GENERATING NEW CLASSES OF ORTHOGONAL POLYNOMIALS

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ABSTRACT. Given a sequence of monic orthogonal polynomials (MOPS), $\{P_n\}$, with respect to a quasi-definite linear functional u, we find necessary and sufficient conditions on the parameters a_n and b_n for the sequence

$$P_n(x) + a_n P_{n-1}(x) + b_n P_{n-2}(x), \; n \geq 1$$
$$P_0(x) = 1, P_{-1}(x) = 0$$

to be orthogonal. In particular, we can find explicitly the linear functional v such that the new sequence is the corresponding family of orthogonal polynomials. Some applications for Hermite and Tchebychev orthogonal polynomials of second kind are obtained.

We also solve a problem of this type for orthogonal polynomials with respect to a Hermitian linear functional.

1 INTRODUCTION

We consider two positive Borel measures μ_1, μ_2 supported on a set $I \subset \mathbb{R}$ with infinitely many points and such that $d\mu_2 = \frac{d\mu_1}{q(x)}$ where $q(x) = \prod_{i=1}^{l}(x - x_i)$ has their zeros outside I. We denote by $\{P_n\}$ and $\{R_n\}$ the MOPS with respect to μ_1 and μ_2, respectively. Then, it is well known that

$$R_n(x) = \frac{\begin{vmatrix} P_n(x) & \ldots & P_{n-l}(x) \\ Q_n(x_1) & \ldots & Q_{n-l}(x_1) \\ & \ldots & \\ Q_n(x_l) & \ldots & Q_{n-l}(x_l) \end{vmatrix}}{\begin{vmatrix} Q_{n-1}(x_1) & \ldots & Q_{n-l}(x_1) \\ & \ldots & \\ Q_{n-1}(x_l) & \ldots & Q_{n-l}(x_l) \end{vmatrix}} \tag{1}$$

where

$$Q_n(s) = \int_I \frac{P_n(t)}{t - s} d\mu_1(t)$$

(see [15] and [18]). This means that

$$R_n(x) = P_n(x) + a_n^{(1)} P_{n-1}(x) + \ldots + a_n^{(l)} P_{n-l}(x) \tag{2}$$

with $a_n^{(l)} \neq 0$.

We are interested in some partial converse of the above result when $l = 2$ as well as some extensions for more general families of orthogonal polynomials.

If u_1 and u_2 denote the linear functionals on the linear space $\mathbb{P}$ of polynomials with complex coefficients, defined respectively by

$$\langle u_\imath, p(x)\rangle = \int_I p(x)\mathrm{d}\mu_\imath(x), \imath = 1, 2$$

for every $p \in \mathbb{P}$, then

$$
\begin{aligned}
\langle q(x)u_2, p(x)\rangle &= \int_I q(x)p(x)\mathrm{d}\mu_2(x) \\
&= \int_I p(x)\mathrm{d}\mu_1(x) = \langle u_1, p(x)\rangle
\end{aligned}
$$

i.e. $qu_2 = u_1$ in the space of linear functionals on $\mathbb{P}$.

If q is a polynomial of degree two and u_1 is a quasi-definite linear functional our first problem is to find necessary and sufficient conditions in order to u_2 defined by $qu_2 = u_1$ be a quasi-definite linear functional.

Maroni studied this problem when p is a polynomial of degree 1 (see [13]) and in the case when p is a polynomial of degree 2 with $x_1 = x_2$ (see [14]). In this last case, he does not give the proof of this result. The conditions that we give here are basically different from that ones.

On the other hand, we consider the following inverse problem:

Given a MOPS $\{P_n\}$ with respect to a quasi-definite linear functional u_1, to characterize the sequences (a_n), (b_n) of real numbers, such that the sequence of monic polynomials

$$R_n = P_n + a_n P_{n-1} + b_n P_{n-2}, n \geq 1 \tag{3}$$

is orthogonal with respect to some quasi-definite linear functional u_2. As an immediate consequence, to find the relation between u_1 and u_2.

A first attempt to propose an inverse problem was made by J.L.Geronimus in [6]. In fact, he obtained information about the quasi-definite linear functionals u and v associated to two MOPS $\{P_n\}$ and (R_n) related by (2). Furthermore, he gave necessary and sufficient conditions on the family $\{R_n\}$ in order to be orthogonal with respect to v.

This problem is also connected with positive quadrature formulae (see [19]).

Geronimus gave also in that paper a proof of the Hahn's theorem for the classical MOPS, i.e. the only MOPS $\{P_n\}$ such that the $\{\frac{P'_{n+1}}{n+1}\}$ is also a MOPS are the classical ones. He used the fact that the classical MOPS satisfy

$$
\begin{aligned}
P_{n+2}(x) &= \frac{P'_{n+3}}{n+3}(x) + a_{n+2}\frac{P'_{n+2}}{n+2}(x) + b_{n+2}\frac{P'_{n+1}}{n+1}(x), n \in \mathbb{N} \\
P_1(x) &= \frac{P'_2}{2}(x) + a_1 P'_1(x), P_0(x) = 1.
\end{aligned}
$$

We have proved that this relation characterize classical MOPS (see [3]).

This problem is connected to another stated by Geronimus (see [7]):

• Construct a MOPS, $\{P_n\}$, which satisfy

$$xP_{n+s+1} = P_{n+s+2} + \beta_{n+s+1}P_{n+s+1} + \gamma_{n+s}P_{n+s}, n \in \mathbb{N}.$$

When $\begin{cases} \beta_{n+s+1} = 0 \\ \gamma_{n+s} = \frac{1}{4} \end{cases}$, $n \in \mathbb{N}$ he gave a representation of the measure with respect to $\{P_n\}$ is orthogonal. This is because $\{P_n\}$ is a finite linear combination of second kind Tchebychev orthogonal polynomials.

Finally, we study an inverse problem for orthogonal polynomials with respect to Hermitian linear functionals on the linear space of Laurent polynomials. This represents a generalization of the theory of orthogonal polynomials with respect to a measure supported on the unit circle.

The structure of this work is the following:

(a) In Section 2 we present the basic tools concerning linear functionals as well as the concept of quasi-orthogonality which will play a central role in our paper.

(b) Section 3 is devoted to the direct problem, i.e. the relation between orthogonal polynomials associated with two quasi-definite linear functionals u, v satisfying a relation

$$u = p(x)v$$

where $\deg p = 2$. Furthermore, we give in Theorem 3 the connection between the corresponding parameters of the three term recurrence relation.

(c) In Section 4 we study the inverse problem (3) and we consider some examples. Theorem 4.1 gives the characterization of such sequences by a constructive approach.

