ A proof of the reverse inclusion, which uses the fullness of $\mathcal{A}$, is equally straightforward

We note that these are essentially the proofs used in [7, Propositions 17, 18] □□□

Recall (see [8, p 332]) that a topological algebra B is said to be regular provided that any weak-* closed subset W of $\triangle(B)$ and point of $\triangle(B)$ disjoint from it may be separated by an element of B It happens that B is regular iff the weak-* and hull-kernel topologies coincide on $\triangle(B)$

PROPOSITION 10. *Suppose that we are given the general data on $\mathcal{A}$, as above. If $\mathcal{A}$ is a regular algebra, then so is each A_x.*

PROOF. Choose $x \in X$ We know that $\triangle(\mathcal{A})$ contains a homeomorphic copy of $\triangle(A_x)$ in the weak-* topology, in particular, $\{x\} \times W = p^{-1}(W)$ is weak*- closed in $\triangle(A_x)$ whenever W is a weak-* closed in $\triangle(A_x)$, where $p : \triangle(\mathcal{A}) \to \triangle(A_x)$ is the continuous projection map. Hence, if $h \in \triangle(A_x) \backslash W$, then $(x, h) \in \triangle(\mathcal{A}) \backslash p^{-1}(W)$, and so there exists $\sigma \in \mathcal{A}$ which separates (x, h) and $p^{-1}(W)$ Then it is evident that $\sigma(x) \in A_x$ separates h and W in $\triangle(A_x)$ □□□

Now, if $x \in X$, and if $I_x \subset A_x$ is an ideal, set $\mathcal{A}(x, I_x) = \{\sigma \in \mathcal{A} : \sigma(x) \in I_x\}$ It is easy to see that $\mathcal{A}(x, I_x)$ is always a closed proper ideal in $\mathcal{A}$ whenever I_x is a closed proper ideal of A_x (In fact, $\mathcal{A}(x, I_x)$ is also a closed $C_b(X)$-submodule of $\mathcal{A}$ when I_x is closed)

PROPOSITION 11. *Let $J \subset \mathcal{A}$ be a closed ideal which is also a $C_b(X)$-submodule of $\mathcal{A}$. Then $J = \bigcap_{x \in X} \mathcal{A}(x, \overline{J_x})$.*

PROOF. Clearly, $J \subset \bigcap_{x \in X} \mathcal{A}(x, \overline{J_x}) = J'$.

To show the reverse inclusion, we use a partition of unity argument similar to that of Theorem 8 of [1] Let $\sigma \in J'$ To show that $\sigma \in J$, it suffices to show that for $K \in \mathcal{K}$, $i \in \mathcal{I}$, and $\epsilon > 0$ there is $\tau \in J$ such that $\rho_{K,i}(\sigma - \tau) < \epsilon$

Fix K, t, and ϵ, and let $x \in K$ be arbitrary Then $\sigma(x) \in \overline{J_x}$, and so there exists $\sigma' \in J$ such that $\nu_t^{x'}(\sigma(x) - \sigma'(x)) < \epsilon$ since the seminorm functions $x' \mapsto \nu_t^{x'}(\sigma(x') - \sigma'(x'))$ is upper semicontinuous, there is a neighborhood U_x of x such that when $x' \in U_x$ we have $\nu_t^{x'}(\sigma(x') - \sigma'(x')) < \epsilon$

Since K is compact, we may choose a cover $U_{x_1}, ..., U_{x_p}$ of K, with corresponding $\sigma'_1, ..., \sigma'_p \in J$ such that $\nu_t^{x'}(\sigma(x') - \sigma'_r(x')) < \epsilon$ whenever $x' \in U_{x_r} (r = 1, ..., p)$ Now, $\{U_{x_r} \bigcap K : r = 1, ..., p\}$ is an open cover of the compact Hausdorff space K, and so there is a partition of unity $\{f_r : r = 1, ..., p\} \subset C(K)$ subordinate to $\{U_{x_r} \bigcap K\}$ In particular, $0 \leq f_r(x) \leq 1 (x \in K)$, $\mathrm{supp}(f_r) \subset U_{x_r} \bigcap K$ for $r = 1, ..., p$, and $\sum_{r=1}^{p} f_r(x) = 1$ for $x \in K$ As in Proposition 1, we may extend f_r to $f_r^* \in C_b(X)$ Then $\tau = \sum_{r=1}^{p} f_r^* \sigma'_r \in J$, and it is easy to check that $\rho_{K,t}(\sigma - \tau) < \epsilon$ □□□

COROLLARY 12. *Suppose that $\mathcal{A}$ has an identity e, and let $J \subset \mathcal{A}$ be a closed ideal. Then $J = \bigcap_{x \in X} \mathcal{A}(x, \overline{J_x})$.*

PROOF. It suffices to note that J is a $C_b(X)$-submodule of $\mathcal{A}$ Let $f \in C_b(X)$ and $\sigma \in J$ Then $f\sigma = f(e\sigma) = (fe)\sigma \in J$ □□□

COROLLARY 13. *Let $J \subset \mathcal{A}$ be a closed proper ideal, and let $\langle J \rangle$ denote the closed $C_b(X)$-submodule in $\mathcal{A}$ generated by J. Then $\langle J \rangle = \bigcap_{x \in X} \mathcal{A}(x, \overline{J_x})$.*

PROOF. This follows immediately from the method of proof in Proposition 11 □□□

We point out in closing the crucial role which the assumptions on the space X play Complete regularity of X allows us to extend the functions appearing in the proofs of Propositions 1 and 11, and provides sufficiently many continuous functions to demonstrate the continuity of the projection map $p : \Delta(\mathcal{A}) \to X$ in Propositions 6 and 9. That X is Hausdorff means that each $K \in \mathcal{K}$ is a compact Hausdorff space, and allows us to use the full power of the cited theorems from [3] in the proof of Proposition 2

REFERENCES

[1] ARHIPPAINEN, J , On the ideal structures of algebras of LMC-valued functions, *Studia Math.* **101** (3) (1992), 311-318.

[2] FELL, J M G , The structure of algebras of operator fields, *Acta Math.* **106** (1961), 233-280

[3] GIERZ, G., Bundles of topological vector spaces and their duality, *Lect. Notes in Math.* **955** (Berlin Springer-Verlag), 1982

[4] GILMAN, L and JERISON, M , *Rings of Continuous Functions* (New York: Springer-Verlag), 1976

[5] KITCHEN, J.W and ROBBINS, D.A., Integral operators on the section space of a Banach bundle, *Internat. J. Math. & Math. Sci.* **16** (1993), 449-458.

[6] KITCHEN, J.W. and ROBBINS, D.A., Bundles of Banach algebras, *Internat. J. Math. & Math. Sci.* **17** (1994), 671-680.

[7] KITCHEN, J.W. and ROBBINS, D.A., Bundles of Banach algebra II, *Houston J. Math.* **20** (1994), 435-451

[8] MALLIOS, A , *Topological Algebras. Selected Topics* (New York Elsevier), 1986

[9] SEDA, A.K., On the categories Sp(X) and Ban(X), *Cahiers Topo. et Géo. Diff.* XXIV (1983), 97-112