Internat. J. Math. & Math. Sci.
VOL. 19 NO. 4 (1996) 733-736

A REPRESENTATION OF BOUNDED COMMUTATIVE BCK-ALGEBRAS

H.A.S. ABUJABAL

Department of Mathematics, Faculty of Science
King Abdul Azız University, P O Box 31464
Jeddah - 21497, SAUDI ARABIA

M. ASLAM

Department of Mathematics
Quaid-i-Azam University
Islamabad, PAKISTAN

A.B. THAHEEM

Department of Mathematical Sciences
King Fahd University of Petroleum and Minerals
P O Box 469, Dhahran 31261, SAUDI ARABIA

(Received April 26, 1993 and in revised form November 13, 1995)

ABSTRACT. In this note, we prove a representation theorem for bounded commutative BCK-algebras

KEY WORDS AND PHRASES: Bounded commutative BCK-algebra, ideal, prime ideal, quotient BCK-algebras, spectral space
1991 AMS SUBJECT CLASSIFICATION CODES: Primary 06D99, Secondary 54A

1. INTRODUCTION

The representation theory of various algebraic structures has been extensively studied The corresponding representation theory for BCK-algebras remains to be developed. Rousseau and Thaheem [1] proved a representation theorem for a positive implicative BCK-algebra as BCK-algebra of self-mappings which apparently does not possess many algebraic properties. Cornish [2] constructed a bounded implicative BCK-algebra of multipliers corresponding to a bounded implicative BCK-algebra, but no representation of these algebras has been studied there. The purpose of this note is to prove a representation theorem for a bounded commutative BCK-algebra We essentially prove that a bounded commutative BCK-algebra X is isomorphic to the bounded commutative BCK-algebra $\widehat{X}$ of mappings acting on the associated spectral space of X Our approach depends on the theory of quotient BCK-algebras as developed by Iséki and Tanaka [3] and the theory of prime deals of commutative BCK-algebras Before we develop our results, we recall some technical preliminaries for the sake of completeness A BCK-algebra is a system $(X, *, 0, \leq)$ (denoted simply by X), satisfying (i) $(x*y)*(x*z) \leq z*y$ (ii) $x*(x*y) \leq y$ (iii) $x \leq x$ (iv) $0 \leq x$ (v) $x \leq y$, $y \leq x$ imply $x = y$, where $x \leq y$ if and only if $x * y = 0$ for all $x, y, z \in X$ If X contains an element 1 such that $x \leq 1$ for all $x \in X$, then X is said to be bounded X is said to be commutative if $x \wedge y = y \wedge x$ for all $x, y \in X$, where $x \wedge y = y*(y*x)$ A non-empty set A of a BCK-algebra X is said to be an ideal of X if $0 \in A$ and $x, y*x \in A$ imply $y \in A$ A proper ideal A of a commutative BCK-algebra X is said to be prime ıf $x \wedge y \in A$ implies $x \in A$ or $y \in A$ It is well-known that every maximal ideal in a commutative BCK-algebra is prime (see e g [4]) The theory of prime ideals plays an important role in the study of commutative BCK-algebras For some information about prime ideals, we refer to [5] which contains further references about the theory of prime ideals A subset S of a commutative BCK-algebra is said to be $\wedge$-closed if $x \wedge y \in S$ whenever $x, y \in S$

We now state the following theorem known as the prime ideal theorem (see [6, Theorem 2 4] and [5, Corollary 3])

THEOREM A. *Let I be an ideal and S be a $\wedge$-closed set of a commutative BCK-algebra X such that $S \cap I = \emptyset$. Then there exists a prime ideal P such that $I \subseteq P$ and $P \cap S = \emptyset$.*

COROLLARY B. *Let I be an ideal of a commutative BCK-algebra X and $a \in X$ such that $a \notin I$. Then there exists a prime ideal P such that $a \notin P$ and $I \subseteq P$.*

The above corollary follows from Theorem A by choosing $s = \{a\}$ If a non-trivial commutative BCK-algebra and $I = \{0\}$, then Corollary B ensures the existence of a prime ideal in X We now recall the definition of a quotient BCK-algebra If X is a BCK-algebra and A is an ideal of X, then we define an equivalence relation $\sim$ on X by $x \sim y$ if and only if $x * y,\ y * x \in A$ Let $C_x = \{y \in X : x * y, y * x \in A\}$ Let $C_x = \{y \in X : x * y, y * x \in A\}$ denote the equivalence class containing $x \in X$ Then one can see that $C_0 = A$ and $C_x = C_y$ if and only if $x \sim y$ Let X/A denote the set of all equivalence classes C_x, $x \in X$. Then X/A is a BCK-algebra (known as quotient BCK-algebra) with $C_x * C_y = C_{x*y}$, and $C_x \leq C_y$ if and only if $x * y \in A$, and $C_0 = A$ is the zero of X/A (see for instance [3-7]). If X is bounded commutative, then X/A is also bounded commutative with C_1 as the unit element For the general theory of BCK-algebras and other undefined terminology and notations used here, we refer to Iséki and Tanaka [3-7] and Cornish [8]

2. A REPRESENTATION THEOREM

Throughout X denotes a bounded commutative BCK-algebra. Let $Spec(X)$ denote the set of all prime ideals of X, called the spectrum of X It has been shown in [5] that $Spec(X)$ is a compact topological space referred to as the spectral space associated with X. It is well-known that $\bigcap_{P \in Spec(X)} P = \{0\}$ (see e.g [8]).