(d) In Section 5 we give necessary conditions on the sequence a_n in order to

$$\psi_n(z) = \phi_n(z) + a_n\phi_{n-1}(z) \tag{4}$$

be a MOPS with respect to an Hermitian linear functional if $\{\phi_n\}$ is a MOPS with respect to an Hermitian linear functional.

2   QUASI-ORTHOGONALITY

We now introduce the algebraic tools that we use in this paper, (see [5] and [12] for more details). Let $\{P_n\}$ be a MOPS with respect to the *quasi-definite (respectively, positive definite) linear functional* u defined on $\mathbb{P}$, i.e.

$$\langle u, P_n(x)P_m(x)\rangle = k_n\delta_{nm} \text{ with } k_n \neq 0 \text{ (respectively, } k_n \geq 0) \text{ for } n, m \in \mathbb{N}, \tag{5}$$

where $\langle .,.\rangle$ means the duality bracket. Furthermore, $\{P_n\}$ satisfies the following three term recurrence relation

$$\begin{aligned} xP_n(x) &= P_{n+1}(x) + \beta_nP_n(x) + \gamma_nP_{n-1}(x) \text{ for } n = 1, 2, \dots \\ P_0(x) &= 1, \ P_1(x) = x - \beta_0. \end{aligned} \tag{6}$$

where (β_n) and (γ_n) be two sequences of complex numbers with $\gamma_{n+1} \neq 0$ in the quasi-definite case and $\gamma_{n+1} > 0$, $(\beta_n) \subset \mathbb{R}$ in the positive definite case, for $n \in \mathbb{N}$.

Since $\{P_n\}$ is an algebraic basis in $\mathbb{P}$, we can define its dual basis (α_n) in $\mathbb{P}^*$ (the algebraic dual space of $\mathbb{P}$) as $\langle \alpha_n, P_m\rangle = \delta_{m,n}$.

If v is an element of $\mathbb{P}^*$, we can express it as $v = \sum_{i=0}^{\infty} \lambda_i\alpha_i$ where $\lambda_i = \langle v, P_i\rangle$, $i \in \mathbb{N}$. As an immediate consequence, if $v \in \mathbb{P}^*$ satisfies $\langle v, P_i\rangle = 0$ for $i \geq l$, then $v = \sum_{i=0}^{l-1} \lambda_i\alpha_i$. In particular, $u = \mu_0\alpha_0$. We can represent the elements of the dual basis in terms of $\{P_n\}$ and u. In fact, since $\langle \alpha_n, P_m\rangle = \delta_{n,m} = \langle \frac{P_n u}{\langle u, P_n^2\rangle}, P_m\rangle$, $n, m \in \mathbb{N}$, we have

$$\alpha_n = \frac{P_n u}{\langle u, P_n^2\rangle}, \ n \in \mathbb{N}. \tag{7}$$

Let define some linear operators on $\mathbb{P}$:

$$p \mapsto (qp)(x) = q(x)p(x), \ q \in \mathbb{P} \tag{8}$$

$$p \mapsto (\theta_c p)(x) = \frac{p(x) - p(c)}{x - c}, \ c \in \mathbb{C} \tag{9}$$

By transposition, we introduce the following linear operators on $\mathbb{P}^*$:

1. $\langle q\alpha, p\rangle = \langle \alpha, qp\rangle$ where

$$\langle q\alpha, x^n\rangle = \sum_{i=0}^{\deg q} a_i\langle \alpha, x^{i+n}\rangle, \ n \in \mathbb{N}$$

for $q(x) = \sum_{i=0}^{\deg q} a_i x^i$ and $\alpha \in \mathbb{P}^*$;

2. $\langle (x-c)^{-1}\alpha, p \rangle = \langle \alpha, \theta_c p \rangle$ where

$$\langle (x-c)^{-1}\alpha, x^n \rangle = \begin{cases} 0 \text{ , if } n = 0 \\ \sum_{i=0}^{n-1} c^{n-1-i}\langle \alpha, x^{i+n} \rangle \text{ , if } n \geq 1 \end{cases}$$

Given a polynomial $p(x) = \prod_{i=1}^{s}(x - x_i)^{m_i}$, for every $f \in \mathbb{P}$, we define

$$(\Theta_X f)(x) = \frac{f(x) - (\mathcal{L}f)(x)}{p(x)} \tag{10}$$

where,

$$(\mathcal{L}f)(x) = \sum_{i=1}^{s} \sum_{k=1}^{m_i} f^{(k-1)}(x_i)L_{ik}(x) \tag{}$$

is the Lagrange-Sylvester (Hermite) interpolation polynomial of f with nodes x_i, $i = 1, \ldots, s$ and the L_{ik} verify the following conditions:

$$L_{ik}^{(\nu)}(x_j) = \begin{cases} 1 \text{ if } \nu = k-1 \text{ and } j = i \\ 0 \text{ elsewhere} \end{cases} \tag{11}$$

for $k = 1, 2, \ldots, m_i$ and $i = 1, \ldots, s$. Then:

DEFINITION 1 $p^{-1}u$ is the linear functional given by

$$\langle p^{-1}(x)u, f \rangle = \langle u, \Theta_X f \rangle. \tag{12}$$

DEFINITION 2 ([5]) Let $\{P_n\}$ be a MOPS with respect to the quasi-definite linear functional u. We say $\{P_n^{(1)}\}$ is the *first associated MOPS* for the MOPS $\{P_n\}$ if it satisfies

$$\begin{aligned} &xP_n^{(1)}(x) = P_{n+1}^{(1)}(x) + \beta_{n+1}P_n^{(1)}(x) + \gamma_{n+1}P_{n-1}^{(1)}(x) \text{ for } n = 1, 2, \ldots \\ &P_0^{(1)}(x) = 1, \ P_1^{(1)}(x) = x - \beta_1. \end{aligned} \tag{13}$$

DEFINITION 3 ([4]) The MOPS defined by

$$P_{n+1}(x; c) = P_{n+1}(x) - cP_n^{(1)}(x), \ n \in \mathbb{N}. \tag{14}$$

is called a *co-recursive* sequence of the MOPS $\{P_n\}$.

Notice that these polynomials verify the same three-term recurrence relation as $\{P_n\}$ with initial conditions $P_0(x; c) = 1$ and $P_1(x; c) = P_1(x) - c$.

