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ON MATRIX CONVEXITY OF THE MOORE-PENROSE INVERSE

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ABSTRACT. Matrix convexity of the Moore-Penrose inverse was considered in the recent literature Here we give some converse inequalities as well as further generalizations

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1. INTRODUCTION

Let A and B be two complex Hermitian positive definite matrices, and let $0 \leq \lambda \leq 1$ Then

$$[\lambda A + (1-\lambda)B]^{-1} \leq \lambda A^{-1} + (1-\lambda)B^{-1} \tag{1 1}$$

where $A \geq B$ means that $A - B$ is a positive semi-definite matrix.

This result, i e., matrix convexity of the inverse function is an old result that appears explicitly in the papers [1,2,3,4,5] (see also the books [6, pp 554-555] and [7, pp. 469-471]).

The related matrix convexity of the Moore-Penrose (generalized) inverse, denoted by A^+, was considered in paper [8,9,10] The following was given in [10]:

Let A and B be two complex Hermitian positive semi-definite matrices of the same order. The inequality

$$[\lambda A + (1-\lambda)B]^{+} \leq \lambda A^{+} + (1-\lambda)B^{+} \tag{1 2}$$

for every $0 \leq \lambda \leq 1$ holds if and only if

$$R(A) = R(B) \tag{1 3}$$

where $R(A)$ is the range of A.

Two converses of (1.1) were obtained in [11]:

If A and B are complex Hermitian positive definite matrices and $0 \leq \lambda \leq 1$ is a real number, then

$$[\lambda A + (1-\lambda)B]^{-1} \geq K\left(\lambda A^{-1} + (1-\lambda)B^{-1}\right) \tag{1 4}$$

and

$$[\lambda A + (1-\lambda)B]^{-1} - \left(\lambda A^{-1} + (1-\lambda)B^{-1}\right) \geq \tilde{K} A^{-1} \tag{1.5}$$

where

$$K = 4 \min_i \frac{\mu_i}{(1+\mu_i)^2}, \quad \tilde{K} = \min_i \frac{\left(\sqrt{\mu_i} - 1\right)^2}{-\mu_i}, \tag{1 6a,b}$$

and the μ_i are the solutions of the equation

$$\det(B - \mu A) = 0. \tag{1 7}$$

In this note, we give analogous converses for (1 2), as well as some related results

2. CONVERSES OF THE MATRIX CONVEXITY INEQUALITY OF THE MOORE-PENROSE INVERSE

Let A and B be two complex Hermitian positive semi-definite matrices of the same order such that (1 3) holds Let P be a unitary matrix such that $A = P \operatorname{diag}(A_1, 0)P^*$ where A_1 is a diagonal positive definite matrix When (1 3) holds, we have $B = P \operatorname{diag}(B_1, 0)P^*$ where B_1 is positive definite

THEOREM 1. Let A and B be two complex Hermitian positive semi-definite matrices of the same order such that (1 3) holds and let $0 \leq \lambda \leq 1$ Then

$$[\lambda A + (1-\lambda)B]^{+} \geq K(\lambda A^{+} + (1-\lambda)B^{+}) \tag{2 1}$$

where K is defined by (1 6a) and the μ_i are the positive solutions of the equation

$$\det(B_1 - \lambda A_1) = 0. \tag{2 2}$$

THEOREM 2. Let A, B be defined as in Theorem 1 Then

$$[\lambda A + (1-\lambda)B^{+}] - (\lambda A^{+} + (1-\lambda)B^{+}) \geq \tilde{K} A^{+} \tag{2 3}$$

where $\tilde{K}$ is defined by (1 6b) and the μ_i are positive solutions of the equation (2 2)

PROOF. By (1 4) and (1 5) we have

$$[\lambda A_1 + (1-\lambda)B_1]^{-1} \geq K\big(\lambda A_1^{-1} + (1-\lambda)B_1^{-1}\big) \tag{2 4}$$

and

$$[\lambda A_1 + (1-\lambda)B_1]^{-1} - \big(\lambda A_1^{-1} + (1-\lambda)B_1^{-1}\big) \geq \tilde{K} A_1^{-1} \tag{2 5}$$

where K is defined by (1 6a), $\tilde{K}$ by (1 6b) and the μ_i are solutions of (2 2) Since $PA^{+}P^{*} = (PAP^{*})^{+}$, (2.1) follows from (2 4) and (2 3) from (2.5)