DEFINITION 2.1. For any $x \in X$, we define a mapping

$$\widehat{x} : Spec(X) \to \bigcup_{P \in Spec(X)} X/P$$

where $\widehat{x}(P)$ denotes the image of x into X/P

It is easy to see that $\widehat{x}(P) = C_0$ if and only if $x \in P$.

We denote by $\widehat{X}$, the set of all mappings $\widehat{x}$, $x \in X$. For any $\widehat{x}$, $\widehat{y} \in \widehat{X}$, we define the following operations on $\widehat{X}$:

$$\widehat{x} * \widehat{y} = \widehat{(x * y)} \text{ and } \widehat{x} \leq \widehat{y} \text{ if and only if } \widehat{x} * \widehat{y} = \widehat{0}.$$

These operations are well-defined because of the properties of quotient algebras. Indeed, as $\widehat{x}(P)$ is the canonical image of x in X/P, namely the class C_x relative to P, and the union $\bigcup_{P \in Spec(X)} X/P$ is disjoint

Routine verifications similar to ones for quotient BCK-algebras (see e g [3]) lead to the following

PROPOSITION 2.2. *$(\widehat{X}, *, \widehat{0})$ is a bounded commutative BCK-algebra.*

We now prove the following representation result.

THEOREM 2.3. *The mapping $\phi : x \in X \to \widehat{x} \in \widehat{X}$ is an isomorphism.*

PROOF. That ϕ is surjective homomorphism follows from the definition (because the mapping $x \in X \to C_x \in X/P$ is the canonical homomorphism) To prove that ϕ is injective it is enough to show that $\phi(x) = \widehat{0}$ if and only if $x = 0$ For any $P \in Spec(X)$, $\phi(x)(P) = \widehat{0}$ implies that $x \in P$ for all $P \in Spec(X)$ and hence $x \in \bigcap_{p \in Spec(X)} P = \{0\}$ Thus $x = 0$ This completes the proof

We provide an example to explain some essential ideas developed above

EXAMPLE 2.4 ([3, p 363]) Let $X = \{0, a, b, 1\}$ be a set Define a binary operation $*$ on X as in Table 1

$*$	0	a	b	1
0	0	0	0	0
a	a	0	a	0
b	b	b	0	0
1	1	b	a	0

Table 1

The $(X, *, 0)$ is a bounded commutative BCK-algebra with $P = \{0, a\}$ and $Q = \{a, b\}$ as prime ideals (cf Table 2)

$\wedge$	0	a	b	1
0	0	0	0	0
a	0	a	0	a
b	0	0	b	b
1	1	a	b	1

Table 2

Then $Spec(X) = \{P, Q\}$, $X/P = \{\{0, a\}, \{b, 1\}\}$, $X/Q = \{\{0, b\}, \{a, 1\}\}$, X/P, X/Q, are disjoint and $\bigcup_{P \in Spec(X)} X/P$ is the disjoint union as defined above The rest of the calculations can easily be made to get the representation of X in this case.

ACKNOWLEDGMENT. The authors are grateful to the referee for his useful suggestions that led to an improvement of the paper One of the authors (A B Thaheem) thanks K F U P M for providing research facilities.

REFERENCES

[1] ROUSSEAU, R. and THAHEEM, A B , A representation of BCK-algebras as algebras of mappings, *Math. Japonica*, **34** (1989), 421-427

[2] CORNISH, W H., A multiplier approach to implicative BCK-algebras, *Math. Sem. Notes*, **8** (1980), 157-169

[3] ISÉKI, K and TANAKA, S., Ideal theory of BCK-algebras, *Math. Japonica*, **21** (1976), 351-366

[4] PALASINNSKI, R , Ideals in BCK-algebras which are lower semilattices, *Math. Japonica*, **26** (1981), 245-250

[5] ASLAM, M and THAHEEM, A.B., On spectral properties of BCK-algebras, *Math. Japonica*, **38** (1993), 1121-1128

[6] ASLAM, M and THAHEEM, A.B , A new proof of the prime ideal theorem for BCK-algebras, *Math. Japonica*, **38** (1993), 969-972

[7] ISÉKI, K and TANAKA, S., An introduction to the theory of BCK-algebras, *Math. Japonica*, **23** (1978), 1-26

[8] CORNISH, W.H., Algebraic structures and applications, *Proc. of the First Western Australian Conference on Algebra*, Marcel-Dekker, Inc , New York (1982), 101-122