Next we introduce a basic definition in our work:

DEFINITION 4 ([14]) Let $\{p_n\}$ be a MPS with $\deg p_n = n$ and u be a linear functional. We say that $\{p_n\}$ is *strictly quasi-orthogonal of order s* if

$$\begin{aligned} &\langle u, p_m p_n \rangle = 0, \ |n - m| \geq s + 1 \\ &\forall r \geq s \ \langle u, p_{r-s}p_r \rangle \neq 0. \end{aligned} \tag{15}$$

THEOREM 5 ([16]) *Let $\{P_n\}$ be a MOPS with respect to the quasi-definite linear functional u; then, $\{p_n\}$ is strictly quasi-orthogonal of order s with respect to u if and only if*

$$p_n(x) = \begin{cases} \sum_{k=0}^{n} a_{n,k}P_k, \ 0 \leq n \leq s-1 \\ \sum_{k=n-s+1}^{n} a_{n,k}P_k, \ n \geq s \end{cases} \text{ where } a_{n,n-s} \neq 0, \ \forall n \geq s. \tag{16}$$

3   DIRECT PROBLEM

This section is motivated by some modifications considered in the literature of orthogonal polynomials. As an example, the Bernstein-Szegö polynomials (see [5], [9] and [10]) are orthogonal with respect to the weight functions

$$w(x) = \begin{cases} (1-x^2)^{-1/2}(\rho(x))^{-1} \\ (1-x^2)^{1/2}(\rho(x))^{-1} \\ (\frac{1-x}{1+x})^{1/2}(\rho(x))^{-1} \end{cases} \qquad (17)$$

where ρ is a positive polynomial in $[-1,1]$ of fixed degree. These polynomials can be represented in an easy way in terms of the Tchebychev Polynomials of first and second kind as can be deduced from (1) (see also [17, Theorem 2.6]).

Here we solve a more general problem:

THEOREM 6 ([2]) *Let u be a quasi-definite linear functional with $u_0 = 1$ and $x_1 \neq x_2 \in \mathbf{C}$. The linear functional v defined by $u = (x - x_1)(x - x_2)v$ is quasi-definite if and only if $v_0 \neq 0$ and v_1 are such that*

$$\begin{array}{c} \frac{x_2 v_0 - v_1}{x_2 - x_1}(x_1 - v_1)^2 + \frac{v_1 - x_1 v_0}{x_2 - x_1}(x_2 - v_1)^2 \neq 1 \\ |d_n| \neq 0,\ n \in \mathbf{N} \end{array} \qquad (18)$$

Here

$$d_n = \left[\begin{array}{cc} P_{n+1}(x_1; -\frac{1}{v_1 - x_2 v_0}) & P_n(x_1; -\frac{1}{v_1 - x_2 v_0}) \\ P_{n+1}(x_2; -\frac{1}{v_1 - x_1 v_0}) & P_n(x_2; -\frac{1}{v_1 - x_1 v_0}) \end{array} \right]$$

and v_0, v_1 are the first and second moments of v, respectively. In this situation the MOPS with respect to v is given by

$$\begin{array}{l} R_{n+2}(x) = P_{n+2}(x) + a_{n+2}P_{n+1}(x) + b_{n+2}P_n(x),\ n \in \mathbf{N} \\ R_1(x) = P_1(x) + a_1 P_0(x) \\ R_0(x) = P_0(x) \end{array} \qquad (19)$$

where

$$\left[\begin{array}{c} a_{n+2} \\ b_{n+2} \end{array} \right] = -d_n^{-1} \left[\begin{array}{c} P_{n+2}(x_1; -\frac{1}{v_1 - x_2 v_0}) \\ P_{n+2}(x_2; -\frac{1}{v_1 - x_1 v_0}) \end{array} \right],\ n \in \mathbf{N} \qquad (20)$$

and $a_1 = \beta_0 - v_1$. The coefficients of the corresponding recurrence relation

$$\begin{array}{l} x R_{n+1}(x) = R_{n+2}(x) + \xi_{n+1}R_{n+1}(x) + \eta_{n+1}R_n(x),\ n \in \mathbf{N} \\ R_1(x) = x - \xi_0,\ R_0(x) = 1 \end{array} \qquad (21)$$

are given by

$$\xi_n = \beta_n - (a_{n+1} - a_n),\ n \in \mathbf{N} \qquad (22)$$

$$\left. \begin{array}{l} \eta_{n+3} = \frac{b_{n+3}}{b_{n+2}}\gamma_{n+1},\ n \in \mathbf{N} \\ \eta_1 = \gamma_1 + a_1(\beta_0 - \xi_1) - b_2 \\ \eta_2 = \gamma_2 + a_2(\beta_1 - \xi_2) - (b_3 - b_2) \end{array} \right\} \qquad (23)$$

LEMMA 7 *Let u, v be the linear functionals defined in the above theorem; then*

$$v = (x - x_1)^{-1}(x - x_2)^{-1}u + \frac{x_2 v_0 - v_1}{x_2 - x_1}\delta_{x_1} + \frac{v_1 - x_1 v_0}{x_2 - x_1}\delta_{x_2} \qquad (24)$$

where δ_{x_i} means the Dirac mass at x_i.

PROOF OF THE LEMMA.   We only need to determine

$$(x - x_1)^{-1}(x - x_2)^{-1}((x - x_1)(x - x_2)v)$$

$$\langle (x - x_1)^{-1}(x - x_2)^{-1}((x - x_1)(x - x_2)v), f(x)\rangle$$

$$= \langle (x - x_1)(x - x_2)v, \frac{\frac{f(x)-f(x_1)}{x-x_1} - \frac{f(x_2)-f(x_1)}{x_2-x_1}}{x - x_2}\rangle$$

$$= \langle v, f(x) + \frac{x - x_2}{x_2 - x_1}f(x_1) - \frac{x - x_1}{x_2 - x_1}f(x_2)\rangle$$

$$= \langle v + \frac{v_1 - x_2v_0}{x_2 - x_1}\delta_{x_1} + \frac{x_1v_0 - v_1}{x_2 - x_1}\delta_{x_2}, f(x)\rangle;$$

so, we get (24).   □

PROOF OF THE THEOREM. From the above lemma, it seems natural that the necessary and sufficient conditions for the quasi-difiniteness of v were given in terms of x_1, x_2, v_0 and v_1.

As $\{P_n\}$ is a basis of $\mathbb{P}$, we have $R_{n+2}(x) = P_{n+2}(x) + \sum_{k=0}^{n+1} a_{n,k}P_k(x)$, where the coefficients $a_{n,k}$ can be obtained as follows:

$$\langle u, P_k^2\rangle a_{n,k} = \langle v, (x - x_1)(x - x_2)R_{n+2}P_k\rangle$$

$$= \begin{cases} 0 & \text{, if } k = 0, 1, \ldots, n - 1 \\ \langle v, R_{n+2}^2\rangle & \text{, if } k = n \\ \langle v, (x - x_1)(x - x_2)R_{n+2}P_{n+1}\rangle & \text{, if } k = n + 1 \end{cases};$$

and so,

$$R_{n+2}(x) = P_{n+2}(x) + a_{n+2}P_{n+1}(x) + b_{n+2}P_n(x), \tag{25}$$

where $a_{n+2} = a_{n,n+1}$ and $b_{n+2} = a_{n,n}$; so, a necessary condition for the quasi-definitnes of the linear functional v is that $b_{n+2} \neq 0$ for $n \in \mathbb{N}$ (by (5)).