3. SOME RELATED RESULTS

Let (Y, B, μ) be a probability space and $A_y, y \in Y$ a collection of positive semi-definite matrices of the same order. Let $A_y = (a_{ijy})$, $1 \leq i, j \leq n$ and $y \in Y$ Assume that a_{ijy} as a function of y is measurable for every $1 \leq i, j \leq n$ The following results were proved in [9,10]

Suppose there exists a set $D \in B$ such that $\mu(D) = 1$ and $A_{y1}A_{y2} = A_{y2}A_{y1}$ for every $y_1, y_2 \in D$ Let $R(A_y)$ be the same for all $y \in D \in B$. Suppose A_y and A_y^{+} as functions of y are integrable with respect to μ Then

$$\left[\int_Y A_y \mu(dy)\right]^{+} \leq \int_Y A_y^{+} \mu(dy). \tag{3 1}$$

By $\int_Y A_y \mu(dy)$ we mean the matrix whose $(i,j)^{th}$ element is $\int_Y a_{ijy}\mu(dy)$.

THEOREM 3. If also all positive eigenvalues of A_y for all $y \in Y$ are in the interval $[m, M]$ where $0 < m < M$, then the following inequalities hold:

$$\int_Y A_y^{+} \mu(dy) \leq \frac{(M+m)^2}{4Mm}\left[\int_Y A_y \mu(dy)\right]^{+} \tag{3 2}$$

and

$$\int_Y A_y^{+} \mu(dy) - \left[\int_Y A_y \mu(dy)\right]^{+} \leq \frac{\left(\sqrt{M} - \sqrt{m}\right)^2}{Mm} I. \tag{3 3}$$

PROOF. As in [9], we have that there exists an orthogonal matrix C such that

$$C^T AC = \operatorname{diag}\{\lambda_{1y}, \lambda_{2y}, \ldots, \lambda_{ny}\}, \quad y \in Y$$

where $\lambda_{1y}, \lambda_{2y}, \ldots, \lambda_{ny}$ are the eigenvalues of A_y Since A_y is positive semi-definite, each $\lambda_{iy} \geq 0$. Let k be the rank of A_y We can assume without loss of generality that

$$\lambda_{1y}, \lambda_{2y}, \ldots, \lambda_{ky} \neq 0 \quad \text{for every} \quad y \in Y, \quad \text{and} \quad \lambda_{k+1,y} = \lambda_{k+2,y} = \ldots \lambda_{ny} = 0 \quad \text{for every} \quad y \in Y.$$

Note that

$$A_y^{+} = C \operatorname{diag}\left\{\frac{1}{\lambda_{1y}}, \frac{1}{\lambda_{2y}}, \ldots, \frac{1}{\lambda_{ky}}, 0, \ldots, 0\right\} C^T$$

so that

$$C^T A_y^+ C = \text{diag}\left\{\frac{1}{\lambda_{1y}}, \frac{1}{\lambda_{2y}}, \dots, \frac{1}{\lambda_{ky}}, 0, \dots, 0\right\}.$$

Thus, we have

$$K\left[\int_Y A_y\mu(dy)\right]^+ - \int_Y \lambda_y^+ \mu(dy) = C\,\text{diag}\left\{K\left(\int_Y \lambda_{1y}\mu(dy)\right)^{-1} - \int_Y \lambda_{1y}^{-1}\mu dy, \dots, K\left(\int_Y \lambda_{ky}\mu(dy)\right)^{-1} - \int_Y \lambda_{ky}^{-1}\mu(dy), 0, \dots, 0\right\}C^T$$

where $K = (M+m)^2/(4Mm)$ The inequality

$$K\left[\int_Y \lambda_{iy}\mu(dy)\right]^{-1} \int_Y \lambda_{iy}^{-1}\mu(dy)$$

is the well-known Kantorovich inequality Hence each diagonal element in the above diagonal matrix is nonnegative This completes the proof of (3 2)