Taking into account theorem 5, the MOPS with respect to v is a strictly quasi-orthogonal MPS of order two with respect to u. Now, if we substitute in definition 4 u by $(x - x_1)(x - x_2)v$ we obtain that v is quasi-definite (because v is strictly quasi-orthogonal of order two with respect to u, i.e $b_{n+2} \neq 0$, $n \in \mathbb{N}$), if and only if:

i.1 $\langle v, R_{n+1}\rangle = 0$ for $n \in \mathbb{N}$,

i.2 $\langle v, xR_{n+2}\rangle = 0$ for $n \in \mathbb{N}$,

i.3 $\langle v, R_1^2\rangle \neq 0$.

Expressing these conditions in terms of the data:

1. From i.1 with $n = 0$ we obtain $a_1 = \beta_0 - v_1$.

2. From i.3 and taking into account the last expression for a_1 we get $\frac{x_2v_0 - v_1}{x_2 - x_1}(x_1 - v_1)^2 + \frac{v_1 - x_1v_0}{x_2 - x_1}(x_2 - v_1)^2 \neq 1$.

3. From $\begin{cases} \langle v, R_{n+2}\rangle = 0 \\ \langle v, xR_{n+2}\rangle = 0 \end{cases}$, $n \in \mathbb{N}$ we conclude that $\begin{cases} \langle v, (x - x_1)R_{n+2}\rangle = 0 \\ \langle v, (x - x_2)R_{n+2}\rangle = 0 \end{cases}$ for $n \in \mathbb{N}$.

These two expressions are enough in order to determine a_{n+2} and b_{n+2}:

- Substituting x by x_1 in (25) and substracting from (25) the equation we have found, we get, after dividing by $x - x_1$

$$\theta_{x_1}R_{n+2}(x) = \theta_{x_1}P_{n+2}(x) + a_{n+2}\theta_{x_1}P_{n+1}(x) + b_{n+2}\theta_{x_1}P_n(x). \tag{26}$$

Applying the quasi-definite functional u to both members of (26) and so on, we get

$$\langle v, (x - x_2)(R_{n+2}(x) - R_{n+2}(x_1))\rangle = P_{n+1}^{(1)}(x_1) + a_{n+2}P_n^{(1)}(x_1) + b_{n+2}P_{n-1}^{(1)}(x_1), n \in \mathbb{N};$$

but we know that $\langle v, (x - x_2)R_{n+2}(x)\rangle = 0$, so

$$(v_1 - x_2v_0)R_{n+2}(x_1) = P_{n+1}^{(1)}(x_1) + a_{n+2}P_n^{(1)}(x_1) + b_{n+2}P_{n-1}^{(1)}(x_1), n \in \mathbb{N}. \tag{27}$$

- Using the same process described above with x_2 instead of x_1 we get

$$(v_1 - x_1v_0)R_{n+2}(x_2) = P_{n+1}^{(1)}(x_2) + a_{n+2}P_n^{(1)}(x_2) + b_{n+2}P_{n-1}^{(1)}(x_2), n \in \mathbb{N} \tag{28}$$

and taking into account (25) we obtain

$$\begin{cases} P_{n+1}(x_1; -\frac{1}{v_1 - x_2 v_0})a_{n+2} + P_n(x_1; -\frac{1}{v_1 - x_2 v_0})b_{n+2} = -P_{n+2}(x_1; -\frac{1}{v_1 - x_2 v_0}) \\ P_{n+1}(x_2; -\frac{1}{v_1 - x_1 v_0})a_{n+2} + P_n(x_2; -\frac{1}{v_1 - x_1 v_0})b_{n+2} = -P_{n+2}(x_2; -\frac{1}{v_1 - x_1 v_0}) \end{cases} \tag{29}$$

and from this, we get (20).
Hence

$$b_{n+2} = -\frac{|d_{n+1}|}{|d_n|}, \ n \in \mathbb{N}. \tag{30}$$

As a conclusion:

- The linear functional v is quasi-definite if and only if (18) is verified.

Concerning the determination of the coefficients of the recurrence relation (21):

Substituting R_{n+1} in (21) by $P_{n+1}(x) + a_{n+1} P_n(x) + b_{n+1} P_{n-1}(x)$ and applying the recurrence relation (6) we get, after comparing the coefficients of the P_k with $k = n - 2, n - 1, n, n + 1$,

   (a) $\eta_{n+1} = \frac{b_{n+1}}{b_n} \gamma_{n-1}, \ n \geq 2$

   (b) $a_{n+1} \gamma_n + b_{n+1} \beta_{n-1} = \xi_{n+1} b_{n+1} + \eta_{n+1} a_n, \ n \geq 1$

   (c) $\gamma_{n+1} + a_{n+1} \beta_n + b_{n+1} = b_{n+2} + \xi_{n+1} a_{n+1} + \eta_{n+1}, \ n \geq 0$

   (d) $\xi_{n+1} = \beta_{n+1} - (a_{n+2} - a_{n+1}), \ n \geq -1$ □

As a limit case we have:

COROLLARY 8 ([2],[14]) *Let u be a quasi-definite linear functional with $u_0 = 1$ and $x_1 \in$* **C**; *the linear functional v defined by $u = (x - x_1)^2 v$ is quasi-definite, if and only if $v_0 \neq 0$ and v_1 are such that*

$$\begin{aligned} \frac{(v_1 - x_1 v_0)^2}{v_0} &\neq 1 \\ |d_n| &\neq 0, \ for \ every \ n \in \mathbb{N} \end{aligned} \tag{31}$$

where d_n is the matrix

$$\begin{bmatrix} P_{n+1}(x_1; -\frac{1}{v_1 - x_1 v_0}) & P_n(x_1; -\frac{1}{v_1 - x_1 v_0}) \\ (v_1 - x_1 v_0)P'_{n+1}(x_1; -\frac{1}{v_1 - x_1 v_0}) + v_0 P_{n+1}(x_1) & (v_1 - x_1 v_0)P'_n(x_1; -\frac{1}{v_1 - x_1 v_0}) + v_0 P_n(x_1) \end{bmatrix}.$$