Similarly,

$$\int_Y A_y^+ \mu(dy) - \left[\int_Y A_y\mu(dy)\right]^+ - \tilde{K}I = C\,\text{diag}\left\{\int_Y \lambda_{1y}^{-1}\mu(dy) - \left(\int_Y \lambda_{1y}\mu(dy)\right)^{-1} - \tilde{K}, \dots, \int \lambda_{ky}^{-1}\mu(dy) - \left(\int_Y \lambda_{ky}\mu(dy)\right)^{-1} - \tilde{K}, -\tilde{K}, \dots, -\tilde{K}\right\}C^T$$

where $\tilde{K} = \frac{\left(\sqrt{M}-\sqrt{m}\right)^2}{Mm}$ The inequality

$$\int_Y \lambda_{iy}^{-1}\mu(dy) - \int_Y \lambda_{iy}\mu(dy)^{-1} \le \tilde{K}$$

is a simple consequence of the following Mond-Shisha inequality [12]

$$\int f - \left(\int f^{-1}\right)^{-1} \le \left(\sqrt{M} - \sqrt{m}\right)^2$$

where $m \le f \le M$, $0 < m < M$. Namely

$$\frac{1}{M} \le \frac{1}{f} \le \frac{1}{m} \qquad \text{so that by substituting } f \to \frac{1}{f}, \text{ we get}$$

$$\int f^{-1} - \left(\int f\right)^{-1} \le \frac{\left(\sqrt{M}-\sqrt{m}\right)^2}{Mm} = \tilde{K}.$$

Thus each diagonal element in the above diagonal matrix is non-positive. This completes the proof

Moreover, we can consider the powers of A and A^+. For simplicity of notation, if $r < 0$, we shall use $A^{(r)}$ for $(A^+)^{-r}$. Note that $(A^+)^{-r} = (A^{-r})^+$

THEOREM 4. Let $R(A_y)$ be the same for all $y \in D \in B$. Suppose A_y^s and $A_y^{(r)}$, $(r < 0 < s)$ as functions of y are integrable with respect to μ Then

$$\left[\int_Y A_y^{(r)}\mu(dy)\right]^s \ge \left[\int_Y A_y^s\mu(dy)\right]^{(r)} \tag{3.4}$$

PROOF. As in the proof of (3.2) and (3 3), we have

$$\left[\int_Y A_y^{(r)}\mu(dy)\right]^s - \left[\int_Y A_y^s\mu(dy)\right]^{(r)} = C\,\text{diag}\left\{\left(\int_Y \lambda_{1y}^r\mu(dy)\right)^s - \left(\int_Y \lambda_{1y}^s\mu(dy)\right)^r, \dots, \left(\int_Y \lambda_k^r\mu(dy)\right)^s - \left(\int_Y \lambda_{ky}^s\mu(dy)\right)^r, 0, \dots, 0\right\}C^T.$$

Each diagonal element in the above diagonal matrix is nonnegative This follows from the fact that if f^s and f^r are positive and integrable, the well-known inequality for means of orders s and r states that

$$\left(\int f^r\right)^{1/r} \le \left(\int f^s\right)^{1/s} \quad (r < 0 < s) \tag{3 5}$$

which is the same as

$$\left(\int f^s\right)^r \le \left(\int f^r\right)^s .$$

Similar consequences of converse inequalities for (3 5) (see [12] and [13], respectively) are the next two theorems

THEOREM 5. Let the conditions of Theorem 4 be satisfied and let all positive eigenvalues of A_y for all $y \in Y$ belong to the interval $[m, M]$ $(0 < m < M)$ Then the following inequality holds

$$\left[\int_Y A_y^s \mu(dy)\right]^{(r)} \ge \Delta \left[\int_Y A_y^{(r)} \mu(dy)\right]^s \tag{3 6}$$

where

$$\Delta = \left\{\frac{r(\gamma^s - \gamma^r)}{(s-r)(\gamma^r - 1)}\right\}^r \left\{\frac{s(\gamma^r - \gamma^s)}{(r-s)(\gamma^s - 1)}\right\}^{-s}, \quad \gamma = M/m. \tag{3 7}$$

THEOREM 6. Let the conditions of Theorem 5 be satisfied Then

$$\left[\int_Y A_y^{(r)} \mu(dy)\right]^s - \left[\int_y A_y^s \mu(dy)\right]^{(r)} \le \Lambda I \tag{3 8}$$

where

$$\Lambda = \max_{\theta \in [0,1]} \{[\theta M^r + (1-\theta) m^r]^s - [\theta M^s + (1-\theta) m^s]^r\}.$$

Of course (3 2) and (3 3) are the special cases $r = -1$, $s = 1$ of (3 6) and (3 8)

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