The corresponding MOPS with respect to v is given by

$$R_{n+2}(x) = P_{n+2}(x) + a_{n+2} P_{n+1}(x) + b_{n+2} P_n(x), \ n \in \mathbb{N}, \tag{32}$$

where

$$\begin{bmatrix} a_{n+2} \\ b_{n+2} \end{bmatrix} = -d_n^{-1} \begin{bmatrix} P_{n+2}(x_1; -\frac{1}{v_1 - x_1 v_0}) \\ (v_1 - x_1 v_0)P'_{n+2}(x_1; -\frac{1}{v_1 - x_1 v_0}) + v_0 P_{n+2}(x_1) \end{bmatrix} \tag{33}$$

and $a_1 = \beta_0 - \frac{v_1}{v_0}$. The coefficients of the recurrence relation that R_n satisfies (21) are related with the ones of (6) by the following formulas

$$\xi_n = \beta_n - (a_{n+1} - a_n), \ n \in \mathbb{N} \tag{34}$$

$$\left. \begin{aligned} \eta_{n+3} &= \frac{b_{n+3}}{b_{n+2}} \gamma_{n+1}, \ n \in \mathbb{N} \\ \eta_1 &= \gamma_1 + a_1(\beta_0 - \xi_1) - b_2 \\ \eta_2 &= \gamma_2 + a_2(\beta_1 - \xi_2) - (b_3 - b_2) \end{aligned} \right\} \tag{35}$$

4   REAL INVERSE PROBLEM

The answer to the main problem is the following:

THEOREM 9 *Let $\{P_n\}$ be a MOPS with respect to u and $\{R_n\}$ be a strictly quasi-orthogonal MPS of order two with respect to $\{P_n\}$; $\{R_n\}$ is a MOPS if and only if, the parameters in (16) $a_{n,n} = a_n$ and $a_{n,n-1} = b_n$ satisfy*

$$a_{n+4} = \frac{a_2}{b_2}\gamma_1 - \beta_2 - \beta_3 + a_3 - \frac{a_{n+3}}{b_{n+3}}\gamma_{n+2} + \beta_{n+3} + \beta_{n+2}$$
$$b_{n+4} = b_{n+3}\left(1 - \frac{\gamma_{n+1}}{b_{n+2}}\right) + \gamma_{n+3} + a_{n+3}(\beta_{n+2} - \beta_{n+3} + a_{n+4} - a_{n+3}) \quad , \; n \in \mathbb{N} \qquad (36)$$

where the initial conditions are

$$a_1 = 0 \text{ and } a_3 = \beta_2 - \beta_0 + a_2 - \frac{a_2}{b_2}\gamma_1$$

(a) a_1, a_2, b_2, b_3 if $\Big\langle$

$$a_1 \neq 0, \; b_2 - a_1 a_2 \neq 0 \text{ and } a_3 = \frac{a_2\gamma_1 + b_2(\beta_0 - \beta_2 - a_2) - a_1(\gamma_2 + a_2(\beta_1 - \beta_2 - a_2)) - b_3 + b_2)}{a_1 a_2 - b_2}$$

(b) a_1, a_2, a_3, b_3 if $a_1 \neq 0$ and $b_2 = a_1 a_2$.

Furthermore, a_1, a_2, a_3, b_2, b_3 verify

$$\left.\begin{array}{l} \gamma_2 + a_2(\beta_1 - \beta_2 + a_3 - a_2) - b_3 + b_2 \neq 0 \\ \gamma_1 + a_1(\beta_0 - \beta_1 + a_2 - a_1) - b_2 \neq 0 \end{array}\right\} \qquad (37)$$

In this case, the coefficients of the recurrence relation that $\{R_n\}$ satisfies are given by

$$\left\{\begin{array}{l} \xi_n = \beta_n - (a_{n+1} - a_n) \\ \eta_{n+1} = \gamma_{n+1} + a_{n+1}(\beta_n - \xi_{n+1}) - (b_{n+2} - b_{n+1}) \end{array}\right. , \; n \in \mathbb{N}$$

with the restrictions $a_0 = b_1 = 0$.

In addition, the linear functional v such that $\{R_n\}$ is the corresponding MOPS is given by

$$u = p(x)v, \qquad (38)$$

where $p(x) = \frac{1}{\langle v, 1 \rangle}\left[(1 + \frac{a_1^2}{\eta_1} + \frac{b_2^2}{\eta_1 \eta_2})P_0(x) + (\frac{a_1}{\eta_1} + \frac{a_2 b_2}{\eta_1 \eta_2})P_1(x) + \frac{b_2}{\eta_1 \eta_2}P_2(x)\right]$.

REMARKS:

1. From (38) and taking into account theorem 6 and its corollary we have determined a_n and b_n:

   - If $\left(\frac{a_1\eta_2}{b_2} - a_2\right)^2 - 4\left(\frac{\eta_1\eta_2}{b_2} + \frac{a_1^2\eta_2}{b_2} + b_2 - \gamma_1\right) = 0$ then (33) appears.

   - If $\left(\frac{a_1\eta_2}{b_2} - a_2\right)^2 - 4\left(\frac{\eta_1\eta_2}{b_2} + \frac{a_1^2\eta_2}{b_2} + b_2 - \gamma_1\right) \neq 0$ then (20) holds.

2. (36) provides an algorithm for the computation of (a_n, b_n). In fact, from the first relation we can calculate the a_{n+4} in terms of $(a_k)_{k=2}^{n+3}$ and $(b_{k+1})_{k=1}^{n+2}$; and substituting a_{n+4} in the second one we get b_{n+4}.

PROOF. From theorem 5, there exist two sequences $(a_n), (b_n) \subset \mathbb{R}$ such that

$$R_{n+2}(x) = P_{n+2}(x) + a_{n+2}P_{n+1}(x) + b_{n+2}P_n(x) \qquad (39)$$

with $b_{n+2} \neq 0$, $n \in \mathbb{N}$. Multiplying the equation (39) by x and using (6) we obtain

$$xR_{n+2} = P_{n+3} + \beta_{n+2}P_{n+2} + \gamma_{n+2}P_{n+1} + a_{n+2}(P_{n+2} + \beta_{n+1}P_{n+1} + \gamma_{n+1}P_n) + b_{n+2}(P_{n+1} + \beta_n P_n + \gamma_n P_{n-1})$$

Applying two times (39) we have

$$
\begin{aligned}
xR_{n+2} &= R_{n+3} + \xi_{n+2}R_{n+2} + \eta_{n+2}R_{n+1} + \\
&\quad (a_{n+2}\gamma_{n+1} - a_{n+1}\eta_{n+2} + b_{n+2}(\beta_n - \xi_{n+2}))P_n + (b_{n+2}\gamma_n - b_{n+1}\eta_{n+2})P_{n-1}
\end{aligned}
$$

where

$$
\begin{cases}
\xi_n = \beta_n - (a_{n+1} - a_n) \\
\eta_{n+1} = \gamma_{n+1} + a_{n+1}(\beta_n - \xi_{n+1}) - (b_{n+2} - b_{n+1})
\end{cases} , \ n \in \mathbb{N}. \tag{40}
$$

So, $\{R_n\}$ is a MOPS if and only if

$$
\begin{cases}
\gamma_{n+1} + a_{n+1}(\beta_n - \xi_{n+1}) - (b_{n+2} - b_{n+1}) \neq 0 \\
a_{n+2}\gamma_{n+1} - a_{n+1}\eta_{n+2} + b_{n+2}(\beta_n - \xi_{n+2}) = 0 \ , \ n \in \mathbb{N} \\
b_{n+3}\gamma_{n+1} - b_{n+2}\eta_{n+3} = 0
\end{cases} \tag{41}
$$

Now if we substitute $\eta_{n+3} = \frac{b_{n+3}\gamma_{n+1}}{b_{n+2}}$ in the second equation of (41) and in the expression of η_{n+1} of (40) we get

$$
\frac{a_{n+3}}{b_{n+3}}\gamma_{n+2} - \frac{a_{n+2}}{b_{n+2}}\gamma_{n+1} + (a_{n+4} - a_{n+3}) - (\beta_{n+3} - \beta_{n+1}) = 0
$$

$$
\gamma_{n+3} + a_{n+3}(\beta_{n+2} - \beta_{n+3} + a_{n+4} - a_{n+3}) - (b_{n+4} - b_{n+3}) = \frac{b_{n+3}\gamma_{n+1}}{b_{n+2}}
$$

and from this, the algorithm (36) follows.

Now $\{R_n\}$ is a MOPS associated with the quasi-definite functional v. We search for a relation between these two functionals:

• If we apply the functional u in (39) we obtain $\langle u, R_{n+2}\rangle = 0$, $n \in \mathbb{N}$. Hence

$$
u = \sum_{i=0}^{2} \lambda_i \alpha_i \tag{42}
$$

where (α_n) is the dual basis associated with $\{R_n\}$ and $\lambda_i = \langle u, R_i \rangle$, $i = 0, 1, 2$.

As $\{R_n\}$ is a MOPS associated with v we have by (7) $\alpha_n = \frac{R_n}{\langle v, R_n^2\rangle}v$ for all $n \in \mathbb{N}$; and so (42) can be rewritten as

$$
u = \left(1 + \frac{a_1}{\eta_1}R_1(x) + \frac{b_2}{\eta_1\eta_2}R_2(x)\right)\frac{v}{\langle v, 1\rangle}. \tag{43}
$$

Taking into account R_n for $n = 1, 2$ we get (38).   □

EXAMPLES. We will construct two MOPS strictly quasi-orthogonal of order 2 with respect to:

(1) Second kind Tchebychev MOPS, $\{U_n\}$.

It is well known that in this case $\begin{cases} \beta_n = 0 \\ \gamma_{n+1} = \frac{1}{4} \end{cases}$, $n \in \mathbb{N}$ (see [5, Exercise 4.9]).

Hence, if we choose as initial conditions $a_1 = a_2 = 2a$ and $b_2 = b_3 = -c \neq 0$, with $a, c \in \mathbb{C}$, we obtain by induction from (36) and considerations about initial conditions,
$\begin{cases} a_{n+3} = 2a \\ b_{n+4} = -c \end{cases}$, $n \in \mathbb{N}$.
It is straightforward to prove that (37) holds.

The coefficients of the three term recurrence relation (ttrr) verified by $\{R_n\}$ are

$$
\begin{cases} \xi_0 = -2a \\ \xi_{n+1} = 0 \end{cases} \quad \text{and} \quad \begin{cases} \eta_1 = \frac{1}{4} + c \\ \eta_{n+2} = \frac{1}{4} \end{cases} , \ n \in \mathbb{N}.
$$

This is an example of Al-Salam and A. Verma (see [1]). There they found a measure such that $\{R_n\}$ is the corresponding MOPS. We can say that the condition for the positive definitness of the new linear functional is $c > -\frac{1}{4}$ and $a \in \mathbb{R}$.

Now, we give the quasi-definite (not positive definite, because $\eta_1 < 0$) linear functional, v, in terms of the second kind Tchebychev linear functional, u, in the particular case $a = 0$ and $c = -1$:

$$-\frac{16}{3v_0}v = (\frac{9}{16} + x^2)^{-1}u + \frac{v_0}{2}(\delta_{-\frac{3}{4}} + \delta_{\frac{3}{4}}).$$

From this representation we get $\langle -\frac{16}{3v_0}v, 1 \rangle = v_0$, i.e $v_0 = -\frac{16}{3}$; hence

$$v = (\frac{9}{16} + x^2)^{-1}u - \frac{8}{3}(\delta_{-\frac{3}{4}} + \delta_{\frac{3}{4}}). \qquad \square$$

(2) Hermite MOPS, $\{H_n\}$.

It is well known that $\begin{cases} \beta_n = 0 \\ \gamma_{n+1} = \frac{n+1}{2} \end{cases}$, $n \in \mathbb{N}$ (see [5, Exercise 1.6]).

Hence, if we choose as initial conditions $a_1 = a_2 = 0$ and $b_2 = b_3 = 1$ we obtain from (36) and considerations about initial conditions $a_3 = 0$, $\begin{cases} a_{n+4} = 0 \\ b_{n+4} = \frac{b_{n+3}}{b_{n+2}}\left(b_{n+2} - \frac{n+1}{2}\right) + \frac{n+3}{2} \end{cases}$, $n \in \mathbb{N}$.

It can be seen $b_{2k+2} = b_{2k+3} = k+1$ for $k \in \mathbb{N}$. In fact,

$k = 0$ $b_2 = b_3 = 1$. Using induction,

if $b_{2k+2} = b_{2k+3} = k+1$ for $k \le p$

then $b_{2k+2} = b_{2k+3} = k+1$ for $k = p+1$. In fact

$$b_{2p+4} = \frac{2p+3}{2} + \frac{p+1}{p+1}\left(p + 1 - \frac{2p+1}{2}\right) = p+2$$

and

$$b_{2p+5} = \frac{2p+4}{2} + \frac{p+2}{p+1}\left(p + 1 - \frac{2p+2}{2}\right) = p+2.$$

(37) can be deduced easily.

The coefficients of the ttrr verified by $\{R_n\}$ are given by

$$\xi_n = 0 \quad \text{and} \quad \eta_{n+1} = \begin{cases} -\frac{1}{2} \text{ if } n = 0 \\ k+1 \text{ if } n = 2k+1 \\ \frac{2k+1}{2} \text{ if } n = 2k+2 \end{cases}, n \in \mathbb{N}.$$

The quasi-definite (not positive definite) functional v is given in terms of the Hermite linear functional, u, by:

$$-\frac{2}{v_0}v = x^{-2}u + v_0\delta_0.$$

From this representation we get $\langle -\frac{2}{v_0}v, 1 \rangle = v_0$, i.e $v_0 = -2$; hence

$$v = x^{-2}u - 2\delta_0. \qquad \square$$

5 COMPLEX INVERSE PROBLEM

Before working out the problem (4) we will state some basic definitions.

Let $\mathcal{M} = \begin{bmatrix} c_0 & c_1 & \cdots \\ \bar{c}_1 & c_0 & \cdots \\ & \cdots & \end{bmatrix}$ be an infinite Toeplitz and Hermitian matrix. A linear functional u on the linear space of Laurent polynomials $\mathcal{L} = \text{span}\{z^n : n \in \mathbb{Z}\}$ is said to be Hermitian if $c_n = \langle u, z^n \rangle$, $n \in \mathbb{N}$ and $\bar{c}_n = \langle u, z^{-n} \rangle$, $n \in \mathbb{N}$. Here $\mathbb{Z}$ denotes the set of integers $\{0, \pm1, \pm2, \ldots\}$. The linear functional u is said to be quasi-definite when the principal submatrices of $\mathcal{M}$ are non-singular.

THEOREM 10 ([8]) $\mathcal{M}$ *represents the Gram matrix for the generalized inner product:*

$$\langle p(z), q(z) \rangle = \langle u, p(z)\bar{q}(\frac{1}{z}) \rangle$$

associated with u in $\mathbb{P}$.

In these conditions, we can define a MOPS $\{\phi_n\}$ with respect to the matrix $\mathcal{M}$ or with respect to the *quasi-definite Hermitian linear functional* u by

$$\phi_n(z) = \frac{1}{\Delta_{n-1}} \begin{vmatrix} c_0 & c_1 & \dots & c_n \\ \bar{c}_1 & c_0 & \dots & c_{n-1} \\ & & \dots & \\ \bar{c}_{n-1} & \bar{c}_{n-2} & \dots & c_1 \\ 1 & z & \dots & z^n \end{vmatrix}$$

where Δ_{n-1} means the principal minor of order n for $\mathcal{M}$. This family will be called a *MOPS on the unit circle T* since the shift operator on $\mathbb{P}$ is unitary with respect to the above inner product.

Now, if we denote $K_n = \frac{\Delta_n}{\Delta_{n-1}}$ for $n \geq 1$ and $K_0 = c_0$, it can be seen that $\frac{K_n}{K_{n-1}} = 1 - |\phi_n(0)|^2$; so $|\phi_n(0)| \neq 1$ for $n \geq 1$.

For a MOPS $\{\phi_n\}$ on the *unit circle, T*, Szegö recurrence relations follows (see [17, T. 11.4.2]), i.e.

$$\phi_{n+1}(z) = z\phi_n(z) + \phi_{n+1}(0)\phi_n^\star(z) \tag{44}$$

$$\phi_{n+1}^\star(z) = \phi_n^\star(z) + \overline{\phi_{n+1}(0)}z\phi_n(z) \tag{45}$$

where $|\phi_{n+1}(0)| \neq 1$ and $\phi_n^\star(z) = z^n\overline{\phi_n}(z^{-1})$ for $n \in \mathbb{N}$. Conversely, if $|\phi_{n+1}(0)| \neq 1$, $n \in \mathbb{N}$ the sequence defined by (44) is orthogonal with respect to a quasi-definite Hermitian linear functional (see [8]). The values $(\phi_n(0))$ are called reflection parameters.

We define a MPS $\{\psi_n\}$ by

$$\psi_n(z) = \phi_n(z) + a_n\phi_{n-1}(z) \text{ for } n \geq 1. \tag{46}$$

But, $\{\psi_n\}$ is a MOPS on the unit circle if and only if verifies the following recurrence relation

$$\psi_{n+1}(z) = z\psi_n(z) + \psi_{n+1}(0)\psi_n^\star(z), \, n \in \mathbb{N} \text{ and } |\psi_{n+1}(0)| \neq 1.$$

So, using (46) we get

$$\phi_{n+1}(z) + a_{n+1}\phi_n(z) = z\phi_n(z) + a_nz\phi_{n-1}(z) + (\phi_{n+1}(0) + a_{n+1}\phi_n(0))(\phi_n^\star(z) + \bar{a}_nz\phi_{n-1}^\star(z))$$

for $n \geq 1$. Applying two times (44) we obtain

$$\left[a_{n+1} - a_n - a_{n+1}|\phi_n(0)|^2\right]z\phi_{n-1}(z) = \left[\phi_{n+1}(0) + a_{n+1}\phi_n(0)\right]\bar{a}_nz\phi_{n-1}^\star(z) \tag{47}$$

for $n \geq 1$. We can state the following result,

THEOREM 11 *Let $\{\psi_n\}$ be a MPS defined by (46). If $\{\psi_n\}$ is orthogonal with respect to a Hermitian linear functional u then $\psi_n(z) = z^{n-2}\psi_2(z)$, $n \geq 2$, where ψ_2 is a polynomial of degree two.*

PROOF. From (47) and taking into account [11, proposition 2.3 (c)] we have

$$a_2(1 - |\phi_1(0)|^2) = (\phi_2(0) + a_2\phi_1(0))\bar{a}_1 + a_1 \tag{48}$$

$$\bar{a}_n(\phi_{n+1}(0) + a_{n+1}\phi_n(0)) = \bar{a}_n\psi_{n+1}(0) = 0, \, n \geq 2 \tag{49}$$

$$a_{n+1}(1 - |\phi_n(0)|^2) = \frac{K_n}{K_{n-1}} = a_n, \, n \geq 2. \tag{50}$$

Now we consider three cases:

(a) If $a_1 = 0 \overset{(48)}{\Longrightarrow} a_2 = 0 \overset{(50)}{\Longrightarrow} a_n = 0$, $n \geq 1$. So $\psi_n(z) = \phi_n(z)$, for $n \in \mathbb{N}$.

(b) If $a_1 \neq 0$ and $a_2 = 0$ then by (48) we get $|\psi_2(0)| = 1$; which contradicts the quasi-definiteness of the linear functional associated with $\{\psi_n\}$.

(c) If $a_1a_2 \neq 0$ then (50) gives $a_{n+1} = \frac{K_{n-1}}{K_n}a_n$; so $a_{n+1} = \frac{K_1a_2}{K_n}$, for $n \geq 2$.

Now, because $a_n \neq 0$ for $n \geq 2$ we have from (49) $\psi_{n+1}(0) = 0$ for $n \geq 2$. Applying the recurrence relation that $\{\psi_n\}$ satisfies we get

$$\psi_{n+1}(z) = z\psi_n(z), \, n \geq 2; \tag{51}$$

and so a representation for these polynomials

$$\psi_n(z) = z^{n-2}\psi_2(z) \tag{52}$$

follows. Hence, in order to determine $\{\phi_n\}$ we need $a_1, a_2, \phi_1(0)$ as initial conditions because $\phi_2(0)$ is defined by (48).

SOME EXAMPLES.

1. If $\phi_1(0) = 0$ we have from (48)

$$a_2 = \phi_2(0)\bar{a}_1 + a_1 \tag{53}$$

then, we can consider two situations:

(a) $a_1 = a_2$

In this case $\phi_2(0)\bar{a}_1 = 0$ and as a consequence $\phi_2(0) = 0$. From (49) $\phi_n(0) = 0$, for $n \geq 1$ and from (50) $a_n = a_1$, $n \geq 1$.

This situation means that

$$\begin{aligned}\phi_n(z) &= z^n \\ a_n &= a_1, n \geq 1\end{aligned}$$

i.e. (ψ_n) take the form $\psi_n(z) = z^{n-1}(z + a_1)$, $n \geq 1$ and is orthogonal with respect to the Poisson measure $d\mu = \dfrac{d\theta}{|\exp(i\theta) + a_1|^2}$, $|a_1| \neq 1$.

(b) $a_1 \neq a_2$

Then, taking conjugates in (53)

$$\begin{aligned}a_2 &= \phi_2(0)\bar{a}_1 + a_1 \\ \bar{a}_2 &= \overline{\phi_2(0)}a_1 + \bar{a}_1\end{aligned}$$

and $a_1 = \frac{a_2 - \bar{a}_2\phi_2(0)}{1 - |\phi_2(0)|^2}$.

In this way, we can deduce

$$\phi_{n+1}(0) = a_{n+1}\phi_n(0) = -\frac{K_1 a_2}{K_n}\phi_n(0)$$

and, thus because $\prod_{i=1}^{n} K_i = \Delta_n$ we have,

$$\phi_{n+1}(0) = (-K_1 a_2)^n \frac{1}{\Delta_n}, \, n \geq 2.$$

Remark that $\phi_2(0) \neq 0$ because of, otherwise, $a_1 = a_2$ in contradiction with our hypothesis. Then $\psi_n(z) = z^{n-2}\psi_2(z)$ with $\psi_2(0) \neq 0$.

If we take $\phi_2(0) = q$ and $a_2 = 1 - q$ then $a_1 = \frac{1-q}{1+q}$. By induction

$$\begin{aligned}a_n &= \frac{1 + (n - 3)q}{1 + (n - 2)q}, \, n \geq 2 \\ \phi_n(0) &= (-1)^n \frac{q}{1 + (n - 2)q}, \, n \geq 2 \text{ and } \phi_1(0) = 0.\end{aligned}$$

2. If $\phi_1(0) \neq 0$, we will consider two cases:

(a) $\phi_2(0) = 0$

From (49) this implies that $\phi_n(0) = 0$, $n \geq 2$. Then, $\phi_n(z) = z^{n-1}\phi_1(z)$, $n \geq 1$. Moreover, $a_n = a_2$ and from (48) $a_2(1 - |\phi_1(0)|^2) = a_2\bar{a}_1\phi_1(0) + a_1$. Taking conjugates

$$\bar{a}_2(1 - |\phi_1(0)|^2) = \bar{a}_2 a_1 \overline{\phi_1(0)} + \bar{a}_1$$

and thus

$$a_1 = \bar{a}_2 \frac{(1 - |\phi_1(0)|^2)(1 - a_2\overline{\phi_1(0)})}{1 - |a_2|^2|\phi_1(0)|^2}$$

Observe that in this situation

$$
\begin{aligned}
\psi_n(z) &= z^{n-1}\phi_1(z) + a_n z^{n-2}\phi_1(z) \\
&= z^{n-2}\phi_1(z)(z + a_2) \\
&= z^{n-2}(z + \phi_1(0))(z + a_2), \, n \geq 2
\end{aligned}
$$

This situation corresponds to the case when $\{\psi_n\}$ is a MOPS with respect to the measure
$$\mathrm{d}\mu = \frac{\mathrm{d}\theta}{|\exp(i\theta) + \phi_1(0)|^2|\exp(i\theta) + a_2|^2}, \, |a_2| \neq 1.$$

(b) $\phi_2(0) \neq 0$

Then $\phi_n(0) \neq 0$, $n \in \mathbb{N}$. This seems to be the more interesting situation.

An algorithm can be deduced in order to compute all the reflection parameters. In fact, from (48) we have

$$\phi_2(0) = \frac{a_2(1 - |\phi_1(0)|^2) - a_1 - a_1\bar{a}_2\phi_1(0)}{\bar{a}_1}.$$

Now, using sucessively (49) and (50) we get

$$a_3 = \frac{a_2}{1 - |\phi_2(0)|^2} \quad \longrightarrow \quad \phi_3(0) = -a_3\phi_2(0)$$

$$\cdots$$

$$a_{n+1} = \frac{a_2}{\prod_{k=1}^{n}(1 - |\phi_k(0)|^2)} \quad \longrightarrow \quad \phi_{n+1}(0) = (-1)^{n+1}\prod_{k=2}^{n} a_k\phi_1(0) \text{ for } n \geq 1.$$

Again, we can consider the particular situation $\phi_2(0) = q$ and $a_2 = 1 - q$. As before, we can deduce

$$
\begin{aligned}
a_n &= \frac{1 + (n-3)q}{1 + (n-2)q} \\
\phi_n(0) &= (-1)^n \frac{q}{1 + (n-2)q}
\end{aligned}
$$

for $n \geq 2$; if we take $\phi_1(0) = \frac{-q}{1-q}$ in (48) then $a_1 = \frac{1-2q}{1-q}$. Hence

$$
\begin{aligned}
a_n &= \frac{1 + (n-3)q}{1 + (n-2)q} \\
\phi_n(0) &= (-1)^n \frac{q}{1 + (n-2)q}
\end{aligned}
$$

for $n \geq 1$. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad\square$

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